MAXIMUM DOMAIN OF ATTRACTION FOR COPULAS*

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ABSTRACT
Maximum attractors of copulas star unimodal (about an (a, b)) are determined. If (a, b) ≠ (1,1) these attractors form a two-parameter family of copulas extending that of Cuadras-Augé whereas if (a, b) = (1,1) they cover all maximum value copulas. Relationship between unimodality and Archimax copulas of Capéraà, Fougères, and Genest [3] is also examined.

Key words: Archimax copula, Archimedean copula, attractor, copula, domain of attraction, extreme value distribution, maximum domain of attraction, star unimodality.

RESUMEN
Atractores maximales de estrellas-copula unimodal (alrededor de un (a,b)) son determinadas. Si (a,b) ≠ (1,1) estos atractores forman una familia bi-paramétrica de copulas extendiendo el de Cuadras-Augé mientras que si (a,b) = (1,1) ellos cubren todas las copulas de valor máximo. Relaciones entre unimodalidad y las cópulas Archimax de Capéraà, Fougères y Genest (J. Mult. Ana., 2000) son examinadas también.

MSC: 60E05

1. INTRODUCTION

An important property of a distribution is unimodality. It is then natural to ask whether copulas are unimodal. This question has been answered for central convex, block, and star unimodality in Cuculescu and Theodorescu (2002). As a follow-up we examine in this paper the maximum domain of attraction for star unimodal copulas.

The paper is organized as follows. Section 2 has an auxiliary character; here we indicate several definitions, notations, and results to be used throughout this paper. In Section 3 we show that the maximum domain of attraction to which copula C star unimodal about (a, b) ≠ (1,1) belongs is an element of a two-parameter family of copulas extending that of Cuadras-Augé. When (a, b) = (1,1) the set of all possible attractors changes dramatically covering all maximum value copulas. As a consequence of the results in Section 3 we examine in Section 4 the relationship between star unimodality and Archimax copulas of Capéraà, Fougères, and Genest [2000]; we show that many of them are hot star unimodal.

2. PRELUDE

We shall use the term probability measure or distribution at our convenience; m is Lebesgue measure, $\otimes$ stand s for measure product, I_A for the indicator function of A, ^{C} for complementation, $\overline{\mu}$ for the ′survival′ function of μ, and f_μ for the measure $\int f\, d\mu$.

2.1 Copulas: maximum domain of attraction

Let I = [0,1]. It is Sklar [1959] in 1959 coined the term *copula* for a distribution on I^2 whose margins are uniform. The notations M, V, and Π stand for the copulas min{u, v}, max{u + v - 1, O}, and uv respectively. For details on copulas we shall refer the reader to the recent book by Nelsen (1999).

A copula C* is said to be the *maximum attractor* of copula C (or C belongs to the *maximum domain of attraction* of C*) if we have

$$\lim_{n\to\infty} C^n(x^{1/n}, y^{1/n}) = C^*(x,y), \qquad x, y \in I. \tag{1}$$

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The concept of maximum attractor is related (Galambos (1987), Theorem 5.2.3, p. 294) to the convergence in distribution of the suitable normed sequence $\{(X_{(n)}, Y_{(n)}): n \geq 1\}$, where $\{(X_i, Y_i) : i \geq 1\}$ are independent pairs of random variables, each having the same joint distribution function and $X_{(n)} = \max\{X_1: 1 \leq i \leq n\}$, $Y_{(n)} = \max\{Y_1: 1 \leq i \leq n\}$. Here (1) is equivalent to

$$\lim_{n\to\infty} n(1 - C(x^{1/n}, y^{1/n})) = -\log C^*(x,y), \qquad x, y \in \mathrm{I}. \tag{2}$$

Since

$$1 - C\{x, y) = (1 - x) + (1 - y) - \overline{C}(x,y), \qquad x, y \in \mathrm{I},$$

where $\overline{C}(x,y) = C((x,1] \times (y,1]))$ is the joint *survival function* of C, (2) is equivalent to

$$\lim_{n\to\infty} n\overline{C}(x^{1/n}, y^{1/n}) = -\log(xy) + \log C^*(x,y), \qquad x, y \in \mathrm{I}. \tag{3}$$

Only the behavior of $\overline{C}$ near the point (1,1) plays a role in deciding whether C belongs or not to the maximum domain of attraction of C*.

Remark 2.1. A copula C_* is said to be the *minimum attractor* of copula C(or C belongs to the *minimum domain of attraction* of C*) if (using a similar argument as for the maximum attractor) we have

$$\lim_{n\to\infty} \overline{C}^n(1 - x^{1/n}, 1 - y^{1/n}) = C_*(x,y), \qquad x, y \in \mathrm{I}.$$

In other words, C belongs to the minimum domain of attraction of C_* if and only if C_* is the maximum attractor of the *survival copula*

$$\hat{C}(x,y) = \overline{C}(1-x, 1-y) = x + y - 1 + C(1-x, 1-y), \qquad x, y \in \mathrm{I}.$$

Therefore any assertion concerning the minimum domain of attraction is equivalent to one concerning the maximum domain of attraction by changing C to $\hat{C}$.

Since the work of Pickands (1982) (see also Tawn (1988)) it is known that C*, also called *extreme value copula*, can be expressed in the form

$$C^*(x,y) = C_A(x,y) = \exp\{\log(xy)A(\log(x)/\log(xy))\}, \qquad x, y \in \mathrm{I}, \tag{4}$$

in terms of a convex *dependence function* A defined on I in such a way that $\max\{t, 1- t\} \leq A(t) \leq 1$ for all $t \in \mathrm{I}$. The bounds 1 and $\max\{t, 1 - t\}$ correspond to copulas Π and M respectively. We denote by A the set of all A' s.

