

A NOTE ON THE DUAL DESCRIPTION OF PROJECTED POLYTOPES

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ABSTRACT

The inequalities which describe the projection Q of a given polytope P onto a subspace are usually generated by an elimination procedure of Fourier-Motzkin type. In this note we give a dual approach for the description of Q . In fact, the vertices of a dual polytope serve as indices for the describing inequalities. Moreover we show how the redundancy of inequalities is connected with the existence of Slater points in the images of a set-valued mapping.

Key words: polytope, linear programming, duality.

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RESUMEN

Las desigualdades que describen la proyección Q de un polítopo dado sobre un subespacio son usualmente generados por el procedimiento de eliminación del tipo Fourier-Motzkin. En esta nota damos un enfoque dual para la descripción de Q . De hecho, los vértices de un polítopo dual sirven como índices para describir desigualdades. Más aún demostramos que la redundancia de las desigualdades están conectadas con la existencia de puntos de Slater en las imágenes de una aplicación conjuntamente evaluadas.

Palabras clave: polítopos, programación lineal, dualidad.

1. INTRODUCTION

For $m, n, p \in \mathbb{N}$ consider an arbitrary function $a: \mathbb{R}^n \rightarrow \mathbb{R}^p$, a $(p \times m)$ – matrix B and the set

$$P = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid a(x) + By \geq 0\},$$

where the inequality is to be understood componentwise. We are interested in a description of the orthogonal projection $\pi(P)$ of P onto the first n variables, i.e. $Q = \pi(P) \subset \mathbb{R}^n$. In contrast to the standard procedure of Fourier-Motzkin type elimination (cf., e.g., Stoer and Witzgall (1970), Telgen (1982) and the references therein) we give a dual description of Q in this note. Unlike the dual approach in Dantzig and Eaves (1973) our description of Q is explicit up to the determination of certain vertices of a dual polytope, as it is shown in Theorems 3 and 7. Our use of set-valued mappings enables us to give a sufficient condition for redundancy of certain vertices in terms of the Slater condition. Moreover, our approach carries over to the case where P is a polytope and π is some arbitrary projection operator, which we study in Theorem 9.

2. THE DUAL DESCRIPTION

Since P is the graph of the set-valued mapping

$$\Gamma: \mathbb{R}^n \rightarrow 2^{\mathbb{R}^m}, x \mapsto \{y \in \mathbb{R}^m \mid a(x) + By \geq 0\},$$

we have $x \in Q$ if and only if $\Gamma(x) \neq \emptyset$. Note that the images of Γ are polyhedra. The following assumption is supposed to hold throughout this note:

Assumption 1 The set-valued mapping Γ has bounded images.

For fixed x we now consider the following (primal) optimization problem:

$$P(x): \max_{(y, z)} z \quad \text{s.t.} \quad a(x) + By \geq z \cdot e,$$

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where $e = (1, \dots, 1)^T \in \mathbb{R}^p$, and z is a scalar. Its dual problem is

$$D(x): \max_{\mu} \mu^T a(x) \quad \text{s.t.} \quad B^T \mu = 0, e^T \mu = 1, \mu \geq 0.$$

Now let

$$Z_P(x) = \{(y, x) \in \mathbb{R}^m \times \mathbb{R}^1 \mid a(x) + By \geq ze\}$$

and

$$Z_D = \{\mu \in \mathbb{R}^p \mid B^T \mu = 0, e^T \mu = 1, \mu \geq 0\}$$

denote the feasible sets of $P(x)$ and $D(x)$, respectively. Observe that Z_D neither depends on x nor on y .

Lemma 1. The following assertions hold:

- (i) Z_D is non-empty and bounded.
- (ii) For each $\bar{x} \in \mathbb{R}^n$, both $P(\bar{x})$ and $D(\bar{x})$ are solvable.

Proof. In assertion (i), Z_D is bounded as a subset of the standard simplex. Assume that Z_D is empty. Then, by the duality theorem of linear programming, for given $\bar{x} \in \mathbb{R}^n$ either $Z_P(\bar{x})$ is empty or the objective function z of $P(\bar{x})$ is not bounded from above on $Z_P(\bar{x})$. As the set $Z_P(\bar{x})$ always contains the point $(y, z) = (0, \min_{1 \leq i \leq p} a_i(\bar{x}))$, it is non-empty. Consequently, there is a sequence (y^v, z^v) with

$$a(\bar{x}) + By^v \geq z^v e \tag{1}$$

and $z^v \rightarrow +\infty$. For some $v_0 \in \mathbb{N}$ and all $v \geq v_0$ we have $y^v \in \Gamma(\bar{x})$ so that, without loss of generality, $y^v \rightarrow \bar{y} \in \Gamma(\bar{x})$ by the compactness of $\Gamma(\bar{x})$. However, then (1) cannot hold for arbitrarily large $v \in \mathbb{N}$. Contradiction. Assertion (ii) follows from (i) in virtue of the duality theorem. ■

Let V denote the vertex set of Z_D , i.e. the set of points in Z_D where p of the active constraints are linearly independent.

