PORTFOLIO OPTIMIZATION
WITH TARGET-SHORTFALL-PROBABILITY VECTOR

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ABSTRACT
Traditional portfolio optimization uses the standard deviation of the returns as a measure of risk. In recent years, the Target-Shortfall-Probability (TSP) was discussed as an alternative measure. From the utility-theoretical point of view, the TSP is not perfect. Furthermore it is criticized due to the insufficient description of the risk. The advantages of the TSP are the usage independent of the distribution and the intuitive understanding by the investor.The use of a TSP-vector reduces an utility-theoretical disadvantage of a single TSP and offers an sufficient description of risk. The developed Mean-TSP-vector model is a mixed-integer linear program. The CPU-Time of the program to get a solution demonstrates that the model is suitable for practical applications. A test of the performance shows, that the average return of the model when used in bear markets is equal to the results of the traditional portfolio optimization but -due to skewness- in bullish markets can achieve better returns.

Key words: optimal portfolio management, Integer mixed models, risk criterion.

MSC: 91B24

RESUMEN
La tradicional solución de la optimización para el portafolio usa la desviación standard como medida del riesgo. En años recientes se discute sobre la probabilidad de obtener menos que el objetivo esperado (Target-Shortfall probability, TSP) como una medida alternativa del riesgo. A partir del punto de vista de la teoría de la utilidad el TSP no es perfecto. Las ventajas del TSP están en el uso independiente de la distribución y en la compresión del inversionista. El uso del vector de TSP reduce una de la desventajas del TSP simple de acuerdo a la teoría de la utilidad y ofrece una descripción suficiente del riesgo. El modelo Vector-Media-TSP es un programa entero lineal mixto. El tiempo CPU para obtener una solución del programa demuestra que el modelo se ajusta a las aplicaciones prácticas. Una prueba de su comportamiento muestra que el retorno promedio del modelo al ser usado en mercados bajistas (bear markets) es igual a los resultados del tradicional modelo de optimización del portafolio pero debido a la deformación en mercados de puja pueden obtenerse mejores retornos.

1. INTRODUCTION

This year we celebrate the 50th birthday of modern portfolio theory. In the seminal work "Portfolio Selection", H. Markowitz (1952) proposed to use the variance of the returns of assets as a risk measure. Important developments in portfolio management were founded on that definition of risk. In the context of this 50th anniversary, we can celebrate the discussion of the Target Shortfall Probability (TSP) as a risk measure, too. Some months after the publication of "Portfolio Selection", A. D. Roy proposed in "Safety-First" that alternative or additional risk measure which reflects better what investors try to avoid. Despite the good reflection of risk and the intuitive understanding of that risk measure by investors, it is not restricted to a special return distribution. Hence it can also be used, when the return distribution is skewed (An empirical research at the Tokyo Stock Exchange exhibited skewness in the return distributions (see Kariya, T., Tsukuda, Y., Maru, J., (1989)). That advantages motivated many researchers to discuss the Target Shortfall Probability (E.g. Roy, A.D. (1952), Telser, L.G. (1955), Kataoka, S. (1963), Leibowitz, M.L. and Henrickson, R.,D (1989), Leibowitz, M.L.; Kogelmann, S. and Bader, L.N. (1996). Like the traditional portfolio optimization, the use of TSP as a risk measure has its disadvantages. First, it is criticized, because of its limited description of risk. Two portfolios with the same TSP can have a very different shape of the return distribution below the target and therefore the investor's utility would be different too. A second disadvantage is the time for computing an optimal solution, due to the mixed integer structure of the TSP based models. To reduce the first handicap, the following model will use a vector of TSPs. The relatively short computing time of several empirical examples shows, that the second handicap is no longer a barrier for first use in practice. After the introduction of the Mean–TSP-vector model, utility-concerned characteristics will be discussed.

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Empirical data were used to exhibit some features of the solutions and their position in the mean-variance-space. A first empirical test indicates an interesting performance.

2. TARGET SHORTFALL PROBABILITY

For the consideration of skewness in portfolio optimization some researchers implement this parameter directly into the model (Cf. King, A.J. (1993), Konno, H. and others, (1993, 1998, 2000), Stone, B. K., (1973). Another way to respect skewness offer the Lower Partial Moments (LPM):

$$LPM^l(R,\tau) = \int_{-\infty}^{\tau}(\tau-r)^l f(r)dr$$

with f(r): Probability density function of the return variable R
    τ: Target-return
    l: order of the LPM

The LPM^l of order $l \leq 3$ was only discussed from a theoretical point of view (Cf. e.g. Schubert, L., (1996), while the order l = 0, 1, 2 was tested in practice. The Target-Shortfall-Probability (TSP) or LPM^0, is a measure of risk, which is controlled and used as a descriptive feature in the asset-liability management today.

