A NEW ALGORITHM TO COMPUTE THE EUCLIDEAN DISTANCE FROM A POINT TO A CONIC

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ABSTRACT
In this paper a new algorithm to compute the Euclidean distance from a point to a conic is presented. This algorithm provides good approximations for the Euclidean distance, even when the point is not very close to the given conic. Furthermore, the approximations may be improved iteratively to attain a prescribed accuracy. Unlike the most commonly known methods to approximate the Euclidean distance, in the proposed method the coordinates of the footpoint for the orthogonal projection of the point on the conic are computed. This particular feature permits to obtain a noteworthy accuracy without increasing too much the computational cost.

Key words: Conics, Euclidean distance from a point to a conic

RESUMEN
En este trabajo se presenta un nuevo algoritmo para calcular la distancia Euclideana de un punto a una cónica. Este algoritmo proporciona una buena aproximación incluso cuando el punto no se encuentra muy próximo a la cónica. Adicionalmente, la aproximación se puede mejorar de forma iterativa hasta alcanzar la precisión deseada. A diferencia de los métodos más conocidos, el método propuesto calcula las coordenadas de la proyección ortogonal del punto externo sobre la cónica. Esto nos permite obtener una notable precisión sin incrementar sustancialmente el costo computacional del algoritmo.

Palabras clave: cónicas, distancia Euclidea desde un punto hasta una cónica.

MSC: 65Y25, 51N35.

1. INTRODUCTION. PREVIOUS WORK

In this paper we study the problem of computing the Euclidean distance from a point on the plane to an arbitrary conic. This subject have been intensively treated in the literature (see [Bookstein (1979), Paulidis (1983), Sampson (1982) and Taubin (1994)]). However, most of the existing methods avoid the computation of the coordinates of the footpoint of the orthogonal projection on the conic, so that there is no control on the accuracy of the obtained approximate distance. Moreover, instead of the exact Euclidean distance a "suitable" approximation is computed, which usually happens to be a good approximation to the Euclidean distance only if the point is very close to the conic. Nevertheless, in some practical problems of computer graphics and vision, 2D robot path planning, pattern recognition and computational mechanics, it is necessary to compute the footpoint coordinates and we may not assume that the point is very close to the conic.

A new algorithm to compute the Euclidean distance from a point to a conic is presented here. This algorithm provides a good approximation for the Euclidean distance, even when the point is far from the given conic. Furthermore, the approximation may be improved iteratively to attain a prescribed accuracy without increasing too much the computational cost.

Let be C a conic with implicit equation

$$f(x,y) = a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00} = 0 \qquad (1)$$

and $q = (x_0, y_0)$ a point on the plane not on C.

By definition, the Euclidean distance from q to C, d(q,C), is given by

$$d(q, C) = \min \{||q - p||:f(p) = 0\} \qquad (2)$$

Thus, to compute the Euclidean distance we have to solve a constrained nonlinear minimization problem. More geometrically, the Euclidean distance from q to C is attained at a point p on C such that the normal of C at p passes through q. Two different approaches to the solution of the problem have been considered, depending on which representation of the conic is used.

1.1 Implicit approach

If the conic is represented by its implicit equation (1) then the coordinates (x,y) of the footpoint p and the Euclidean distance d may be computed as the solution of the following nonlinear system of polynomial equations,

$$\begin{cases} f_1(x,y,d) := (x-x_0)^2 + (y-y_0)^2 - d^2 = 0 \\ f_2(x,y,d) := f(x,y) = 0 \\ f_3(x,y,d) := \dfrac{\partial f}{\partial y}(p)(x-x_0) - \dfrac{\partial f}{\partial x}(p)(y-y_0) = 0 \end{cases} \tag{3}$$

To be more precise, p and d are solutions of (3), but this system may have four solutions (depending on the position of q up to four normals to the conic may pass through q) and we are interested in that solution which gives us the global minimum (2). Thus, to compute the actual coordinates of footpoint p and the Euclidean distance d, we need a good initial approximation of them. Up to the moment, general methods for estimating such good initial approximations for x,y and d are not reported in the literature. Alternatively, some other approaches have been studied.

Using elimination theory Kriegman and Ponce [1990] and Ponce **et al.** [1992] eliminate the variables x and y (whose initial approximations are more difficult to estimate) and obtain a single polynomial equation on d, $\Phi(d) = 0$, whose minimal positive root d*, is the Euclidean distance from q to C. Unfortunately, the coefficients of $\Phi(d)$ are complicated polynomial expressions in the coefficients of f and in the coordinates of the external point q, even when the conic have been reduced to the canonical form. In the Annex we show the expression of $\Phi(d)$ in this particular case.

Hence, the computation of the coefficients of $\Phi(d)$ may be expensive and the problem of finding its roots, using floating point arithmetic, may be numerically unstable.