Remark 2.2. Let $A \in A$ and set

$$\widetilde{A}(t) = \left[t^r + (1-t)^r\right]^{1/r} A^{1/r}\left(\frac{t^r}{t^r + (1-t)^r}\right), \quad t \in \mathrm{I}. \tag{5}$$

Then $\widetilde{A} \in A$ for $r \geq 1$. Actually for r = 1, $\widetilde{A} = A$; so we assume r > 1. Further we consider copula C_A and we observe that $C_A^q(x,y) = C_A(x^q, y^q)$ is a bivariate distribution function for any q > 0. Hence, for any probability measure ν on $(0, \infty)$, $C_o = \int_0^\infty C_A^q d\nu(q)$ is a bivariate distribution function which may be written as $\psi(-\log C_A)$, where ψ is the Laplace transform of ν. We now take ν to be the stable distribution with characteristic exponent 1/r and $\psi(s) = \exp\{-s^{1/r}\}$. Both margins of C_o coincide and are equal to $F(x) = \exp\{-(-\log(x))^{1/r}\}$ with $x \in \mathrm{I}$. It follows that $C_1(x,y) = C_O\{F^{-1}(x), F^{-1}(y))$ is a copula and

$$C_1(x,y) = \exp\{\log(xy)\tilde{A}(\log(x)/\log(xy))\}, \qquad x, y \in I.$$

Since $C_1(x^{1/n}, y^{1/n})^n = C_1(x, y)$ we conclude that C* belongs to its own maximum domain of attraction; thus $\tilde{A} \in$ A. A different argument for the proof of this assertion is in Capéraà, Fougères, and Genest [3, p. 37].

A dependence function A which will occur in the sequel is

$$A_{\theta_1,\theta_2}(t) = \max\{1 - \theta_1(1 - t), 1 - \theta_2 t\}, \theta_1, \theta_2 \in (0,1], \tag{6}$$

and $A_{0,0} = 1$ (we observe that $A_{0,0}$, $A_{\theta_1,1}$ and A_{1,θ_2} are extreme elements of A). Such an A_{θ_1,θ_2} leads to the copula

$$C_{\theta_1,\theta_2}(x,y) = \begin{cases} xy^{1-\theta_1} & \text{for } x^{1/\theta_1} \le y^{1/\theta_2}, \\ x^{1-\theta_2}y & \text{for } y^{1/\theta_2} \le x^{1/\theta_1}, \end{cases}$$

and $C_{0,0} = \Pi$. This two-parameter family of copulas is an extension of the one-parameter Cuadras-Augé (Nelsen (1999), p. 12 and p. 47) family of copulas ,

$$C_\theta(x,y) = \begin{cases} xy^{1-\theta} & \text{for } x \le y, \\ x^{1-\theta}y & \text{for } y \le x, \end{cases}$$

where $\theta \in I$.

2.2. Unimodality

In what follows we shall be concerned with the following notion of bivariate unimodality:

Star unimodality Dharmadhikari and Joag-dev (1988), p. 38, Bertin, Cuculescu, and Theodorescu (1997), p. 72 :a distribution C is said to be *star unimodal* about $x \in R^2$ if it belongs to the closed convex hull of the set of all uniform distributions on sets which are star-shaped about x (i.e. which contain together with an y the whole segment joining x to y).

A distribution C is star unimodal about (a, b) if and only if it is a mixture of the form

$$C = \int \sigma_{(a,b),(u,v)} d\mu(u,v), \tag{7}$$

where the probability measure μ on R^2 is unique, $\sigma_{(a,b),(a,b)} = \varepsilon_{(a,b)}$ (ε_w stands for the point mass at w), $\sigma_{(a,b),(u,v)}$, for $(u, v) \neq (a, b)$, is concentrated on the segment joining (a, b) to (u, v) and has with respect to the uniform distribution a probability density function f(u', v') which is proportional to the distance between (u', v') and (a, b).

In the sequel we shall use the following result (Cuculescu and Theodorescu (2002), (Proposition 3.3]) concerning star unimodal copulas.

Proposition 2.3. A copula C star unimodal about a point $(a, b) \in I^2$ is a mixture of the form

$$\overline{C} = \int_{I^2} \overline{\sigma}_{(a,b),(u,v)} d\mu(u,v), \tag{8}$$

with the unique probability measure

$$\mu = \sum_{\alpha,\beta\in\{0,1\}} c_{\alpha,\beta}\varepsilon_{(\alpha,\beta)} + d_0^1\varepsilon_0 \otimes (f_0^1 m) + d_1^1\varepsilon_1 \otimes (f_1^1 m) + d_0^2(f_0^2 m) \otimes \varepsilon_0 + d_1^2(f_1^2 m) \otimes \varepsilon_1 + c\xi, \tag{9}$$

where $c = \sum_{\alpha,\beta\in\{0,1\}} c_{\alpha,\beta}$, the remaing c's and d's are nonnegative such that

$$c_{00} + c_{01} + d_0^1 = a/2, \quad c_{10} + c_{11} + d_1^1 = (1-a)/2,$$
$$c_{00} + c_{10} + d_0^2 = b/2, \quad c_{01} + c_{11} + d_1^2 = (1-b)/2, \tag{10}$$

and f_α^i are probability density functions on I satisfying

$$(d_0^1 f_0^1 + d_1^1 f_1^1)m + c\xi_2 = (d_0^2 f_0^2 + d_1^2 f_1^2)m + c\xi_1 = 1_I m/2,$$

ξ_1 and ξ_2 being the margins /of the distribution ξ.

