AN M/M/1 QUEUEING SYSTEM WITH INTERRUPTED ARRIVAL STREAM

K.C. Madan, Department of Statistics, College of Science, Yarmouk University, Irbid, Jordan

ABSTRACT
We study an M/M/1 queueing system in which the arrival stream is 'shut off' from time to time. The time-dependent probability generating functions of the number in the system have been found. The corresponding steady state results have been derived. The mean number in the system as well as server's utilization time and idle time have been obtained explicity. In a particular case, some well-known results have been derived.

Key Words: interrupted arrival stream, variable batch arrivals, exponential service, probability generating function, steady state, utilization time, idle time, mean number in the system.

RESUMEN
Estudiaremos un sistema de colas M/M/1 en el que la corriente de arribos se desembaraza de vez en vez. Han sido halladas las funciones generatrices de probabilidades dependientes del tiempo. Los resultados correspondientes para el estado previo han sido derivados. El número promedio en el sistema así como del tiempo de la utilización del servidor y del tiempo ocioso son obtenidos explícítamente. Algunos resultados bien conocidos son derivados para un caso particular.

MSC 60K25.

1. INTRODUCTION

Queueing systems with vacations or service interruptions or breakdowns have been studied by numerous authors including Keilson and Servi [1986], Scholl and Kleinrock [1983], Cramer [1989], Shanthikumar [1988], Doshi [1986] and Madan [1992, 1995]. In this paper we investigate a queueing system in which there are no server vacations or breakdowns. Instead, the arrivals into the system are interrupted in the sense that the arrival stream gets 'shut off' from time to time. Such a situation could have definite effect on queue characteristics including queue length and server's utilization time. Some examples where such a situation could be encountered are the following. Aircrafts may stop landing at an airport for some time due to bad weather conditions. Branches of an assembly line could stop from time to time thereby blocking the input to the next branch. Flow of oil into or out of a refinery or flow of water into or out of a reservoir could experience random stoppages due to some reasons. Similarly flow of vehicles in a large network of a traffic system could stop for some time and then may become normal again and so on.

2. THE MATHEMATICAL MODEL

Customers arrive at the system in batches of variable size. Given that the arrival stream (input source) is 'on' at time t, let $\lambda c_i\, dt$ ($\lambda > 0$) be the first order probability that a batch size i customers arrives during the time interval (t, t+dt) where $0 < c_i < 1$ and $\sum_{i=1}^{\infty} c_i = 1$.

Customers are served one by one by the server with their service times having negative exponential distribution function $B(x) = 1 - e^{-\mu t}$, $t > 0$, where $\frac{1}{\mu}$ ($\mu > 0$) is the mean service time.

Given that the arrival stream is 'on' at time t, let ξdt ($\xi > 0$) be the first order probability that the arrival stream will be 'off' during the short interval of time (t, t+dt).

Given that the arrival stream is 'on' at time t, let ηdt ($\eta > 0$) be the first order probability that the arrival stream will be 'on' during the short interval of time (t, t+dt).

Further, we assume that the various stochastic processes involved in the system are independent of each other.

3. DEFINITIONS AND SYSTEM EQUATIONS

Let $p_n(t)$, ($n \geq 0$) be the probability that at time t there are n units in the system including one in service, if any, and the input source is 'on' and let $q_n(t)$, ($n \geq 0$) be the probability that at time t there are n units in the system including one in service, if any and the input source is 'off'.

The system is governed by the following set of forward differential-difference equations

$$\frac{d}{dt} p_n(t) + (\lambda + \mu + \xi)p_n(t) = \sum_{i=1}^{n} \lambda c_i p_{n-i}(t) + \mu p_{n+1}(t) + \eta q_n(t) \qquad (n \geq 0) \qquad (1)$$

$$\frac{d}{dt} p_0(t) + (\lambda + \xi)p_0(t) = \mu p_1(t) + \eta q_0(t) \qquad (2)$$