To overcome these difficulties, other approximations of the Euclidean distance have been considered. The simplest is the algebraic distance, $d_a(q,C)$ given by

$$d_a(q,C) = |f(x_0, y_0)| \tag{4}$$

To compute the algebraic distance is very cheap, but it is a poor approximation of the Euclidean distance. More recently, G.Taubin introduced in [Tau2] several approximations of the Euclidean distance from a point to an implicit curve f(x,y) = 0, if the function f(x,y) has continuous partial derivatives in a neighborhood of q.

Taubin's approximate distance of first order, δ_1 is given by,

$$\delta_1 = \frac{|F_0|}{\|F_1\|} = \frac{|f(x_0, y_0)|}{\|\nabla f(x_0, y_0)\|} \tag{5}$$

while Taubin's approximate distance of second order, δ_2, is the unique positive root of the quadratic polynomial,

$$F_q^2(\delta) = |F_0| - \|F_1\|\,\delta - \|F_2\|\,\delta^2$$

where

$$F_0 = f(x_o, y_o)$$

$$F_1 = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0)\right)$$

and

$$F_2 = (\frac{\partial^2 f}{\partial x^2}(x_0, y_0), \frac{1}{2}\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0), \frac{\partial^2 f}{\partial y^2}(x_0, y_0))$$

Hence,

$$\delta_2 = \frac{-\|F_1\| + \sqrt{\|F_1\|^2 + 4|F_0|\ \|F_2\|}}{2\|F_2\|} \tag{6}$$

All previous mentioned approximations of the Euclidean distance are given by closed expressions. Moreover, their computation is cheap. Nevertheless, they have the following disadvantages:

1. They give us a good approximation of the Euclidean distance, only when the external point is very close to the conic.

2. They do not provide us the coordinates of the normal footpoint. Hence, we may not improve the accuracy of the computed distance solving (3), because we don't have an initial approximation of the footpoint.

In our previous paper [Hernández et.al.(1997)] we have proposed a method to compute the Euclidean distance from a point to a conic given in implicit form, which is inspired in the method developed by Sampson in [1982]. This method is computationally cheap and provides the coordinates of the footpoint. Hence, we may improve the accuracy of the approximation of the Euclidean distance solving the nonlinear system (8). However, if the point is not close to the conic, we can not assure that our method converges to the orthogonal projection which realizes the Euclidean distance.

1.2 Parametric approach

An alternative solution to the problem of computing the Euclidean distance from a point to a conic, is to use a parametric representation of the conic.

We may represent the conic by two rational polynomials

$$\begin{cases} x(t) = \dfrac{p(t)}{r(t)} \\[2ex] y(t) = \dfrac{q(t)}{r(t)} \end{cases}$$

where p,q, and r are polynomials of degree two. Then, the implicit equation of the normal to the conic at the point $(x(t_p), y(t_p))$ is,

$$x'(t_p)\,(x - x(t_p)) + y'(t_p)\,(y - y(t_p)) = 0$$

Thus, according to the definition of the Euclidean distance, we have to compute a point $p = (x(t_p), y(t_p))$ such that the normal to C at p passes through the external point $q = (x_0,y_0)$ i.e.,

$$\Gamma(t) = x'(t_p)\,(x_0 - x(t_p)) + y'(t_p)\,(y_0 - y(t_p)) = 0 \tag{7}$$

Since $x(t_p)$ and $y(t_p)$ are rational functions with the same denominator, (7) is a polynomial equation of degree four in t_p. The coefficients of $\Gamma(t)$ are complicated polynomial functions of the coefficients of x(t) and y(t) and the coordinates of the external point (x_0, y_0). For instance, if we use the Bezier representation of a conic with control polygon $b_0 = (b_{0x}, b_{0y})$, $b_1 = (b_{1x}, b_{1y})$, $b_2 = (b_{2x}, b_{2y})$ and $w_1 \in R$ [Far]

$$x(t) = \frac{b_{0x}(1-t)^2 + 2w_1 b_{1x} t(1-t) + b_{2x} t^2}{(1-t)^2 + 2w_1 t(1-t) + t^2}$$

$$y(t) = \frac{b_{0y}(1-t)^2 + 2w_1 b_{1y} t(1-t) + b_{2y} t^2}{(1-t)^2 + 2w_1 t(1-t) + t^2}$$

then the polynomial $\Gamma(t)$ is shown in the Annex.

Again, the computation of the coefficients of this polynomial may be expensive, and the problem of computing its real roots numerically unstable.

2. THE NEW ALGORITHM

2.1 Theoretical results

Let's return to the system of nonlinear equations (3). If we were able to compute the coordinates (x,y) of the footpoint, then the Euclidean distance d may be obtained immediately from the first equation. Hence, we may restrict ourselves to the solution of the nonlinear system,

$$\begin{cases} f(x,y) = 0 \\ \dfrac{\partial f}{\partial y}(p)(x - x_0) - \dfrac{\partial f}{\partial x}(p)(y - y_0) = 0 \end{cases} \tag{8}$$

in the coordinates of the footpoint p =(x,y).

This system may be efficiently solved by the Newton's method if:

a) The Jacobian matrix is nonsingular in a neighborhood of the solution.

b) A good initial approximation to the solution is known.

In connection with the first point we proved the following result.