Remark 2.4. If in Proposition 2.3 copula C is star unimodal about (1,1) the representation (9) of μ becomes

$$\mu = c[\varepsilon_{(0,0)} + \xi] + (0.5 - c)\left[\varepsilon_0 \otimes (f_0^1 m) + f_0^2 m) \otimes \varepsilon_0\right], \tag{11}$$

where $c \in [0, 0.5]$ and $(0.5 - c) f_0^1 m + c\xi_2 = (0.5 - c) f_0^2 m + c\xi_1 = 1_I m/2$.

Remark 2.5. If C is star unimodal about $(a, b) \in I^2$ then $\hat{C}$ is star unimodal about (1 - a, 1 - b). With selfexplanatory notations we can translate Proposition 2.3 in 'hat' terms. More precisely we can write $\hat{C} = C \circ \rho^{-1}$, where $C \circ \rho^{-1}I$ is the image of C by the map $(x, y) \mapsto \rho(x, y) = (1 - x, 1 - y)$ and similarly for the other quantities. For instance $\hat{c}_{11} = c_{00}$.

3. ASYMPTOTICS OF EXTREMES

Let copula C be star unimodal about $(a, b) \neq (1,1)$.

Proposition 3.1. Copula C belongs to the maximum domain of attraction of copula C_{θ_1, θ_2} with $\theta_1 = 2c_{11}/(1 - b)$, $\theta_2 = 2c_{11}/(1 - a)$ for a, b < 1 and $\theta_1 = \theta_2 = 0$ for a = 1 or b = 1.

Proof. (a) We have

$$\overline{\mu}(x, y) - c_{11} \le k((1 - x) + (1 - y)), \tag{12}$$

where k is a constant; (12) shows that only the term $c_{11}\varepsilon_{1,1)}$ in (9) plays an important part. Inequality (12) holds since (Proposition 2.3)

$$d_j^i(\overline{f_j^i m})(z) \le (1 - z)/2,$$

$$c\overline{\xi}(x, y) \le c\overline{\xi}_1(x) + c\overline{\xi}_2(y) \le ((1 - x) + (1 - y))/2,$$

$$\overline{\xi}_{(\alpha,\beta)}(x, y) = 0, \quad (\alpha, \beta) \neq (1,1).$$

(b) Further we take into account representation (8) and inequality (12) to evaluate $\overline{C}(x, y)$ when x, y are sufficienly clase to 1. Namely we suppose in the remaining of the proof that min {x, y} > min {a, b}. With this purpose in mind, we diminish the domain of integration I^2. Indeed if $(a, b) \neq (1,1)$ then $\overline{\sigma}_{(a,b)(u,v)}(x, y) = 0$ for $(u, v) \notin (x', 1] \times (y', 1])$, where x' = x'(x, y) and y' = y'(x, y) are defined by: x' = x for $a \le x$ and $\frac{a - x'}{a - x} = \frac{1 - b}{y - b}$ for $x < a$ and y' = y for $b \le y$ and $\frac{b - y'}{b - y} = \frac{1 - a}{x - b}$ for $y < b$. Thus

$$\lim_{x,y\to 1}(1-x')/(1-x) = \lim_{x,y\to 1}(1-y')/(1-y) = 1.$$

Therefore we can write

$$\overline{C}(x,y) = \int_{(x',1]\times(y',1]} \overline{\sigma}_{(a,b),(u,v)}(x,y)d\mu(u,v) = c_{11}\overline{\sigma}_{(a,b),(1,1)}(x,y) + R; \tag{13}$$

here,

$$\begin{aligned} |R| &\le \sup\{\overline{\sigma}_{(a,b),(u,v)}(x,y) : (u,v) \in (x',1]\times(y',1]\}k((1-x')+(1-y')) \\ &\le \sup\{\overline{\sigma}_{(a,b),(u,v)}(x,y) : (u,v) \in (x',1]\times(y',1]\}k_1((1-x)+(1-y)), \end{aligned} \tag{14}$$

where k_1 is a constant.

(c) In order to evaluate the upper bound in (14) we denote by $S_{A,B}$ the segment with endpoints A and B; for (w, z) $\in S_{(a,b),(u,v)}$ we observe that

$$\sigma_{(a,b),(u,v)}(S_{(w,z),(u,v)} = 1-\left(\frac{w-a}{u-a}\right)^2 = 1-\left(\frac{z-b}{v-b}\right)^2. \tag{15}$$

Since in (15) we have two equal values for the mass of $S_{(w,z),(u,v)}$ we shall use the notations $S_{(*,z),(*,v)}$ and $S_{(w,*),(u,*)}$ when missing coordinates do not play any role in calculations. For (u, v) $\in$ (x, 1] $\times$ (y, 1] we have (by (15))

$$\overline{\sigma}_{(a,b),(u,v)}(x,y) \le \min\{\sigma_{(a,b),(u,v)}(S_{(x,*),(1,*)}, \sigma_{(a,b),(u,v)}(S_{(*,y),(*,1)})\}$$

$$= \min\left\{1-\left(\frac{x-a}{u-a}\right)^2, 1-\left(\frac{y-b}{v-b}\right)^2\right\}.$$

For (u, v) $\in$ (x', x] $\times$ (y, 1] we obtain (by (15))

$$\begin{aligned} \overline{\sigma}_{(a,b),(u,v)}(x,y) &\le \sigma_{(a,b),(u,v)}(S_{(*,y),(*,1)}) \\ &= 1-\left(\frac{y-b}{v-b}\right)^2. \end{aligned}$$

