RANDOM DEMANDS: OPTIMUM LOT SIZE AND THE NEWSBOY PROBLEM

Sira Allende and Carlos Bouza[1], Facultad de Matemática y Computación, Universidad de La Habana

ABSTRACT
The determination of the optimum lot size is a stochastic problem because of the randomness of the demands. The usual approaches consider that the involved distributions are known. We consider the case in which they are unknown. The optimization problem is probabilistic constraint program. The demands are modeled by an autoregressive process and the needed quantiles are derived. The newsboy problem is revisited using the derived results.

Key words: scenario analysis, probabilistic constrained optimization, Cornish-Fisher expansion.

RESUMEN
La determinación del tamaño óptimo del lote es un problema estocástico dada la aleatoriedad de las demandas. Usualmente se consideran conocidas las funciones de distribución que entran en la modelación. Consideramos el caso en que ellas son desconocidas. El. problema de optimización es un programa con restricciones probabilísticas. Utilizando los resultados obtenidos el problema del newsboy es reanalizado.

Palabras clave: análisis de escenarios, optimización con restricciones probabilísticas, expansión de Cornish-Fisher.

MSC: 90C15

1. INTRODUCTION

In many applications we need to determine the optimum lot size [ols]. Large inventories determine increases in the management costs. Therefore the problem to be solved is an ols one. When the demands are considered random the involved optimization problem is stochastic. Lasserre-Bes-Roubellat [1985] studied it considering that the distribution is known. The problem is to determine the lot size during k periods. Assuming the standardization of the involved variables they denoted, for a fixed period t, the inventory level [lot size] by X_t and by f_t the corresponding cost function. Similarly s_t is the production level [control variable] and g_t the corresponding cost function. The demand w_t is a random variable. They assumed that the demands are iid and that the system is an open loop. The program is

$$\text{P1: min } E\left[\sum_{t=1}^{k} f_t(X_t) + g_t(s_t)\right] = M' \qquad [1.1]$$

subject to:

$$X_t = X_{t-1} + s_t - w_t \qquad [1.2]$$

$$s_t \in [0, s'_t] \qquad [1.3]$$

$$\text{Prob}\{X_t \le x'_t\} \ge \alpha_t \qquad [1.4]$$

$$\text{Prob}\{X_t \ge x''_t\} \ge \beta_t \qquad [1.5]$$

t = 1,..,k.

[1.2] is the evolution function of the Stochastic Dynamic Model P1. The constraints [1.4] and [1.5] establish that it is a probabilistic constrained problem. A deterministic equivalent program is:

[1]**Email**: sira@matcom.uh.cu and bouza@matcom.uh.cu

$$\text{P2: min } E\left[\sum_{t=1}^{k} f_t^{'}(y_t) + g_t(s_t)\right] = M \qquad [1.6]$$

subject to:

$$y_t = y_{t-1} + s_t - w_{t0}$$

$$s_t \in \left[0, s_t^{'}\right],\ y_t \in \left[y_t^{''}, y_t^{'}\right]$$

Where

$$y_0^{'} = x_0,\ y_t^{'} = x_t^{'} + \lambda_{\alpha t} - \lambda_{1-\beta t},\ y_t$$

$$y_t = x_t,\ f_t^{'}[y_t] = E[f_t(y_t - \lambda_{t\beta t} - w_t)]$$

t = 1,...,k

and

$$w_{t_0} = \lambda_{t\beta t} \text{ if } t = 1\ (\lambda_{t\beta t} - \lambda_{[t-1]\beta t} \text{ if } t > 1).$$

λ_p is the quantile of order p of the known distribution function F. Suppose that it is unknown and that the DM is able to fix $F^* = \{F_1,...,F_n,....\}$ as the family of distribution functions where F belongs. Then we need to estimate M. If M_n is an estimate of M derived from a sample the approximation error [AE]

$$e_n = |M_n - M| = e_n[z']$$

is a non decreasing function. Using this assumptions Allende-Bouza [1998] derived the deterministic equivalent of P1 and the convergence of e_n was obtained using results of Birgé [1991].

In this paper we drop the hypothesis that the demands are i.i.d and assume that the sums of the demands D_t conform a linear autoregressive process. An approximation by Edgeworth Series [ES] is used for deriving an approximation to the involved quantiles. A discretization of the interval where they supposedly belong permit to compute a solution using scenario analysis. The accuracy is related with the interval's width.

The newsboy problem [NP] is studied as a particular case: a single item inventory problem with k-periods.

2. MAIN RESULTS

Consider that the demand at moment j+1 is modeled by the linear autoregressive process

$$D_{j+1} = \sum_{i=0}^{l-1} \lambda_i D_{j+1} + e_{j+l}$$

where $\{e^t, t+1,2... \}$ is an idd sequence of random errors with null expectation. It reflects the responses and the model establishes that the demand at time t is related with at most l previous periods. The λ_i's are unknown parameters such that

$$\sum_{i=0}^{l-1} \lambda_i z_i + Z^l \qquad [2.1]$$

belongs to the zero unit circle.