Theorem: Let be C a conic with implicit equation (1) and q = (x_0, y_0) a point not on C, then there exists a set of points on the plane Z(C) (with Lebesgue measure 0) such that:

i) If $q \notin Z(C)$ then the Jacobian matrix of the system (8) is nonsingular in any of its solutions p.

ii) If $q \in Z(C)$ then the Jacobian matrix of the system (8) may be singular at some of its solutions p.

Proof:

Lets denote by $f_1(x,y) = 0$ and $f_2(x,y) = 0$ the equations of the system (8). We may rewrite f_2 as

$$f_2(x,y) = f_{1y}(x-x_0) - f_{1x}(y-y_0) = 0 \tag{9}$$

Then, the Jacobian matrix of (8) is given by,

$$\begin{bmatrix} f_{1x} & f_{1y} \\ f_{1xy}(x - x_o) + f_{1y} - f_{1xx}(y - y_o) & f_{1yy}(x - x_o) - f_{1x} - f_{1xy}(y - y_o) \end{bmatrix}$$

Assuming that (9) holds, the determinant of the Jacobian matrix, $f_3(x,y)$, reduces to,

$$f_3(x,y) = f_{1x}f_{1yy}(x-x_0) + f_{1y}f_{1xx}(y-y_0) - f_{1x}^2 - f_{1y}^2 \tag{10}$$

Hence, the Jacobian of the system (8) is singular in a solution p = (x,y), if and only if the polynomial system

$$\begin{cases} f_1(x,y) = 0 \\ f_2(x,y) = 0 \\ f_3(x,y) = 0 \end{cases} \tag{11}$$

has a solution. Using elimination theory, we may eliminate the variable x from f_1 and f_2 as well as from f_1 and f_3 obtaining two polynomial expressions in the variable y. Finally, eliminating y from this two last equations, we obtain a polynomial expression $\Psi(x_0,y_0)$ of degree twelve in x_0 and y_0, whose coefficients depend on the coefficients of the conic C and with the property that (11) has a solution if (x_0,y_0) is a root of Ψ. Thus, if the Jacobian of the system (8) is singular in a solution p = (x,y) then (x_0,y_0) belongs to the algebraic curve Z (C) : $\Psi(x_0,y_0) = 0$. Since $f_1 = 0$, $f_2 = 0$, and $f_3 = 0$ are quadratic equations in x and y, after Bezout's theorem [Walker(1978)] any pair of them has at most four common solutions. ■

As a consequence of the previous result, only for a subset of points on the curve Z(C) the Jacobian matrix of the system (8) is singular and it is not possible to compute the Euclidean distance solving (8) by the classic Newton's method. Nevertheless, this situation occurs with zero probability.

In any case, we need a good initial approximation to the footpoint, since we are interested in the global minimum of (2). Exploiting the geometry of conics, we may find a conic section containing the actual orthogonal projection p.

2.2 Computing a good initial approximation

- How to compute a good initial approximation of the footpoint?

- Instead of considering the family of normal lines to the points on the conic, we are going to select from the pencil of lines passing through the external point q the ones which are normal lines to some point on the conic (see Fig. 1).

This approach is simpler, since the parameter space for the pencil of lines is a circle, while in the direct approach it is a general conic. In order to simplify the analysis, let's suppose that the conic has been reduced to the canonical form.

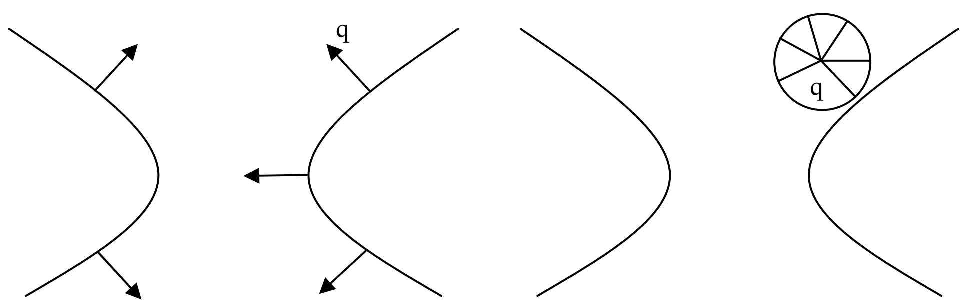

Figure 1. Direct Approach Our Approach

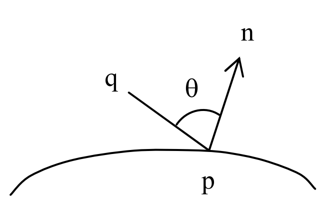

Figure 2. Definition of the angle θ(p)

Let be r a line through the point q intersecting the conic at a point p and consider the angle $\theta(p)$ between $\vec{pq}$ and the normal $\vec{n}$ to C at p (see Figure 2). Then, p is an orthogonal projection of q on C if and only if,

$$\mathrm{sen}\theta(p) = 0 \tag{12}$$

Thus, a good initial approximation to the footpoint p is an approximate solution of the equation (12), i.e. a point p Є C, such that $|\mathrm{sen}\theta(p)| < \varepsilon$, $\varepsilon > 0$. On the other hand, to compute the solutions of (12), it is sufficient to find an interval $[\theta_1,\theta_2]$, such that $\mathrm{sen}\theta_1\mathrm{sen}\theta_2 < 0$. More geometrically, we have to find an arc of conic limited by two points P_1 and P_2 such that the footpoint is contained in that arc, i.e. $\mathrm{sen}\theta(P_1)\mathrm{sen}\theta(P_2) < 0$ where $\theta(P_i)$ i = 1, 2 is the angle between the normal to the conic at P_i and the line passing through P_i and q.