In the case of normal distributed returns, i.e. $R \sim N(\mu, \sigma)$, a TSP restriction can be represented as a line in the μ-σ-space of the traditional portfolio chart. Figure 1 shows the TSP restriction. Every portfolio on that TSP-line has the probability α to achieve a return smaller than the target: $P(R < \tau) \leq \alpha$. The factor z_α in the linear inequality $(\mu - \tau - z_\alpha\sigma)$ is the abscissa value of the N(0, 1) probability distribution corresponding with the probability α.
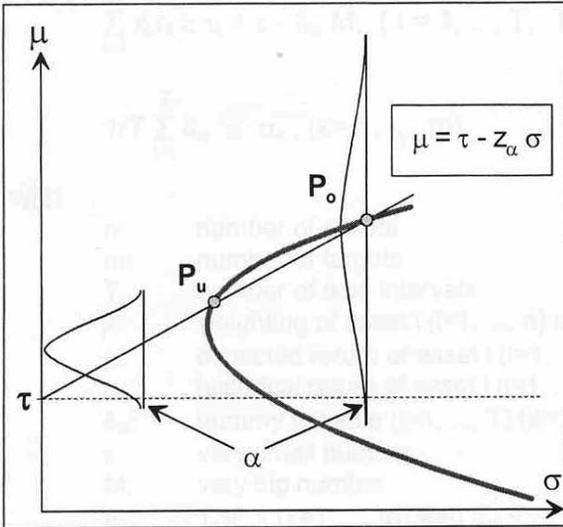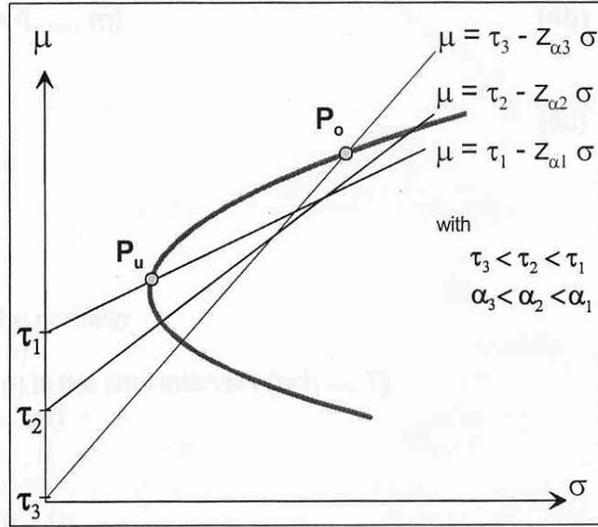

Figure 1. TSP.   Figure 2. TSP-vector.

One TSP is not sufficient for the description of risk. Therefore the TSP is criticized (Cf. e.g. Harlow, W. V., (1991). The use of a vector with m TSPs

TSP-vector $[\tau, \alpha]$ with $P(R < (\tau_k) \leq \alpha_k$ (1)

        and $\tau_k < \tau_{k'} \Leftrightarrow \alpha_k < \alpha_{k'}, k, k' = 1,...,m, k \neq k'$

can reduce this disadvantage (To avoid an insufficient description of risk, it would be consistent, if the mallest target τ_m would have the probability $\alpha_m = 0$. In Figure 2 the linear restrictions of a TSP-vector with m = 3 elements is displayed. Under the assumption of normal distribution, only the portfolios in the area between the Mean – Variance-efficient frontier and the three lines are feasible. Generally a portfolios can be called **TSP-vector feasible**, if it holds the probability conditions in (1).

3. MEAN – TSP-vector PORTFOLIO

The Mean – TSP-vector Model is geared to a portfolio manager, who maximizes the expected return under the restriction, that the portfolios should be TSP-vector feasible. The computation of the optima is not based on the parameter of the return distribution like it is in traditional approaches. Instead of parameters the model uses the historical returns directly (cf. stochastic programming). This way of portfolio selection is implemented in the following linear mixed integer program (Cf. Engesser, K., Schubert L., (1997):