The counterpart of [1.4] is

$$\text{Prob } \{x_0 + S_t - D_t \le x_t^{'}\} \ge \alpha_t$$

x_0 is the initial inventory level and

$$\sum_{j=1}^{t} s_j = S_t \qquad [2.2]$$

Take the quantile

$$F_t^{-1}[q_t] = \lambda_{qt}$$

Then an equivalent constraint is

$$x_0 + S_t - x_t^{'} \leq \lambda_{1-\alpha t} \qquad [2.3]$$

Similarly for [1.5] we obtain

$$x_0 + S_t - x_t^{''} \geq \lambda_{\beta t} \qquad [2.4]$$

The following proposition establishes the conditions needed for using the normal distribution for obtaining adequate approximations for the unknown quantiles.

Proposition 2.1. Take U* as a class of Borel sets of R such that for $e_n \to 0$ and for some $a > 0$

$$\mathrm{Sup}\ \{A \in U^*\} \int_{[\partial A]} \phi_\sigma[h]dh = 0[e^a],$$

holds. ϕ_σ is the density function of the normal distribution $N[0,\sigma]$ and

$$\sigma^2 = \sigma_{00} + 2\sum_{j\geq 1}\sigma_{0j} \geq 0$$

defining

$$\sigma_{0j} = E[X_1\, X_{j+1}] - E[X_j]E[X_{j+1}]$$

and

$$\sigma_{00} = E[X_j^2] - E[X_j]^2 .$$

Take z_p as the quantile of order p of the N[0,1], $\pi_{(j[2])}[\lambda_q]$ as a polynomial of degree not larger than 3j - 1 depending on cumulants of F_t and

$$\Delta_{qt} = \sum_{j\geq 1} \pi_{(j[2])}[\lambda_q]t^{-j/2} \in (\Delta_q^{'}, \Delta_q^{''}) .$$

If

1) All zeros of (2.1) lie within the circle $Z^* = \{z \in C\ |z| < 1\}$

2) $Y_t = (D_t,...,D_{t+l-1})$, t > 0 and $E[|D|^{m+1}] > 0$ for some t > 2.

3) $\lim\sup_{|\eta| \to \infty} E[\exp(i\eta e_1)] < 1$

are satisfied then, the two constraints

$$x_0 + S_t - x_t^{''} \geq z_{\beta t} - \Delta_{\beta t}^{''} \qquad [2.5]$$

$$x_0 + S_t - x_t^{'} \leq z_{1-\alpha t} - \Delta_{1-\alpha t}^{'} \qquad [2.6]$$

determine a constraint set which is more restrictive than that associated to [2.3] - [2,4].

Proof:

The sequence $Y_t = (D_t,...,D_{t+l-1})$, $t > 0$ is Markovian, see Friedst and Gray (1997), and only an initial distribution τ on R^l is admitted by it because all the zeros of [2.1] belong to Z*. Hence Y_n is a stationary ergodic [SE] Markov process resulting that D_n is a SE sequence. From the boundness conditions of the expected value of some absolute values of D_t and the Cramer's condition fixed by 3) the hypothesis of the example (1.1) of Götze-Hipp (1983) hold. Therefore the Edgeworth expansion for F_t is valid, see Bhattacharya (1987).

Then for $t \to \infty$

$$\mathrm{Sup}_{a\in U^*}\left| F_t(A) - \left(\int_A \left[[1 + \sum_{j=1}^{m-2} \pi_{(j[2])}[\lambda_q][t^{-j/2}} \right] \right) \right| = o(t^{-(m-2)/2})$$

and the corresponding Cornish-Fisher expansion is easily derived. The expansion of the quantiles are

$$z_{\beta t} = \lambda_{\beta t} + \sum_{j=1}^{m-2} \pi_{(j[2])}[\lambda_q] t^{-j/2} = \lambda_{\beta t} + \Delta_{\beta t}$$

and

$$z_{\beta t} - \Delta''_{\beta t} = z''_{\beta t} \le \lambda_{\beta t}$$

Similarly

$$z_{1-\alpha t} - \Delta'_{1-\alpha t} = z'_{1-\alpha t} \ge \lambda_{1-\alpha t}$$

Then we have

$$y_t = x_0 + S_t - z''_{\beta t} \quad \text{if } t = 1 \quad \left(y_{t-1} + S_t - \left[z''_{\beta t} - z''_{\beta t-1} \right] \text{ if } \quad t = 2,...,k \right) \qquad [2.8]$$

$$y_t \in (x''_t) \in \left(x''_t, x'_t + \left[z_{1-\alpha t} - z'_{1-\alpha t-1} \right] \right) \qquad [2.9]$$

$$S_t \in (0, S'_t) \qquad [2.10]$$

It is clear that this constraint set is more restrictive than its counterpart in the original problem. Because of the convergence of the Cornish-Fisher expansion we expect that the solution of this problem be close to the real one.