The method to compute P_1 and P_2 depends on the position of the point q, i.e. if q is an “external” or “internal” point to the conic.

Lemma: Let be C an irreducible conic, q a point on the plane and R_q the pencil of lines passing through q. Then, exactly one of the following statements hold,

i) q is on C and there is only one line tangent to C passing through q.
ii) q is not on C and there are two lines tangent to C passing through q.
iii) q is not on C and there are not lines tangentt to C passing through q.

Proof

The pencil R_q of lines passing through q = (x_0, y_0) consists of the lines,

$$r_k : y = y_0 + k(x\text{-}x_0) \tag{13}$$

Inserting (13) in the implicit equation of the conic, we obtain a polynomial of degree 2 in the variable x, $p_2(x) := a_{2k}x^2$ `+ $a_{1k}x + a_{0k}$ whose coefficients depend on the slope k of r_k,

$$\begin{aligned} p_2(x) &= (a_{11}k + a_{20} + a_{02}k^2)x^2 \\ &\quad + (2a_{02}ky_0 - 2a_{02}k^2x_0 + a_{10} + a_{01}k + a_{11}y_0 - a_{11}kx_0)x \\ &\quad + a_{00} + a_{02}y_0^2 - 2a_{02}y_0kx_0 + a_{02}k^2x_0^2 + a_{01}y_0 - a_{01}kx_0 \end{aligned}$$

Let be $d(k) := a_{1k}^2 - a_{2k}a_{0k}$ the discriminant of $p_2(x)$ with respect to x,

$$\begin{aligned} d(k) &= (8a_{20}a_{02}y_0x_0 + 4a_{20}a_{01}x_0 - 4a_{11}a_{00} - 2a_{11}a_{01}y_0 \\ &\quad - 2a_{11}^2y_0x_0 - 2a_{10}a_{11}x_0 + 2a_{10}a_{01} + 4a_{10}a_{02}y_0)k \\ &\quad + a_{11}^2y_0^2 + 2a_{11}y_0a_{10} + a_{10}^2 - 4a_{20}a_{00} - 4a_{20}a_{02}y_0^2 \\ &\quad - 4a_{20}a_{01}y_0 + (-4a_{20}a_{02}x_0^2 + 2a_{11}a_{01}x_0 - 4a_{02}a_{00} \\ &\quad - 4a_{02}x_0a_{10} + a_{11}^2x_0^2 + a_{01}^2)k^2 \end{aligned} \tag{14}$$

Surprisingly, d(k) happens to be a polynomial of degree two in k, instead of degree four! Since the line r_k is tangent to the conic if (for its slope k), $p_2(x)$ has a double root, i.e. if d(k) = 0, we must consider the discriminant Δ of d(k) with respect to k,

$$\Delta = \begin{aligned} &-16(x_0^2a_{20} + a_{02}y_0^2 + a_{00} + x_0y_0a_{11} + x_0a_{10} + a_{01}y_0) \\ &(-a_{00}a_{11}^2 + 4a_{02}a_{20}a_{00} - a_{20}a_{01}^2 - a_{02}a_{10}^2 + a_{01}a_{10}a_{11}) \end{aligned}$$

There are only three possibilities,

i) $\Delta = 0$, i.e. d(k) has a double real root. Since the conic is irreducible by hypothesis, the last factor of Δ is different from zero. Hence, Δ vanishes if and only if q Є C.

ii) $\Delta > 0$, i.e. d(k) has two different real roots k_1 and k_2. Then, $q \notin C$ and the lines r_{k1} and r_{k2} are tangent to C.

iii) $\Delta < 0$, i.e. d(k) has two complex conjugate roots. Thus, $q \notin C$ and no line in R_q is tangent to C.

■

Definition: Let be C a conic and q a point not on C. We call q an internal point to C if any member of R_q intersects transversely the conic. Otherwise, q is called an external point to C.

Let's return to the problem of computing the points P_1 and P_2. After the previous definition and lemma, if q is an external point to C, then there are two lines tangent to C passing through q. Here, we must consider two cases:

i) If the intersection points of these lines with C belong to the same connected component of C, we may take P_1 and P_2 equal to these points, (see Fig 3i). In this case holds $\text{sen}\theta(P_1)\text{sen}\theta(P_2) = -1$.

ii) Otherwise (it may occur only when C is an hyperbola! see Fig. 3ii), we may take P_1 equal to the intersection point of the tangents with C which is closest to q. Let be r_a be the asymptote of C closest to q and denote by P_{1p} and q_p the orthogonal projections of P_1 and q on r_a respectively. Set P_c equal to the closest point to the origin between P_{1p} and q_p and set P_f equal to the other point.Then, P_2 is the point on r_a which is symmetric to P_c with respect to P_f. Strictly speaking, P_2 is not on C, but since it is on one of its asymptotes and is not close to the origin, we may assume that approximately it is on C and also that the tangent line to C at P_2 is r_a.