For (u, v) $\in$ (x, 1] $\times$ (y', y] we are led to (by (15))

$$\begin{aligned} \overline{\sigma}_{(a,b),(u,v)}(x,y) &\le \sigma_{(a,b),(u,v)}(S_{(x,*),(1,*)}) \\ &= 1-\left(\frac{x-a}{u-a}\right)^2. \end{aligned}$$

It is not possible that x' < x and y' < y since this implies x < a, y < b, and min{x,y} < min{a,b}. Por u > a we have u $\notin$ (x',x] and

$$1-\left(\frac{x-a}{u-a}\right)^2 < 1-\left(\frac{x-a}{1-a}\right)^2 = \left(1+\frac{x-a}{1-a}\right)\frac{1-x}{1-a} \le 2\frac{1-x}{1-a} \tag{16}$$

while for v > b we have v $\notin$ (y', y] and

$$1-\left(\frac{y-b}{v-b}\right)^2 < 2\frac{1-y}{1-b}. \tag{17}$$

Since (a, b) # (1,1), for x, y sufficiently close to 1, inequalities (16) and (17) lead to

$$\sup\{\overline{\sigma}_{(a,b),(u,v)}(x,y) : (u,v) \in (x',1] \times (y',1]\} \le k_2((1-x)+(1-y)), \tag{18}$$

where k_2 is a constant. Hence, in view of (13), (14), and (18), we conclude that

$$\left|\overline{C}(x,y) - c_{11}\overline{\sigma}_{(a,b)(1,1)}(x,y)\right| \le k_3(1-x)^2 + (1-y)^2), \tag{19}$$

where k_3 is a constant.

(d) By virtue of (10) for a = 1 or b = 1 we have C_{11} = O, while for a, b < 1 and x > a, y > b

$$\overline{\sigma}_{(a,b),(1,1)}(x,y) = \begin{cases} 1-\left(\frac{x-a}{1-a}\right)^2 & \text{for } \frac{x-a}{1-a} \ge \frac{y-b}{1-b}, \\ \\ 1-\left(\frac{y-b}{1-b}\right)^2 & \text{for } \frac{x-a}{1-a} \le \frac{y-b}{1-b}. \end{cases} \tag{20}$$

Rewriting (20) we obtain

$$\overline{\sigma}_{(a,b),(1,1)}(x,y) = \begin{cases} 2(1+\eta_1(x))(1-x)/(1-a) & \text{for } \frac{x-a}{1-a} \ge \frac{y-b}{1-b}, \\ \\ 2(1+\eta_2(x))(1-y)/(1-b) & \text{for } \frac{x-a}{1-a} \le \frac{y-b}{1-b}, \end{cases} \tag{21}$$

where $\eta_1(z)$, $\eta_2(z) \to 0$ as $z \to 1$.

(e) We now fix O < x, y < 1. We consider the inequality

$$\frac{x^{1/n-a}}{1-a} > \frac{y^{1/n-b}}{1-b}.$$

Since $n(1 - x^{1/n}) \to \log(l/x)$ as $n \to \infty$ the inequality is valid for $n \ge n_0(x, y)$ if

$$\frac{\log(1/x)}{1-a} < \frac{\log(1/y)}{1-b},$$

i.e. if $x^{1-b} > y^{1-a}$; the opposite inequality is valid for $n \ge n_1(x, y)$ if $x^{1-b} < y^{1-a}$. We obtain (by (19) and (21))

$$\lim_{n\to\infty} n\overline{C}(x^{1/n}, y^{1/n}) = \begin{cases} \log(1/x)\frac{2c_{11}}{1-a} & \text{for } x^{1-b} > y^{1-a}, \\ \\ \log(1/y)\frac{2c_{11}}{1-b} & \text{for } x^{1-b} > y^{1-a}. \end{cases}$$

These two limits coincide for $x^{1-b} = y^{1-a}$. Thus the limit relations are also valid for $x^{1-b} = y^{1-a}$. Hence (3) holds with $C^* = C_{\theta_1, \theta_2}$. □

Remark 3.2. Copula C_{θ_1, θ_2} is not star unimodal except when $\theta_1, \theta_2 = 1$ (i.e. $C_{1,1} = C_{A_{1,1}} = M$) and $\theta_1, \theta_2 = 0$ (i.e. $C_{0,0} = \Pi$). This is the consequence of the fact that the singular part of a star unimodal (about (a, b)) copula is concentrated on a union of half-lines originating in (a, b) (Cuculescu and Theodorescu [(2002), Remark 4.3]). If we leave M and Π out, copula C_{θ_1, θ_2} has a singular part concentrated on $I^2 \cap \{(x, y): x^{1/\theta_1} = x^{1/\theta_2}\}$ which charges every arc of this curve. To show this property we observe that the conditional distribution function $C_{\theta_1,\theta_2}(\cdot|x)$ has the probability density function $(1 - \theta_2)x^{-\theta_2}$ on the interval $(0, x^{\theta_2/\theta_1})$ and $(1 - \theta_1)y^{-\theta_1}$ on $(x^{\theta_2/\theta_1},1)$, and the masses of these two intervals sum up to $x^{\theta_2/\theta_1-\theta_2}(1 - \theta_2) + 1 - x^{(1 - \theta_1)\theta_2/\theta_1} = 1 - \theta_2 x^{(1-\theta_1)\theta_2/\theta_1}$, i.e. $C_{\theta_1, \theta_2}(\cdot|x)$ has an atom of mass $\theta_2 x^{(1-\theta_1)\theta_2/\theta_1}$ at x^{θ_2/θ_1}.