Lemma 2. For each $\bar{x} \in \mathbb{R}^n$ the following are equivalent:

- (i) $\Gamma(\bar{x})$ is non-empty.
- (ii) $\min_{\mu \in V} \mu^T a(\bar{x}) \geq 0$.

Proof. The set $\Gamma(\bar{x})$ is non-empty if and only if there exists a point $(y, z) \in Z_P(\bar{x})$ with $z \geq 0$. As $P(\bar{x})$ is solvable by Lemma 1(ii), the latter holds if and only if the optimal value of $P(\bar{x})$ is non-negative. The duality theorem now implies the equivalence of (i) and

$$\min_{\mu \in Z_D} \mu^T a(\bar{x}) \geq 0. \tag{2}$$

By the vertex theorem of linear programming, (2) implies (ii). Since $D(\bar{x})$ is solvable by Lemma 1(ii), assertion (ii) also implies (2). ■

As we have $Q = \{x \in \mathbb{R}^n \mid \Gamma(x) \neq \emptyset\}$, the next assertion follows immediately from Lemma 2:

Theorem 3. For the set $Q = \pi(P)$ the following description holds:

$$Q = \bigcap_{\mu \in V} \{x \in \mathbb{R}^n \mid \mu^T a(x) \geq 0\}. \quad (3)$$

Theorem 3 gives a description of Q by finitely many inequalities, where V serves as the finite index set. Observe that the defining functions for the inequalities are linear combinations of the functions a_i , $1 \leq i \leq p$. In order to obtain an explicit description of Q , a vertex enumeration algorithm can be applied to the polytope Z_D , Avis and Fukuda (1992). A similar description of Q is given in Nemhauser and Wolsey (1988), but there the function a is supposed to be affine linear. Moreover, our approach via a set-valued mapping gives rise to a sufficient criterion for redundant constraints, which we investigate in the next section.

3. REDUNDANT CONSTRAINTS

Like in the Fourier-Motzkin elimination procedure, some of the inequalities corresponding to vertices $\mu \in V$ may be redundant for the description of Q . We will not dwell on a minimal representation of Q in this note Balas (1998) and Telgen (1982), but we only give a sufficient condition for redundancy in this section.

Definition 4. For given $\bar{x} \in \mathbb{R}^n$ we call \bar{y} a Slater point of $\Gamma(\bar{x})$ if $a(\bar{x}) + B\bar{y} > 0$.

Lemma 5. For each $\bar{x} \in \mathbb{R}^n$ the following are equivalent:

- (i) $\Gamma(\bar{x})$ possesses a Slater point.
- (ii) $\min_{\mu \in V} \mu^T a(\bar{x}) > 0$.

Proof. The set $\Gamma(\bar{x})$ possesses a Slater point if and only if there exists a point $(y, z) \in Z_p(\bar{x})$ with $z > 0$. The equivalence of (i) and (ii) now follows with the same arguments as in the proof of Lemma 2. ■

Lemmas 2 and 5 immediately yield:

Corollary 6 . For each $\bar{x} \in \mathbb{R}^n$ the following are equivalent:

- (i) $\Gamma(\bar{x})$ is non-empty and does not possess a Slater point.
- (ii) $\min_{\mu \in V} \mu^T a(\bar{x}) = 0$.

Subsequently let the following assumption hold:

Assumption 2. The function a is lower semi-continuous.

For given $\bar{x} \in Q$ define the active index set

$$V_0(\bar{x}) = \{\mu \in V \mid \mu^T a(\bar{x}) = 0\}.$$

If $V_0(\bar{x})$ is non-empty, then in a neighborhood of (\bar{x}) the set Q coincides with the set

$$Q_{(\bar{x})} = \bigcap_{\mu \in V_0(\bar{x})} \{x \in \mathbb{R}^n \mid \mu^T a(x) \geq 0\}.$$

in view of Assumption 2. In the case when $V_0(\bar{x})$ is empty, \bar{x} lies in the topological interior of Q . Consequently, at least the vertices $\mu \in V$ which do not belong to

$$\tilde{V} = \bigcup_{x \in Q} V_0(x),$$

are redundant for the description of Q . To be more precise, they are strongly redundant in the sense that they are not active at any point in Q ([6]). In view of Lemma 2 and Corollary 6, a point $\bar{x} \in Q$ possesses a non-empty active index set $V_0(\bar{x})$ if and only if $\Gamma(\bar{x})$ is non-empty and does not possess a Slater point. This proves the following result.

Theorem 7. In the characterization (3) the set V can be replaced by its subset

$$\tilde{V} = \bigcup_{x \in \tilde{Q}} V_0(x),$$

where

$$\tilde{Q} = \{x \in \mathbb{R}^n \mid \Gamma(x) \text{ is non-empty and does not possess a Slater point}\}.$$

The following example shows that \tilde{V} may be a proper subset of V .

Example 8.