If q is an internal point (see Fig 4) and C is an ellipse, then P_1 and P_2 may be selected as the vertices of the ellipse located in the same quadrant as q. If C is an hyperbola or parabola, then P_1 is the vertex of the conic contained in the same quadrant as q and P_2 is the intersection between C and the line orthogonal to the axis containing P_1 and passing through q.

Once the interval $[\theta_1,\theta_2]$, $\theta_i = \theta(P_i)$ i = 1, 2 has been determined, we may compute an approximation of the solutions of (12) using some bisection method. Instead of the classical bisection of the angle $\theta(p)$, which is computationally more expensive, we compute the intersections of the conic with the line passing through q and the middle point of the segment P_1P_2. Since any line intersects a conic at most in two points with real coordinates (see[Walker (1978)]), in each step of the bisection process we have to select the intersection point which is located in the interior of the triangle defined by q, P_1 and P_2. We only need to find an approximation to the footpoint laying in the conic arc which contains the global minimum, therefore we finish the bisection process after some few steps or when $|\text{sen}\theta(p)| < \varepsilon$, $\varepsilon > 0$. Then, the position (x_p,y_p) of the orthogonal projection or footpoint p may be improved using the Newton's method to solve the system (8).

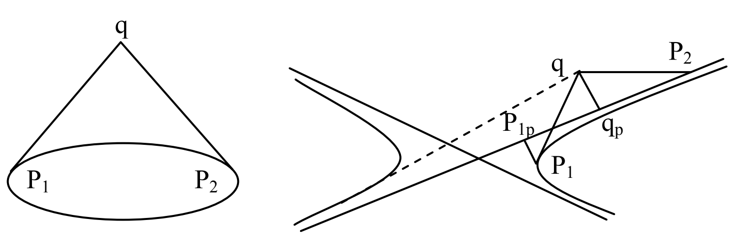

Figure 3. Case external point: i) left ii) right

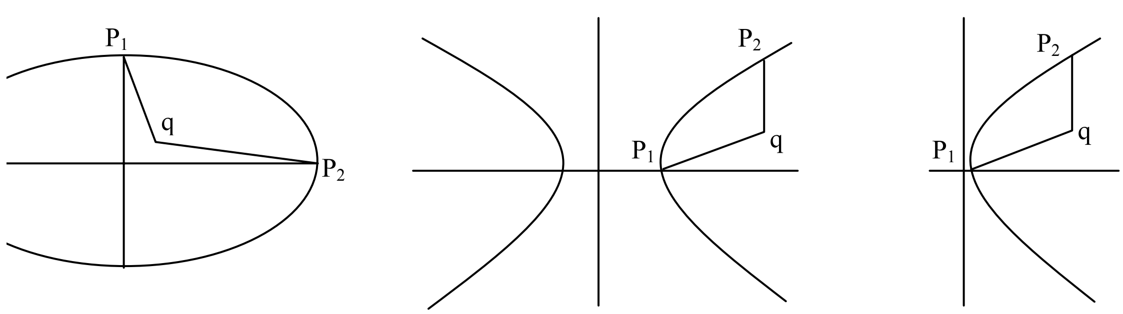

Figure 4. Case internal point

2.3 The algorithm

In this section we resume our algorithm to compute the Euclidean distance from a point $q = (x_0, y_0)$ to an irreducible conic C, both in the same plane.

Input: The vector of coefficients $a = (a_{20}, a_{11}, a_{02}, a_{10}, a_{01}, a_{00})$ of the conic C, a point $q = (x_0, y_0)$ and the termination criteria N_1, ε_1, N_2, ε_2.

Output: The Euclidean distance d from q to C and the coordinates of the orthogonal projection p of q on C.

1. Compute the lineal change of coordinates T(x,y) that reduces C to the canonical form C'.

(a) Obtain the list a' of the coefficients of the C after T(x,y).

(b) Find the image q' of q after the change of coordinates, $q' = T(x_0, y_0)$.

2. Compute Δ and decide if q' is an internal or external point.

(a) If q' is an external point, compute k_1 and k_2 as the real roots of (14) and P_1, P_2 using the method described at the end of the previous section.
(b) If q' is an internal point, find P_1 and P_2 using the method described at the end of the previous section.

3. **Bisection process** (solve (12))

For $j = 1, \dots N_1$ do

(a) Find the slope k_{new} of the line passing through to q' and the middle point of the segment $P_1 P_2$.

(b) Obtain the intersections (two) between C' and the line of $R_{q'}$ with slope k_{new}. Select between them the point p' which is closed to the middle point of the segment $P_1 P_2$.