Remark 3.3 For every copula C_{θ_1, θ_2} there exists a star unimodal (about (a, b) $\neq$ (1,1)) copula C such that the assertion in Proposition 3.1 holds.

The following result deals with the case (a, b) = (1,1) which was left out in the preceding proposition. In what follows copula G is related to the measure μ by (7).

Proposition 3.4. Let copula G be star unimodal about (1,1). The following are equivalent:

(I) G belongs to the maximum domain of attraction of C_A (given by (4)) for some dependence function A.

(II) For all x, y $\in$I there exists

$$\lim_{n\to\infty} n(1-\mu(x^{1/n}, y^{1/n})) = h(x, y). \tag{22}$$

Moreover

$$h(x,y) = -0.5 \log C_A(x,y), \qquad x,y \in I.$$

Proof. Part (II) $\to$ (I)

(a) We note that $h(x^z, y^z) = zh(x, y)$ for $z > 0$. This result follows immediately for $z = 1/k$, then for rational z; the fact that h is nonincreasing in x and y extends the property to all z > O.

We have to show (2), i.e.

$$\lim_{n\to\infty} n(1-C(x^{1/n}, y^{1/n})) = \lim_{n\to\infty} n\int_{I^2} (1-\sigma_{(1,1),(u,v)}(x^{1/n}, y^{1/n}))d\mu(u,v) = 2h(x,y).$$

(b) We observe that

$$\sigma_{(1,1),(u,v)}(x,y) = \begin{cases} 1-\max\left\{\left(\dfrac{1-x}{1-u}\right)^2, \left(\dfrac{1-y}{1-v}\right)^2\right\} & \text{for } u \le x \text{ and } v \le y \\ 0 & \text{for } u \ge x \text{ or } v \ge y \end{cases}$$

Hence

$$\sigma_{(1,1),(1-(1-x)/z,v)}(x,y) = \sigma_{(1,1),(u,1-(1-y)/z)}(x,y) = 1 - z^2$$

for $v \le 1- (1 - y)/z$ and $u \le 1- (1- x)/z$ with $z \in [\max\{1 - x, 1 - y\}, 1]$.

(c) Now we set

$$\lambda_{x,y}(t)=\begin{cases}1-\mu(1-(1-x)/t,1-(1-y/t) & \text{for } t\in[\max\{1-x,1-y\},1],\\ 1 & \text{for } t<\max\{1-x,1-y\}.\end{cases}$$

We note that the function $t\mapsto\lambda_{x,y}(t)$ is nonincreasing. For the sake of simplicity, we assume that x > y. Then

$$1-C(x,y)=\int_{([0,x]\times[0,y])^c}(1-\sigma_{(1,1),(u,v)}(x,y))d\mu(u,v)+\int_{[0,x]\times[0,y]}(1-\sigma_{(1,1),(u,v)}(x,y))d\mu(u,v)$$

For the second integral we can write

$$\int_{[0,x]\times[0,y]}(1-\sigma_{(1,1),(u,v)}(x,y))d\mu(u,v)=\int_{[0,x]\times[0,y]}\left[\min\left\{\frac{1-x}{1-u},\frac{1-y}{1-v}\right\}\right]^2 d\mu(u,v).$$

The set

$$\left\{(u,v):\min\left\{\frac{1-x}{1-u},\frac{1-y}{1-v}\right\}>z\right\}=[0,1-(1-x)/z]\times[0,1-(1-y)/z]$$

has μ-mass $\lambda_{x,y}(z)$. Consequently integrating by parts we obtain

$$\begin{aligned}1-C(x,y)&=1-\mu(x,y)-\int_{[1-y,1]}z^2 d\lambda_{x,y}(z)\\ &=\lambda_{x,y}(1)-\lambda_{x,y}(1)+(1-y)^2+\int_{1-y}2z\lambda_{x,y}(z)dz.\end{aligned}\tag{23}$$

(d) Since $n(l-y^{1/n})^2\to 0$ as $y\mapsto 1$ we can write

$$\lim_{n\to\infty}n(1-C(x^{1/n},y^{1/n}))=\lim_{n\to\infty}2\int_{1-y^{1/n}}^{1}zn\lambda_{x^{1/n},y^{1/n}}(z)dz.$$

Further we set $t_{n,z}=[1-(1-t^{1/n})/z]^n$. Then $t_{n,z}\to t^{1/z}$ as $n\to\infty$. Hence

$$\lim_{n\to\infty}n\lambda_{x^{1/n},y^{1/n}}(z)=\lim_{n\to\infty}n(1-\mu(x_{n,z}^{1/n},y_{n,z}^{1/n}))=h(x^{1/z},y^{1/z})=h(x,y)/z.$$

Also

$$zn\lambda_{x^{1/n},y^{1/n}}(z)\le nz(1-x_{n,z}^{1/n}+1-y_{n,z}^{1/n})=n(1-x^{1/n}+1-y^{1/n}).$$

The last quantity, having as limit - log(xy) as $n\to\infty$, is bounded; therefore Dominated Convergence Theorem applies yielding

$$\lim_{n\to\infty}n(1-C(x^{1/n},y^{1/n}))=2\int_0^1 h(x,y)dz=2h(x,y).$$

Part (I) $\to$ (II)

(a) Let x > y be fixed. From (23) we obtain

$$\lim_{n\to\infty}\int_{1-y^{1/n}}^{1} zn\lambda_{x^{1/n},y^{1/n}}(z)dz = h(x,y) = -0.5\log C_A(x,y), \qquad x,y\in I. \tag{24}$$

(b) Let s, $1 - y < s < 1/(1 - x)$ be fixed and set $u = 1 - (1 - x)/s$, $v = 1 - (1 - y)/s$. For $t \in (1 - u, 1)$ we have $1 - (1 - u)/t = 1 - (1 - x)/(st)$, $1 - (1 - v)/t = 1 - (1 - y)/(st)$; thus $\lambda_{u,v}(t) = \lambda_{x,y}(st)$.