For $m = 1$, $n = 2$, $p = 3$ put $a(x) = \left(-x_1^2 - x_2^2, -\frac{1}{2}x_1, 1\right)^T$ and $B = (1, 1, -1)^T$, i.e. we consider

$$P = \{(x, y) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \leq y \leq 1, x_1 \leq 2y\}.$$

Assumptions 1 and 2 are clearly satisfied. After a short calculation we obtain

$$Z_D = \left\{ \left(\lambda, \frac{1}{2} - \lambda, \frac{1}{2} \right)^T, \lambda \in \left[0, \frac{1}{2} \right] \right\}$$

and

$$V = \left\{ \frac{1}{2}(0, 1, 1)^T, \frac{1}{2}(1, 0, 1)^T \right\},$$

so that (3) yields

$$\begin{aligned} Q &= \{x \in \mathbb{R}^2 \mid (0, 1, 1)a(x) \geq 0, (1, 0, 1)a(x) \geq 0\} \\ &= \{x \in \mathbb{R}^2 \mid x_1 \leq 2, x_1^2 + x_2^2 \leq 1\}. \end{aligned}$$

Obviously the first inequality, corresponding to the vertex $\frac{1}{2}(0, 1, 1)^T \in V \setminus \tilde{V}$, is strongly redundant for the description of Q .

4. ARBITRARY PROJECTION OPERATORS

The results of Theorems 3 and 7 hold particularly in the case where the functions $a_i(x) = c_i^T x + d_i$, $1 \leq i \leq p$, are affine-linear and P is bounded, i.e. when P is a polytope.

Assumptions 1 and 2 are then clearly satisfied. Moreover, in this case our method works for any projection operator after a suitable linear change of coordinates. An endomorphism $\sigma : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is called projection operator $\sigma \circ \sigma = \sigma$. We denote by $\text{Im } \sigma$ and $\text{Ker } \sigma$ the image and the kernel of σ , respectively.

Theorem 9. Let σ be a projection operator on \mathbb{R}^N , and let

$$P = \{u \in \mathbb{R}^N \mid \Phi u + \varphi \geq 0\}$$

be a polytope with a $(p \times N)$ – matrix Φ and $\varphi \in \mathbb{R}^p$. Then we have:

$$\sigma(P) = \bigcap_{\mu \in V} \{\xi \in \text{Im } \sigma \mid \mu^T(\Phi\xi + \varphi) \geq 0\},$$

where V denotes the vertex set of the polytope

$$Z_D = \{\mu \in \mathbb{R}^p \mid \Phi^T \mu \in (\text{Ker } \sigma)^\perp, e^T \mu = 1, \mu \geq 0\}.$$

Proof. Let the columns of the $(N \times n)$ – matrix T_1 and the columns of the $(N \times m)$ – matrix T_2 form a basis of $\text{Im } \sigma$ and $\text{Ker } \sigma$, respectively. The columns of T_1 are eigenvectors of σ to the eigenvalue 1, whereas the columns of T_2 are eigenvectors to the eigenvalue 0. As $\mathbb{R}^N = \text{Im } \sigma \oplus \text{Ker } \sigma$, we have $N = n + m$, and the columns of the matrix $T = (T_1, T_2)$ form a basis of \mathbb{R}^N . Hence, σ is diagonalizable with $\sigma \circ T = T \circ \pi$. Here π

possesses the matrix representations $\begin{pmatrix} E_n & 0 \\ 0 & 0_m \end{pmatrix}$, where E_n and 0_m denote the $(n \times n)$ – identity and the

$(m \times m)$ – zero matrix, respectively. Next, it is easily seen that $\sigma(P) = T(\pi(P))$, where

$$P = \{v \in \mathbb{R}^N \mid (\Phi T)v + \varphi \geq 0\}.$$

P is again a polytope (cf., e.g., Ziegler (1995)), and we can apply Theorem 3 with $x = (v_1, \dots, v_n)$, $y = (v_{n+1}, \dots, v_{n+m})$, $a(x) = \Phi T_1 x + \varphi$, and $B = \Phi T_2$, in order to determine $Q \times \{0\} = \pi(P)$. We arrive at

$$\begin{aligned} \sigma(P) &= T(\pi(P)) = T_1(Q) = \bigcap_{\mu \in V} \{T_1 x \mid x \in \mathbb{R}^N, \mu^T(\Phi T_1 x + \varphi) \geq 0\} \\ &= \bigcap_{\mu \in V} \{\xi \in \text{Im } \sigma \mid \mu^T(\Phi\xi + \varphi) \geq 0\}, \end{aligned}$$

where V denotes the vertex set of

$$Z_D = \{\mu \in \mathbb{R}^N \mid T_2^T \Phi^T \mu = 0, e^T \mu = 1, \mu \geq 0\}.$$

This shows the assertion. ■

5. FINAL REMARKS

An algorithmic implementation of the description for Q in Dantzig and Eaves (1973) obviously relies on an efficient vertex enumeration method for the determination of the vertex set V of the polytope Z_D . Here, the reverse search algorithm of Avis and Fukuda (1992) can be applied, preferably in a version adapted to the special structure of Z_D .

This implementation as well as a comparison of our dual description method and Fourier-Motzkin-type methods with respect to efficiency and generation, respectively detection, of redundant constraints is beyond the scope of this note and will be subject of future research.

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