(c) Compute $\sin(\theta(p'))$ where $\theta(p')$ is the angle between the normal to C' at p' and the line through p' and q'.

(d) If $|\sin(\theta(p'))| < \varepsilon_1$ then go to **4** else

if $\sin(\theta(p'))\sin(\theta(P_1)) < 0$ then set $P_2 = p'$
else set $P_1 = p'$

(e) Set $j = j + 1$ and return to **3**(a).

4. **Newton's method** (to solve (8) from the initial approximation $p^0 = p'$)

For $j = 0 \dots N_2$ do

(a) Compute Δp^j as the solution of the linear system $J(p^j)\, \Delta p^j = -F(p^j)$ where J is the Jacobian matrix of (8) and $F = (f_1, f_2)^t$ is the vector of the left side of (8)

(b) Correct the position of p^j, $p^{j+1} = p^j + \Delta p^j$

(c) Obtain the relative error $e^j = \dfrac{\|\Delta p^j\|}{\|p^{j+1}\|}$

(d) If $e^j < \varepsilon_2$ then **END** else set $j = j + 1$ and return to **4**(a).

5. Set $p = T^{-1}(p')$

6. Compute $d(q,C) = \| q - p \|$

3. NUMERICAL RESULTS

In this section we compare the relative errors of the computed distance from a point to a conic associated with different approximations of the Euclidean distance. In the following table the second column Eud is the exact Euclidean distance from the points of the third column to the conics of the first column.

The fourth column eal, shows the relative errors associated to the algebraic distance, the two following, etau1 and etau2 correspond to the relative errors associated to Taubin's first and second order approximations respectively.

The final column, enew, shows the relative errors of our approximation.

The implicit equations of the conics selected for the numerical experiment are the following:

Parabola: $4x^2 + 4xy + y^2 - 4x = 0$

Ellipse: $x^2 + xy + y^2 - 4x = 0$

Hyperbola: $-x^2 - xy + y^2 - 4x = 0$

The termination criteria considered in all cases were $N_1 = 1$, $\varepsilon_1 = 0.01$ for the bisection process and $N_2 = 4$, $\varepsilon_2 = 10^{-5}$ in the Newton's method.

Conic	Eud	Point	eal	etau1	etau2	enew
Parab.	0.05	(0.050099,0.01948)	2.7215	0.0544	0.0533	$*6.95\text{x}10^{-6}$
	0.5	(0.25,1)	1.5	0.3066	0.5475	$*2.22\text{x}10^{-16}$
	0.05	(0.762337,0.155396)	3.5343	0.0488	0.0414	$*1.35\text{x}10^{-6}$
Ellipse	0.5	(4.556152,-1.61575)	3.4341	0.1859	0.1150	*0.0027
	1.5	(3.879868,-2.60398)	1.5257	0.4346	0.4119	*0.0077
	0.7	(1.501967,-1.806558)	3.5739	0.3035	0.1301	*0.0044
Hyper.	1.5	(-7.737018,0.95532)	12.739	0.0374	0.2207	$*5.26\text{x}10^{-4}$
	2.3	(4.188879,-6.87369)	17.147	0.0328	0.2375	$*4.61\text{x}10^{-4}$
	0.6	(3.539518,-2.962992)	11.365	0.0083	0.1162	$^*0.0027$

Observe that our approximation, marked with (*), performs very well, even when the external point is far from the conic.Taubin's approximate distances are better that the algebraic distance, but both are poor approximations when the external point is not very close to the conic.

4. FINAL REMARKS

The new algorithm gives a good approximation of the Euclidean distance from a point to a conic, even when the external point is far from the conic.

In fact, using this method it is possible to compute the Euclidean distance with a prescribed precision, since we may correct the position of the footpoint (solving a nonlinear system).

Furthermore, the Euclidean distance is computed in an stable way, since the algorithm is based on the solution of some equations of second degree to obtain an approximation of the footpoint and solving systems of nonlinear equations of size 2x2 by Newton's method, starting from a good initial approximation. However, compared to the approximations given by the algebraic distance or the ones given by Taubin, we don't have a closed expression to compute the Euclidean distance and the new algorithm is computationally more expensive.

Finally, we wish to recall, that if the point q and the conic C are not in the same plane, it is also possible to compute the Euclidean distance from q to C using the method proposed in this paper (and also the coordinates of the footpoint). In fact, if q' denotes the orthogonal projection of q on the plane Π containing C and d' is the Euclidean distance from q' to C, then after Pythagoras, the distance from q to the conic C may be computed in terms of d' and the distance from q to Π.