(c) Denoting $1 - (1 - x^{1/n})/s = u_{x,n}^{1/n}$ we have $u_{x,n} \to x^{1/s}$ as $n \to \infty$. Consequently

$$\lim_{n\to\infty}\int_{1-y^{1/n}}^{s} zn\lambda_{x^{1/n},y^{1/n}}(z)dz = s^2\lim_{n\to\infty}\int_{1-u_{y,n}^{1/n}}^{1} zn\lambda_{u_{x,n}^{1/n},u_{y,n}^{1/n}}(z)dz = s^2h(x^{1/s},y^{1/s}) = sh(x,y).$$

(d) Since for $z \le 1$ we have

$$\lambda_{x^{1/n},y^{1/n}}(z) \ge \lambda_{x^{1/n},y^{1/n}}(1) = 1-\mu(x^{1/n},y^{1/n})$$

we obtain, taking s < 1 in (b),

$$0.5(1-s^2)\limsup_{n\to\infty}\left(1-\mu(x^{1/n},y^{1/n})\right) \le \lim_{n\to\infty}\int_{s}^{1} zn\lambda_{x^{1/n},y^{1/n}}(z)dz = (1-s)h(x,y),$$

while for s > 1

$$0.5(s^2-1)\liminf_{n\to\infty}\left(1-\mu(x^{1/n},y^{1/n})\right) \ge (s-1)h(x,y).$$

We conclude the proof by letting $s \uparrow 1$ and $s \downarrow 1$ in the two preceding formulas. □

We now consider representation (11). If c = 0 then C = Π.

Corollary 3.5. If $c \in (0,0.5]$ then for any dependence function A there exists a measure μ of the form (11) such that copula C (given by (7)) belongs to the maximum domain of attraction of C_A (given by (4)).

Proof. (a) We have to find a μ of the form (11) satisfying (22) with

$$h(x, y) = -\tfrac{1}{2}\log(xy)A(\log(x)/\log(xy)).$$

In other words,

$$\lim_{n\to\infty} n\overline{\mu}(x^{1/n},y^{1/n}) = -\tfrac{1}{2}\log(xy)[1-A(\log(x)/\log(xy))]. \tag{25}$$

(b) In view of (11) $\overline{\mu} = c\overline{\xi}$. So finding μ amounts to finding ξ and $f_0^i, i = 1,2,$ as in (11) such that $\overline{\mu} = c\overline{\xi}$ satisfies (25).

With this purpose in mind we take ζ to be a copula which belongs to the maximum domain of attraction of C_A. For c = 0.5 we take $\xi = \zeta$ in the representation (11) of μ. For $c \in (0,0.5)$ we take $f_0^1 = f_0^2$ to be the uniform probability density function on the interval (0,1 - 2c) and

$$\xi = \Big[1_{[1-c,1]\times[1-c,1]}\zeta + (1_{[1-c,1]\times[0,1-c]}\zeta)_1 \otimes (1_{[1-2c,1-c]}m)/c$$
$$+ \ (1_{[1-2c,1-c]}m) \otimes (1_{[0,1-c]\times[1-c,1]}\zeta)_2 /c$$
$$+ \ \overline{\zeta}(1-c,1-c)(1_{[1-2c,1-c]}m) \otimes (1_{[1-2c,1-c]}m)/c^2\Big]\Big/(2c);$$

here subscripts 1, 2 stand for margins. It follows that

$$\xi_1 = \Big[(1_{[1-c,1]\times[1-c,1]}\zeta)_1 + (1_{[1-c,1]\times[0,1-c]}\zeta)_1 + (1_{[1-2c,1-c]}m)\zeta([0,1-c]\times[1-c,1])/c$$
$$+ \ \overline{\zeta}(1-c,1-c)(1_{[1-2c,1-c]}m)/c\Big]\Big/(2c)$$
$$= \ \Big[(1_{[1-c,1]\times[0,1]}\zeta)_1 + (1_{[1-2c,1-c]}m)\zeta([0,1]\times[1-c,1])/c\Big]\Big/(2c)$$
$$= \ \Big[1_{[1-c,1]}m) + (1_{[1-2c,1-c]}m)\Big]\Big/(2c) = 1_{[1-2c,1]}m]/(2c).$$

In a similar manner we obtain $\xi_2 = \xi_1$. Hence

$$(0.5 - c)\, f_0^1 m \ + c\xi_2 = (0.5 - c)\, f_0^2 m + c\xi_1 = 1_{[0,1-2c]}m/2 + 1_{[1-2c,1]}m/2 = 1_I m/2,$$

i.e. (11) is satisfied with these elements. Moreover $c\overline{\xi} = 0.5\overline{\zeta}$. □

Remark 3.6. Why is the behavior of star unimodal copulas so different for (a, b) ≠ (1,1) and for (a, b) = (1,1) as far as the maximum domain of attraction is concerned? For (a, b) ≠ (1,1) Part (b) in the proof of Proposition 3.1 shows that for every given (u, v) ≠ (1,1) we have $\overline{\sigma}_{(a,b),(u,v)}(x,y) = 0$ if x, y are sufficiently close to 1. This does not happen when (a, b) = (1,1). In this case if the point (u, v) is away from (1,1) then the contribution of $\overline{\sigma}_{(1,1),(u,v)}$ to $\overline{C}$ becomes negligeable when x, y → 1, so only the contribution of $\overline{\mu}$ is pertinent.