Maximize

$$\sum_{i=1}^{n} x_i \mu_i \tag{2}$$

under the restrictions

$$\sum_{i=1}^{n} x_i = 1 \text{ with } x_i \geq 0, \ (i = 1,...,n) \tag{3}$$

$$\sum_{i=1}^{n} x_i r_{it} \leq (1 - \delta_{tk}) M + \tau_k, \ (t = 1,...,T, \ k = 1,...,m) \tag{4a}$$

$$\sum_{i=1}^{n} x_i r_{it} \leq \tau_k + \varepsilon - \delta_{tk} M, \ (t = 1,...,T, \ k = 1,...,m) \tag{4b}$$

$$1/T \ (\sum_{t=1}^{T} \delta_{tk} \leq \alpha_k, \ (k = 1,...,m) \tag{4c}$$

with

        n: number of assets

        m: number of targets

        T: number of time intervals

        x_i: weighting of asset i (i = 1, ..., n) in the portfolio

        (μ_i: expected return of asset i (i = 1, ..., n)

        r_{it}: historical return of asset i (i =1, ..., n) in the time interval t (t = 1, ..., T)

        (δ_{tk}: dummy variable (t = 1, ..., T) (k = 1, ..., m)

        (ε: very small number

        M: very big number

        (α_k:TSP k (k = 1, ..., m) with $\alpha_m < ... < \alpha_2 < \alpha_1$

        (τ_k: target k (k = 1, ..., m) with $\tau_m < ... < \tau_2 < \tau_1$.

The inequalities (4a) and (4b) contain dummy variables. A dummy variable (δ_{tk} must be 1, if the kth TSP-restriction is not fulfilled in a time interval t. The kth inequality (4c) counts the cases where the kth TSP-restriction is not fulfilled. The ratio of these cases compared with all time intervals T may not be greater than the probability of α_k.

3.1 Utility Theory

For the return distribution R and two targets, the utility function u(r) of a Mean – TSP-vector oriented investor must be

$$u(r) = \begin{cases} r & \text{if} & \infty \geq r \geq \tau_1 \\ r - g_1 & \text{if} & \tau_1 > r \geq \tau_2 \\ r - g_1 - g_2 & \text{if} & \tau_1 > r \geq -\infty \end{cases}$$ (see Figure 3).

The factor g_k (k = 1, ..., m) indicates the specific loss of utility at target τ_k if the return is below this target. For risk averse investors, the algebraic sign of g_k must be positive. Negative sign would indicate, that the investor likes risks and aims for high returns. The utility function u(r) is not continuous which can be criticized from the utility theoretical point of view.

The expected value of the utility function is

$$E(u(R)) = \mu - \sum_{k=1}^{m} g_k \, \alpha_k$$ (see Appendix).

Traditional portfolio models suppose risk averse investors. The Mean –TSP-vector model is not restricted to a risk averse investor. The investor himself fixes by selecting targets and shortfall-probabilities his risk-return-relationship (see Figure 4).
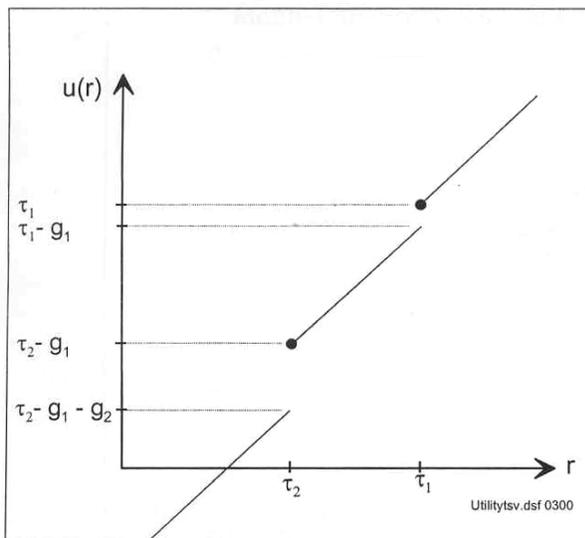

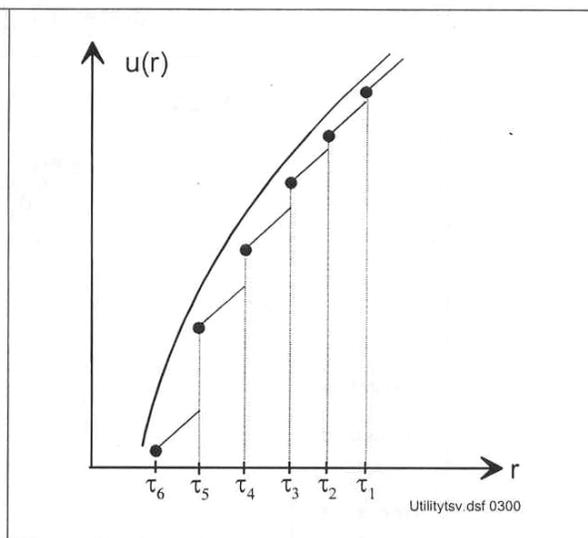
Figure 3. Utility function of a TSP-vector investor.   **Figure 4**. Utility function of a risk averse investor.