$$\frac{d}{dt} q_n(t) + (\mu + \eta)q_n(t) = \mu q_{n+1}(t) + \xi p_n(t) \qquad (n \geq 0) \qquad (3)$$

$$\frac{d}{dt} q_0(t) + \eta q_0(t) = \mu q_1(t) + \xi p_0(t) \qquad (4)$$

4. THE GENERATING FUNCTIONS OF THE NUMBER IN THE SYSTEM

We define $p(z,t) = \sum_{0}^{\infty} p_n(t)z^n$ and $q(z,t) = \sum_{0}^{\infty} q_n(t)z^n$ as the probability generating functions of the numbers in the system when the input source is 'on' and 'off' respectively. Let $c(z) = \sum_{i=1}^{\infty} c_i z^i$ denote the probability generating function for the sequence c_i of arrivals.

Let us assume that the system starts when the input source is 'on' and there are j units in the system, so that the initial condition is

$p_n(0) = \delta_{n,j}$ where δ_{nj} is the Kronecker's delta. (5)

Taking Laplace transform of equations (1) through (4) and using initial condition (5), we have

$$(s + \lambda + \mu + \xi)p_n(s) = \delta_{n,j} + \sum_{i=1}^{n} \lambda c_i p_{n-i}(s) + \mu p_{n+1}(s) + \eta q_n(s) \qquad (n \geq 0) \qquad (6)$$

$$(s + \lambda + \xi)p_0(s) = \mu p_1(s) + \eta q_0(s) \qquad (7)$$

$$(s + \mu + \eta)q_n(s) = \mu q_{n+1}(s) + \xi p_n(s) \qquad (n \geq 0) \qquad (8)$$

$$(s + \eta)q_0(s) = \mu q_1(s) + \xi p_0(s) \qquad (9)$$

Multiplying equations (6) through (9) by suitable powers of z yields

$$[(s + \lambda + \mu + \xi)z - \mu - \lambda c(z)z]p(z,s) = z^{j+1} + \mu(z - 1)p_0 + \eta z q(z,s) \tag{10}$$

$$[(s + \mu + \eta)z - \mu]q(z) = \xi z p(z,s) + \mu(z - 1)q_0(s) \tag{11}$$

Using equation (11) in (10), we obtain

$$p(z,s) = \frac{[z^{j+1} + \mu(z-1)p_0(s)][(s+\mu+\eta)z-\mu] + \mu\eta z(z-1)q_0(s)}{[(s+\mu+\eta)z-\mu][(s+\lambda+\mu+\xi)z-\mu-\lambda c(z)z] - \xi\eta z^2} \tag{12}$$

Further, equation (11) can be written as

$$q(z,s) = \frac{\xi z p(z,s) + \mu(z-1)q_0(s)}{(s+\mu+\eta)z-\mu} \tag{13}$$

Now for z = 1, equations (12) and (13) respectively yield

$$p(1,s) = \frac{s+\eta}{(s+\eta)(s+\xi)-\xi\eta} = \frac{s+\eta}{s(s+\xi+\eta)} \tag{14}$$

$$q(1,s) = \frac{\xi p(1,s)}{(s+\eta)} = \frac{\xi}{s(s+\xi+\eta)} \tag{15}$$

We note that $p(1,s) + q(1,s) = \frac{1}{s}$ as it should be.

Inverting the transforms (14) and (15), we obtain the respective probabilities that at time t the input source is 'on' and 'off' as follows

$$p(t) = \frac{\eta}{\xi+\eta} = \frac{\xi}{\xi+\eta}e^{-(\xi+\eta)t} \tag{16}$$

$$q(t) = \frac{\xi}{\xi+\eta} = \frac{\xi}{\xi+\eta}e^{-(\xi+\eta)t} \tag{17}$$

Obviously, p(t) + q(t) = 1 as it should be.