4. MORE ON UNIMODALITY AND ARCHIMAX COPULAS

Results in Section 3 allow us to infer that certain copulas are not star unimodal.

Let $\phi: I \to [0, \infty]$ with $\phi(1) = 0$ be a continuous, convex, and strictly decreasing function and denote by $\phi^{[-1]}$ its *pseudo-inverse* given by

$$\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t) & \text{for} \quad 0 \le t \le \phi(0), \\ 0 & \text{for} \quad \phi(0) \le t \le \infty. \end{cases}$$

If $\phi(0) = \infty$ then $\phi^{[-1]} = \phi^{-1}$. For the sake of simplicity we shall use only the notation ϕ^{-1}. Further we consider a dependence function A. A copula C is Archimax if

$$C_{\phi,A}(u,v) = \phi^{-1}\left[(\phi(u)+\phi(v))A\left(\frac{\phi(u)}{\phi(u)+\phi(v)}\right)\right], \quad u,v \in \mathbf{I}.$$

The function ϕ is its generator. For $\phi(t) = \log(1/t)$ we are led the extreme value copula (4) and for A ≡ 1 we obtain the *Archimedean* copula (Nelsen [1999, p. 90]). Archimax copulas were introduced by Capéraà, Fougères, and Genest (2000). The llame 'Archimax' was chosen to reflect the fact that the new family includes both the maximum value distributions and the Archimedean copulas.

We observe that for any generator ϕ we have $C_{\phi A_{1,1}} = M$, where $A_{1,1}$ is given by (6).

According to Cuculescu and Theodorescu (2002), [Propositions 6.1 and 6.2] Archimedean copulas are not star unimodal except Π and W. Maximum value copulas are neither except Π and M (Proposition 4.2).

We start with a preliminary result.

Lemma 4.1. Let copula C be star unimodal about (a, b) $\neq$ (0,0). Then

$$\lim_{n\to\infty} nC(x/n, y/n) = \min(\theta_2 x, \theta_1 y), \quad x, y \in \mathrm{I}, \tag{26}$$

with $\theta_1 = 2c_{oo}/a$, $\theta_2 = 2c_{oo}/b$ for a, b > 0 and $\theta_1 = \theta_2 = 0$ for a = 0 or b = 0.

Proof. We have nC(x/n, y/n) = $n\overline{\hat{C}}(1 - x/n, 1 - y/n)$. Since $\hat{C}$ is star unimodal about (1 - a, 1 - b) $\neq$ (1,1) we deduce (Proposition 3.1) that the survival copula $\hat{C}$ belongs to the maximum domain of attraction of C_{θ_1,θ_2}. i.e. (by (3)).

$$\lim_{n\to\infty} n\overline{\hat{C}}(u^{1/n}, v^{1/n}) = -\log(uv) + \log C_{\theta_1\theta_2}(u,v) = \min\{-\theta_2 \log(u) - \theta_1 \log(v)\}, \qquad u,v \in \mathrm{I}.$$

We set 1 - x/n = $u_{x,n}^{1/n}$ and observe that $u_{x,n} \to e^{-x}$ as $n \to \infty$. By substitution we obtain our result. □

Proposition 4.2. A maximum extreme value copula is not star unimodal about any (a, b) $\in \mathrm{I}^2$ except Π and M.

Proof. Suppose that C_A is star unimodal about (a, b).

(a) If (a, b) $\neq$ (1,1) then, since C_A belongs to its own domain of attraction, it follows (Proposition 3.1) that $C_A = C_{\theta_1,\theta_2} = 0$ for some θ_1, θ_2. We conclude (Remark 3.2) that C_{θ_1,θ_2} is not star unimodal unless $\theta_1 = \theta_2 = 0$, i.e. $C_A = \Pi$, or $\theta_1 = \theta_2 = 1$ when C_A = M.

(b) If (a, b) = (1,1) we have (Proposition 2.3) $c_{0,0} \neq 0$ otherwise $C_A = \Pi$. But

$$\lim_{n\to\infty} nC_A(x/n,\ y/n) = \lim_{n\to\infty} n\exp\{\log(xy/n^2)A(\log(x/n)/\log(xy/n^2))\}.$$

Let CA $\neq$ M; for some $\eta > 0$ we have A(t) > ½ + η for all t. Since $\log(xy/n^2) < 0$ we obtain

$$n\exp\{\log(xy/n^2)A(\log(x/n)/\log(xy/n^2))\} \le n(xy/n^2)^{1/2+\eta} = (xy)^{1/2+\eta} n^{-2\eta} \to 0$$

as $n \to \infty$. Contradiction (Lemma 4.1). □

Remark 4.3. A by-product of the proof of Proposition 4.2 is: for every dependence function A $\neq$ 1, $A_{1,1}$ we have $\hat{C}_A \neq C_B$ for all dependence functions B. The case A $\neq$ 1 implies $\lim_{n\to\infty} n\,\hat{C}_A(x/n, y/n) \neq 0$ since, denoting $u_{x,n} = (1 - x/n)^n \to e^{-x}$ as $n \to \infty$, this limit equals (by (3) and (4))

$$\lim_{n\to\infty} n\overline{C_A}(u_{x,n}^{1/n}, u_{x,n}^{1/n}) = (x+y)[1 - A(x/x+y)].$$

For A $\neq A_{1,1}$ (i.e. $\hat{C}_A \neq$ M) it was seen in the proof of Proposition 4.2 that

$$\lim_{n\to\infty} nC_B = (x/n, y/n) = 0$$

for $C_B \neq$ M.