3.2 Mean-TSP-Portfolios in the μ–σ-space

For the empirical analysis of the Mean-TSP-model data from the Japanese capital market were used. The **data base** were the 86 biggest Japanese stocks which were listed in the stock exchange in Tokyo throughout the period from September 5th 1988 until November 1st 1999. For this period gliding i.e. moving annual rates of return for every month were calculated. The number T of annual rates of return which are available out of this database amounts to 123.

On this base Mean-TSP efficient Portfolios were determined. Chosen as the only target was the value –5 in Figure 5. To scan the efficient line in the μ-σ-space the TSP α was varied step-by-step by 0.01. The Minimal-Variance-Point (MVP) was sketched only for orientation in Figure 5. To the target of τ = - 5 could Portfolios be located from α = 0.11 to α = 0.23.

The average **increase of the standard deviation** for the Mean-TSP-Portfolios amounts to about 6 % (compare e.g. Figure 5). In Mean-Absolute-Deviation-Portfolios the increase of the standard deviation is compared to the efficient Mean-Variance-Portfolios estimated at over 10 % (See Konno, H., Shirakawa, H., Yamazaki, H., (1993), p. 211).
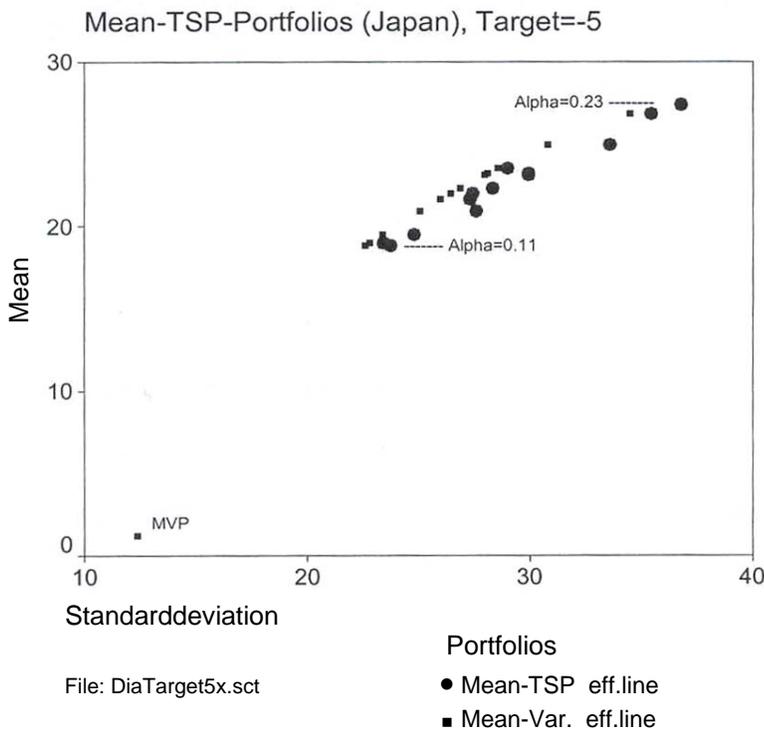
Figure 5. Mean-TSP-Portfolios

The grouping of m = 4 TSPs to one TSP-vector is illustrated in Figure 6. The efficient portfolios to the individual TSPs are signified by triangles, the **TSP-vector-portfolio** by a bold spot. The four elements of the vector $[\tau, \alpha]$ are: [0, 0.25], [-5, 0.20], [-10, 0.10] and [-20, 0.02]. The group of portfolios in the upper right corner were calculated without a restriction relating the fraction of the budget invested in a single stock, the ones in the lower left corner developed under the circumstance that the invested share in a single stock is limited to a maximum of 10 % of the budget.