When $f_t[x_t] = c_t x_t$, $c_t > 0$, we have that $f'_t[x_t] = c_t[y_{t-1} + s_t + D_{t0}] - E[D_t]$. If t is sufficiently large for accepting that [2.7] holds

$$E[D_{t_0}] = \int_R D^*_t \phi_\sigma [D_t] dD_t = \sum_{j=1}^{m-2} t^{-j/2} \left[\int_R D^*_t \phi_\sigma d \right] Dt \to 0$$

where

$$D^*_t = D_t \left[1 + \sum_{j=1}^{m-2} \pi_{(j[2])}[\lambda_q] t^{-j/2} \right]$$

When $t \to \infty$ it has zero expectation because of the standardization of D_t.

An adequate approximation for the original problem is

$$P3[t]: \min G[t] = \sum c_t y_t + s_t - z^*_{t\beta_t}$$

subject to : [2.8] - [2.10]

taking

$$z^*_{t\beta_t} \in \left\{ z''_{t\beta_t} + \Delta_{\beta_t} \mid \Delta_{\beta_t} \in \left[\Delta'_{\beta_t}, \Delta''_{\beta_t}\right] = I\beta_t \right\}$$

The DM fixes the intervals I_{β_t}. These intervals can be discretized and a smaller number of quantiles needs to be computed. The nature of this approach suggests that we can use scenario analysis, within the theoretical frame as proposed by Rockafellar-Wets (1991) for example. For the obtention of a 'well hedged' solution to the underlying problem.

3. AN APPLICATION: THE RISK AVERSE NEWSBOY PROBLEM

The newsboy problem is a single-item inventory problem. We can model this problem for k periods. The original problem deals with the determination of the number of newspapers to buy. If he buys a small quantity a profit is missed out. When the quantity is too large a penalty is charged. The newsboy may want to maximize the expected profit. This model fit many economic problems. See Dohoi-Watanabe-Osaki (1994) and Eechoudt-Gollier-Schlesinger (1995) for a detailed discussion.

The risk averse newsboy problem is a single item is considered in each period. $Q_t \geq 0$ is the amount and W_t the demand of the items at a fixed period. $S_t \in [0, S'_t]$ is the production. The sale is $S^*_t = \min\{Q_t + S_t , W_t\}$. Q_t is ordered and delivered at a cost C per unit and the selling price is R. If $W_t < Q_t + S_t$ the seller sells at a price V at the final of the period. We assume that $R > C > V > 0$ and V' is the shortage penalty per unit

The profit is a random variable

$$Y(Q_t) = RS^*_t + V \text{ Max } \{0, Q_t - W_t\} - V' \text{ Max}\{0, W_t - Q_t\} - CQ_t = (R - V + V')S_t - V'W_t(C-V)Q_t \qquad [3.1]$$

For a fixed Q_t

$$E[W_t \mid Q_t] = \int_0^{Q_t} w_t f(w_t) dw_t$$

The probem to be solved is

$$P4: \text{Max}_{Q_t \geq 0} \; E[Y(Q_t)]$$

Subject to [3.1] and $R > C > V > 0$.

An important feature is that $E[Y(Q_t)]$ is an unimodal function of Q_t with only one solution if the newsboy is risk averse. See Dohoi-Watanabe-Osaki (1994) for examples.

Take

$$- f(X_t) = (R - V + V') S^*_t - (C - V)Q_t$$

and the constraints

$$Q_{t-1} + S_{t-1} - W_{t-1} = X_{t-1}$$

$$X_t = X_{t-1} + S_t - W_t$$

$$S_t \in [0, S_t^{'}]$$

$$P[X_t \leq x_t^{'}] \geq \alpha_t$$

$$P[X_t \geq x_t^{''}] \geq \beta_t$$

Then the approach proposed in Section 2 may be used by the seller for fixing an optimal strategy for k periods.

REFERENCES

ALLENDE, S. and C. BOUZA, C. (1998): "Minimum lot size determination under unknown distribution of the demands", Read at the International Congress of Mathematics, Berlin.

BHATTACHARYA, R.N. (1987): "Some aspects of Edgeworth expansions in statistics and probability", In **New perspectives in Theoretical and Applied Statistics**, Wiley, N. York.

DOHOI, T.; A. WATANABE and S. OSAKI (1994): "A note on the risk averse newsboy", **Recherche Opèrationalle**, 28, 198-202.

EECHOUDT, L.; C. GOLLIER and H. SCHELESINGER (1995): "The risk averse (and prudent) newsboy", **Management Sc**. 41, 786-94.

FRIESDT, B. and L. GRAY (1997): **A Modern Approach to Probability Theory**. Birkhäuser, Boston.

GÖTZE, F.N. and C. HIPP (1983): "Asymptotic expansions for sums of weakly dependent random vectors", **Z. Wahrsch. Verw. Gebeite**. 64, 211-239.

LASERRE, J.B.; C. BESS and F. ROUBELLAT (1985): "The stochastic discrete dynamic lot size problem: an open loop solution", **Operations Res**. 33, 684-689.

ROCKAFELLAR, R. and R.J.B. WETS (1991): "Scenario and policy aggregation in optimization under uncertainty", **Mathematics of Operations Res**. 16, 139-147.