Since in general neither Archimedean nor maximum value copulas are star unimodal let us look closer at Archimax copulas.

Proposition 4.4. Suppose that A $\neq A_{1,1}$. If $\phi(1 - 1/t)$ is regularly varying at infinity with degree – r for some r > 1 then copula $C_{\phi,A}$ is not star unimodal about any (a, b) $\neq$ (1,1).

Proof. According to Capéraà, Fougères, and Genest [(2000), Proposition 4.1] copula $C_{\phi,A}$ belongs to the maximum domain of attraction of $C_{\tilde{A}}$ with $\tilde{A}$ given by (5). If $C_{\phi,A}$ is star unimodal about (a, b) $\neq$ (1,1) then (Proposition 3.1) $\tilde{A} = A_{\theta_1,\theta_2}$ for some θ_1, θ_2. Hence

$$[t^r + (1-t)^r]^{1/r} A^{1/r}\left(\frac{t^r}{t^r + (1-t)^r}\right) = \max\{1-\theta_1(1-t), 1-\theta_2 t\}, \qquad t \in I.$$

For $u = t^r/(t^r + (1-t)^r)$ we obtain

$$A(u) = \left[\max\{u^{1/r} + (1-\theta_1)(1-u)^{1/r}, (1-\theta_2)u^{1/r} + (1-u)^{1/r}\}\right]^r.$$

Suppose θ_1, θ_2 > O. Then $A^{1/r0}(0) = \max\{1 - \theta_1, 1\}$ and therefore there exists $\eta > 0$ such that $A(u) = [(1-\theta_2)u^{1/r} + (1-u)^{1/r}]^r$ for $u < \eta$. If $\theta_2 < 1$ then the derivative at 0 of $(1-\theta_2)u^{1/r} + (1-u)^{1/r}$ is $+\infty$; hence A cannot be a dependence function. In the same way we deal with the case $\theta_1 < 1$. For $\theta_1 = \theta_2 = 0$ we have $A(u) = [u^{1/r} + (1-u)^{1/r}]^r$ and the same argument applies. □

Lema 4.5. Let copula $C_{\phi,A}$ be an Archimax copula. We assume that $\phi(0) = \infty$ and $\phi(l/t)$ is regularly varying at infinity with degree k for some k > 0. Then

$$\lim_{n\to\infty} nC_{\phi,A}(x/n,\ y/n) = (x^{-k} + y^{-k})^{-1/r} A^{1/r}\left(\frac{y^k}{x^k + y^k}\right), \qquad x,y \in I.$$

Proof. We write

$$\lim_{n\to\infty} nC_{\phi,A}(x/n,\ y/n) = \lim_{n\to\infty} n\phi^{-1}(\gamma n(x,y)), \qquad x,y \in I.$$

where

$$\gamma_n(x,y) = (\phi(x/n) + \phi(y/n))A\left(\frac{\phi(x/n)}{\phi(x/n)+\phi(y/n)}\right).$$

From the regularity of ϕ we obtain

$$\frac{\phi(x/n)}{\phi(x/n)+\phi(y/n)} = \frac{1}{1+\phi(y/n)/\phi(x/n)} \to \frac{1}{1+(x/y)^k} = \frac{y^k}{x^k+y^k},$$

therefore

$$\gamma_n(x,y)/\phi(x/n) \to (1+(x/y)^k)A\left(\frac{y^k}{x^k+y^k}\right)$$

as $n \to \infty$. We now observe that ϕ^{-1} is regularly varying at infinity with degree -1/k.

Hence

$$\lim_{n\to\infty} nC_{\phi,A}(x/n,\ y/n) = \lim_{n\to\infty} x\phi^{-1}(\gamma_n(x,y))/\phi^{-1}(\phi(x/n))$$

$$= x(1+(x/y)^k)^{-1/k} A^{-1/k}\left(\frac{y^k}{x^k+y^k}\right). \qquad □$$

Proposition 4.6. Suppose that $A \neq A_{1,1}$. If $\phi(0) = \infty$ and ϕ (1/t) is regularly varying at infinity with degree k for some k > 0 then copula $C_{\phi,A}$ is not star unimodal about any (a,b) ≠ (0,0).

Proof.. If $C_{\phi,A}$ is star unimodal about (a, b) ≠ (0,0) then from Lemmas 4.1 and 4.5 we deduce

$$\min(\theta_2 x, \theta_1 y) = (x^{-k} + y^{-k})^{-1/k} A^{-1/k}\left(\frac{y^k}{x^k + y^k}\right), \; x,y \in \mathbf{I}. \tag{27}$$

Thus $\theta_1, \theta_2 \neq 0$. Formula (27) involves only x/y = u:

$$\min(\theta_2 u, \theta_1) = u(u^k + 1)^{-1/k} A^{-1/k}\left(\frac{1}{u^k + 1}\right).$$

For $t = 1/(u^k + 1)$ we obtain

$$A(t) = \max\{(1-t)\,\theta_1^{-k}, t\theta_2^{-k}\}.$$

But $A(t) \leq 1$ so we are led to $\theta_1 = \theta_2 = 1$. Contradiction since $A \neq A_{1,1.}$ □

From Propositions 4.4 and 4.6 we deduce

Corollary 4.7. Suppose that $A \neq A_{1,1}$. Under the regularity conditions in Propositions 4.4 and 4.6 copula $C_{\phi,A}$ is not star unimodal about any (a, b) $\in \mathbf{I}^2$.

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