The denominator of the right hand side of equation (12) has 2 zeroes inside the unit circle |z| = 1. (For a proof, one can use Rouche's theorem, see Madan [4]). Let these two zeroes be designated as z_1 and z_2. Then since p(z,s) is regular inside |z| = 1, the numerator of the right hand side of equation (12) must vanish for each of these zeroes, thus yielding the following two equations

$$[z_1^{j+1} + \mu(z_1 - 1)p_0(s)]\,[(s + \mu + \eta)z_1 - \mu] + \mu\eta z_1(z_1 - 1)q_0(s) = 0 \tag{18}$$

$$[z_2^{j+1} + \mu(z_2 - 1)p_0(s)]\,[(s + \mu + \eta)z_2 - \mu] + \mu\eta z_2(z_2 - 1)q_0(s) = 0 \tag{19}$$

The unknowns $p_0(s)$ and $q_0(s)$ can be determined by solving equations (18) and (19) simultaneously. Thus p(z) and q(z) can be completely determined.

5. STEADY STATE SOLUTION

Let p_n and q_n define the steady state probabilities corresponding to the time dependent probabilities $p_n(t)$ and $q_n(t)$ defined above and let p(z) and q(z) denote the respective steady state probability generating functions for the number in the system. Then to derive the steady state results we apply the well-known Tauberian property $\lim_{s\to 0} sf(s) = \lim_{t\to\infty} f(t)$ to equations (12) through (15) and obtain

$$p(z) = \frac{\mu(z-1)[\mu(z-1)+\eta z]p_0 + \mu\eta z(z-1)q_0}{[\mu(z-1)+\eta z][\lambda+\mu+\xi)z-\mu-\lambda c(z)z]-\xi\eta z^2} \tag{20}$$

$$q(z) = \frac{\xi z p(z) + \mu(z-1)q_0}{\mu(z-1)+\eta z} \tag{21}$$

At z = 1 equations (20) is indeterminate of the 0/0 form. Therefore, we use L'Hopital's rule to find limit of p(z) as $z \to 1$. Thus we have

$$p(1) = \frac{\mu\eta(p_0+q_0)}{\mu(\xi+\eta)-\eta\lambda E(I)} \tag{22}$$

$$q(1) = \frac{\mu\xi(p_0+q_0)}{\mu(\xi+\eta)-\eta\lambda E(I)} \quad \text{with} \quad \lambda E(I) < \frac{\mu(\xi+\eta)}{\eta} \tag{23}$$

where $c'(1) = \sum_{i=1}^{\infty} ic_i = E(I)$ is the mean batch size of arriving customers.

Equations (22) and (23) respectively give the probabilities that the arrival stream is 'on' or 'off '.

Since p(1) + q(1) = 1, adding (22) and (23), we obtain

$$p_0 + q_0 = \frac{\mu(\xi+\eta)-\eta\lambda E(I)}{\mu(\xi+\eta)} \quad \text{with} \quad \lambda E(I) < \frac{\mu(\xi+\eta)}{\eta} \tag{24}$$

Since equation (24) gives the probability that the server is idle, no matter whether the arrival steam is 'on' or 'off ', we can find its complement to get the system's utilization factor as

$$\rho = \frac{\lambda\eta E(I)}{\mu(\xi+\eta)} \tag{25}$$

We will continue the analysis for the case of single arrivals. In that case $c_1 = 1$ and $c_i = 0$ for $i \neq 1$. Consequently c(z) = z. Then the denominator of the right side of equation (20) can be written as

$[-\lambda(\mu + \eta)z^3 + (\mu^2 + 2\lambda\mu + \lambda\eta + \eta\mu + \mu\xi)z^2 - \mu(\lambda + \xi + \eta + 2\mu)z + \mu^2]$ which can be factored as $(z-1)\,[-\lambda(\mu + \eta)\, z^2$ +
$\mu(\lambda + \mu + \xi + \eta)z - \mu^2]$ so that now the factor (z - 1) can be canceled out with that of the numerator. Therefore, the expression in (20) can be re-written as

$$p(z) = \frac{\mu[\mu(z-1)+\eta z]p_0 + \mu\eta z q_0}{-\lambda(\mu+\eta)z^2 + \mu(\lambda+\mu+\xi+\eta)z - u^2} \tag{26}$$