5. ANNEX

Expression of $\Phi(d)$ when C is a parabola $y^2 - 2px = 0$

$$
\begin{aligned}
\Phi(d) = {} & -4d^6 + (4y_0^2 + 16py_0 + 12x_0^2 - 8p^2)d^4 \\
& +(-8x_0^2 y_0^2 - 12x_0^4 + 4px_0^2 y_0 - 4p^4 - 8y_0p^2 - 20p^2x_0^2 + 16p^3y_0 - 16py_0^3)d^2 \\
& +(4y_0^2 - 4py_0 + p^2 + 4x_0^2)(-x_0^2 + 2py_0)
\end{aligned}
$$

Expression of $\Phi(d)$ when C is a central conic $a_{20}x^2 + a_{02}y^2 + a_{00} = 0$

$$
\begin{aligned}
\Phi(d) = {} & a_{20}^2 a_{02}^2 (a_{02} - a_{20})^2 d^8 + 2a_{02}(a_{02} - a_{20})a_{20}(a_{20}^2 a_{02}y_0^2 - a_{20}^2 a_{00} + 2a_{20}^2 a_{02}x_0^2 - a_{20}a_{02}^2 x_0^2 \\
& - 2a_{20}a_{02}^2 y_0^2 + a_{02}^2 a_{00})d^6 + (a_{00}^2 a_{20}^4 + 6a_{20}^4 a_{02}^2 x_0^4 - 6a_{20}^4 a_{00}a_{02}x_0^2 + 6a_{20}^4 a_{02}^2 x_0^2 y_0^2 \\
& + 2a_{20}^4 a_{00}a_{02}y_0^2 + a_{20}^4 a_{02}^2 y_0^4 - 10a_{20}^3 a_{02}^3 x_0^2 y_0^2 - 6a_{20}^3 a_{02}^3 y_0^4 - 6a_{20}^3 a_{02}^3 x_0^4 - 4a_{20}^3 a_{00}a_{02}^2 y_0^2 \\
& + 8a_{20}^3 a_{02}^2 x_0^2 a_{00} + 2a_{20}^3 a_{02}a_{00}^2 - 6a_{20}^2 a_{00}^2 a_{02}^2 + 6a_{20}^2 a_{02}^4 x_0^2 y_0^2 + a_{20}^2 a_{02}^4 x_0^4 + 6a_{20}^2 a_{02}^4 y_0^4 \\
& - 4a_{20}^2 a_{00}a_{02}^3 x_0^2 + 8a_{20}^2 a_{00}a_{02}^3 y_0^2 + 2a_{20}a_{00}^2 a_{02}^3 - 6a_{20}a_{00}a_{02}^4 y_0^2 + 2a_{20}a_{02}^4 x_0^2 a_{00} + a_{00}^2 a_{02}^4)d^4 \\
& + (-4a_{20}^4 a_{02}^2 x_0^6 + 2a_{00}^3 a_{20}^3 + 2a_{00}^3 a_{02}^3 - 2a_{00}^3 a_{20}^2 a_{02} - 2a_{00}^2 a_{02}^4 y_0^2 - 2a_{00}^3 a_{02}^2 a_{20} - 2a_{20}^4 a_{02}^2 x_0^2 y_0^4 \\
& - 2a_{02}^4 x_0^4 y_0^2 a_{20}^2 + 2 a_{02}^3 x_0^6 a_{20}^3 + 2 a_{02}^3 y_0^6 a_{20}^3 - 4a_{02}^4 y_0^6 a_{20}^2 - 2a_{00}^2 a_{20}^4 x_0^2 + 2a_{02}^3 x_0^4 a_{20}^3 y_0^2 \\
& + 2a_{02}^3 y_0^4 a_{20}^3 x_0^2 - 4a_{20}^4 a_{00}a_{02}y_0^2 x_0^2 - 6a_{20}^4 a_{02}^2 x_0^4 y_0^2 + 6a_{20}^4 a_{00}a_{02}x_0^4 + 6a_{00}^2 a_{02}^3 x_0^2 a_{20} \\
& - 4a_{00}a_{02}^4 x_0^2 a_{20}y_0^2 - 8a_{00}^2 a_{02}^2 x_0^2 a_{20}^2 + 6 a_{00}a_{02}^3 x_0^2 a_{20}^2 y_0^2 + 6 a_{00}a_{02}^3 x_0^4 a_{20}^2 \\
& - 6a_{02}^4 x_0^2 y_0^4 a_{20}^2 - 8a_{00}^2 a_{20}^2 a_{02}^2 y_0^2 + 4a_{00}^2 a_{20}^3 a_{02} x_0^2 + 6a_{00}^2 a_{20}^3 a_{02} y_0^2 \\
& + 4a_{00}^2 a_{02}^3 a_{20} y_0^2 + 6a_{02}^4 y_0^4 a_{20} a_{00} - 10a_{00} a_{02}^3 y_0^4 a_{20}^2 + 6a_{00} a_{02}^2 y_0^2 a_{20}^3 x_0^2 \\
& + 6a_{00} a_{02}^2 y_0^4 a_{20}^3 - 10 a_{02}^2 x_0^4 a_{20}^3 a_{00})d^2 + (-2 a_{00} a_{02}^2 y_0^2 a_{20} - 2a_{00}^2 a_{20} a_{02} \\
& + 2 a_{00} a_{02}^2 x_0^2 a_{20} + 2 a_{02}^2 x_0^2 y_0^2 a_{20}^2 + a_{02}^2 y_0^4 a_{20}^2 + 2 a_{00} a_{02} y_0^2 a_{02}^2 + a_{00}^2 a_{20}^2 + a_{02}^2 x_0^4 a_{20})
\end{aligned}
$$