Now, the denominator of the right hand side of equation (26) has two zeroes given by $\frac{\mu(\lambda+\mu+\xi+\eta)\mp\mu\sqrt{(\lambda+\mu+\xi+\eta)^2-4\lambda(\mu+\eta)}}{2\lambda(\mu+\eta)}$. One of these zeroes is inside the unit circle |z| = 1. Let this zero be denotes as z*. The denominator of the right hand side of (26) must vanish for this zero, giving

$$q_0 = \frac{[\mu-(\mu+\eta)z^*]p_0}{\eta z^*} \tag{27}$$

Since in the case of single arrivals E(I) = 1, equation (24) becomes

$$p_0 + q_0 = \frac{\mu(\xi+\eta)-\lambda\eta}{\mu(\xi+\eta)} \quad \text{where } \lambda < \frac{\mu(\xi+\eta)}{\eta} \tag{28}$$

Solving (27) and (28), we obtain

$$p_0 = \frac{[\mu(\xi+\eta)-\lambda\eta]\eta z^*}{\mu^2(\xi+\eta)(1-z^*)} \tag{29}$$

which is the steady state probability that the server is idle, even though the arrival stream is 'on'.

$$q_0 = \left[\frac{\mu(\xi+\eta)-\lambda\eta]}{\mu(\xi+\eta)}\right]\left[\frac{\mu-(\mu+\eta)z^*}{\mu(1-z^*)}\right] \tag{30}$$

which is the probability that the server is idle when the arrival stream is 'off '.

6. THE MEAN NUMBER IN THE SYSTEM

Let r(z) = p(z) + q(z) be the probability generating function of the number in the system no matter whether the arrival stream is 'on' or 'off ', where p(z) and q(z) are given by equations (26) and (21) respectively. Then the mean number, L, in the system is given by $L = \frac{d}{dz}r(z)$ at z = 1. Carrying out the computations and simplifying, we have

$$L = \left[\frac{(\lambda(\mu+\eta)-\mu^2)\mu\eta q_0 + \mu(\xi\mu^2+\lambda\eta^2)p_0}{(\mu(\xi+\eta)-\lambda\eta)^2}\right]\left[1+\frac{\xi}{\eta}\right] - \frac{\xi\mu}{\eta(\xi+\eta)} \tag{31}$$

7. A PARTICULAR CASE

If we assume $\xi = 0$, which means that there are no interrumptions in the arrival stream, then the denominator of the right side of (26) becomes $-\lambda(\mu+\eta)z^2 + \mu(\lambda+\mu+\eta) - \mu^2$. It is easy to see that one zero of this polynomial which is inside the unit circle |z| = 1 is $\frac{\mu}{\mu+\eta}$. Setting $z^* = \frac{\mu}{\mu+\eta}$ in equation (30) we get $q_0 = 0$ as it should be and the (29) gives $p_0 = 1 - \frac{\lambda}{\mu}$ which is a well-known result of the queueing system M/M/1. Further, with $\xi = 0$, $q_0 = 0$ and $p_0 = 1 - \frac{\lambda}{\mu}$, equation (26) becomes

$$p(z)=\frac{\mu[\mu(z-1)\eta z]\left[1-\frac{\lambda}{\mu}\right]}{-\lambda(\mu+\eta)z^2+\mu(\lambda+\mu+\eta)z-\mu^2} \tag{32}$$

In we divide the numerator and the denominator of the right side of equation (32) by η and take limit as $\frac{1}{\eta}\to 0$ and simplify, we obtain

$$p(z)=\frac{\mu\left(1-\frac{\lambda}{\mu}\right)}{\mu-\lambda z} \tag{33}$$

which is again a well-known result giving the probability generating function of the number in the M/M/1 queueing system.

Again, letting $\xi = 0$ and using $q_0 = 0$ and $p_0 = 1 - \frac{\lambda}{\mu}$ in equation (31), we have the mean queue length in this particular case as

$L = \frac{\lambda}{\mu-\lambda}$ which is also a well-known result. (34)

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