Expression of the polynomial $\Gamma(t)$ when we use the Bezier representation of the conic,

$$
\begin{aligned}
\Gamma(t) = {} & 2w_1(b_{1x}b_{0x} - b_{1y}y_0 + b_{0y}y_0 + b_{0x}x_0 - b_{0x}^2 - b_{1x}x_0 + b_{1y}b_{0y} - b_{0y}^2) \\
& + (-2b_{2y}y_0 + 8w_1b_{1x}x_0 + 2b_{2x}b_{0x} - 8b_{0y}w_1y_0 - 4w_1^2 b_{1x}x_0 + 4w_1^2 b_{1x}^2 \\
& + 8w_1b_{1y}y_0 - 4b_{0x}w_1^2 b_{1x} + 8b_{0x}^2 w_1 + 8b_{0y}^2 w_1 + 2b_{0x}x_0 + 2b_{0y}y_0 \\
& + 4b_{0x}w_1^2 x_0 + 4b_{0y}w_1^2 y_0 + 2b_{2y}b_{0y} - 8w_1b_{1x}b_{0x} - 8b_{0x}w_1x_0 \\
& + 4w_1^2 b_{1y}^2 - 2b_{2x}x_0 - 4b_{0y}w_1^2 b_{1y} - 4w_1^2 b_{1y}y_0 - 2b_{0x}^2 - 2b_{0y}^2 - 8w_1b_{1y}b_{0y})t \\
& +(-12w_1^2 b_{1x}^2 + 6b_{2x}w_1b_{1x} - 12w_1^2 b_{1y}^2 - 6b_{2x}b_{0x} + 12b_{0y}w_1^2 b_{1y} \\
& + 12b_{0x}w_1^2 b_{1x} + 6b_{2y}w_1b_{1y} - 6b_{2y}b_{0y} - 12w_1b_{1x}x_0 + 18b_{0x}w_1x_0 \\
& + 18b_{0y}w_1y_0 - 12w_1b_{1y}y_0 - 6b_{0y}y_0 + 6b_{2x}x_0 - 6b_{0x}x_0 + 6b_{2y}y_0 \\
& - 12b_{0y}w_1^2 y_0 - 6b_{2y}y_0w_1 + 12w_1^2 b_{1y}y_0 - 6b_{2x}x_0w_1 - 12b_{0x}w_1^2 x_0 + 12w_1^2 b_{1x}x_0 \\
& - 12b_{0x}^2 w_1 + 6w_1b_{1x}b_{0x} - 12b_{0y}^2 w_1 + 6w_1b_{1y}b_{0y} + 6b_{0x}^2 + 6b_{0y}^2)t^2 \\
& + (8w_1^2 b_{1x}^2 + 4b_{2x}w_1^2 b_{1x} - 12b_{2x}w_1b_{1x} + 8w_1^2 b_{1y}^2 + 4b_{2x}b_{0x} - 12b_{0y}w_1^2 b_{1y} \\
& + 4b_{2y}w_1^2 b_{1y} - 12b_{0x}w_1^2 b_{1x} - 12b_{2y}w_1b_{1y} + 4b_{2y}b_{0y} + 8w_1b_{1x}x_0 - 20b_{0x}w_1x_0 \\
& - 20b_{0y}w_1y_0 + 8w_1b_{1y}y_0 + 8b_{0y}y_0 - 8b_{2x}x_0 + 8b_{0x}x_0 - 8b_{2y}y_0 + 12b_{0y}w_1^2 y_0 \\
& - 4b_{2y}w_1^2 y_0 + 12b_{2y}y_0w_1 - 8w_1^2 b_{1y}y_0 - 4b_{2x}w_1^2 x_0 + 12b_{2x}x_0w_1 + 12b_{0x}w_1^2 x_0 \\
& - 8w_1^2 b_{1x}x_0 + 8b_{0x}^2 w_1 + 4w_1b_{1x}b_{0x} + 8b_{0y}^2 w_1 + 4w_1b_{1y}b_{0y} - 6b_{0x}^2 + 2b_{2y}^2 \\
& + 2b_{2x}^2 - 6b_{0y}^2)t^3 + 2(w_1 - 1) (-2b_{2y}w_1b_{1y} - 2b_{2x}w_1b_{1x} + 2w_1b_{1y}b_{0y} + 2b_{2y}y_0w_1 \\
& + 2w_1b_{1x}b_{0x} - 2b_{0x}w_1x_0 - 2b_{0y}w_1y_0 + 2b_{2x}x_0w_1 + b_{2y}^2 + b_{2x}^2 + 2b_{0y}y_0 - b_{0y}^2 - 2b_{2y}y_0 - 2b_{2x}x_0 - b_{0x}^2 + 2b_{0x}x_0)t^4
\end{aligned}
$$

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