In both cases the Mean-TSP-vector-portfolio is not identical with one of the portfolios respecting only one of the TSP (like it is under the supposition of continuous normal distributed returns (cf. Figure 6 and P_0 in Figure 2). Obviously, the usage of a TSP-vector reduces variance.
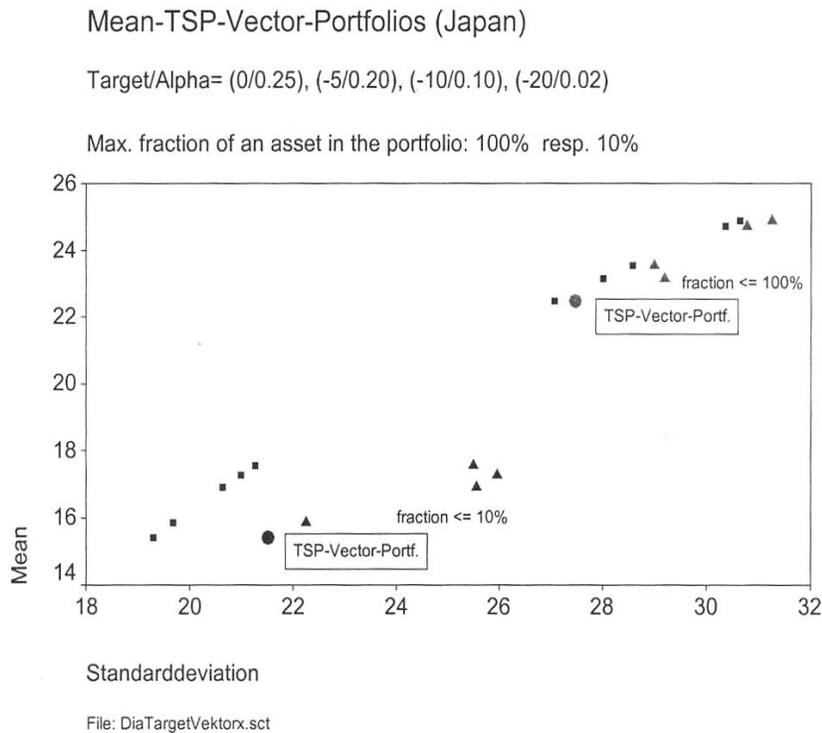

Figure 6. Mean-TSP-Vector-Portfolios

3.3 CPU-time

Due to the mixed integer variables in the program, it is difficult to calculate the CPU-time for finding an optimal solution. In the case of T = 123 and only one target (m = 1) a Mean – TSP-vector efficient portfolio can be determined within a minute. Using T = 266 time intervals for the same set of assets, the CPU-time will be extremely elevated (cf. Tab. 2). With the same time budget it is possible, to compute Mean – TSP-vector

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efficient portfolios with the parameters T = 123, m = 4 targets and n = 700 assets (Other models which respect skewness in the portfolio optimization are also strong time consuming. In Konno H., Suzuki, T., Kobayashi, D., (1998) an example with T = 24 return intervals and n = 100 assets. For finding the optimal solution 6455 sec. were needed).

Table 1. CPU time for T = 123 and T = 266 (in sec).

T = 123	Target	-5.00	-5.00	-5.00	-5.00	-5.00	-5.00	-5.00	-5.00	-5.00	-5.00	-5.00	-5.00	-5.00	-5.00
	Alphas	0.23	0.22	0.21	0.20	0.19	0.18	0.17	0.16	0.15	0.14	0.13	0.12	0.11	0.10 unfeasible
CPU-time	(CPLEX 7.0)	1	1	7	11	11	12	17	22	22	26	41	62	20	50

T = 266	Target	-5.00	-5.00	-5.00	-5.00	-5.00	-5.00	-5.00	-5.00	-5.00	-5.00	-5.00	-5.00	-5.00	-5.00
	Alphas	0.25	0.24	0.23	0.23	0.21	0.20	0.19	0.18	0.17	0.16	0.15	0.14	0.13	0.12 unfeasible
CPU-time	(CPLEX 7.0)	1	4	7	92	527	1121	1866	3040	1300	3877	1325	955	2517	18828

3.4 Number of stocks within the portfolio

The usage of the TSP as a risk criterion results in the number of stocks within a portfolio normally lying between 5-10. This feature was also observed, when positive skewness of a return distribution has to be diversified away (Cf. Simkowitz, M.A., Beedles, W. L. (1978), Duvall, R., Quinn, J. L., (1981), Kane, A., (1982).

This feature seems to be independent of the size of the number of stocks which are the base for selecting the portfolios.

Since often legal conditions limit the weights of the stock within a stock fund resp. portfolio in that case it is necessary to increase the number of stocks within the portfolio. The introduction of a limit for a single stock to a maximum share of the capital budget of q forces a portfolio of at least 1/q stocks. As experience shows this number 1/q is exceeded by 5-10 stocks.

An alternative usage of the TSP-restriction could be the integration in other linear portfolio approaches. (E.g. in the model of Konno, H., Yamazaki, H., (1991) resp. Feinstein, C. D., Thapa, M. N., (1993)). This way the number of stocks within the portfolio would increase.

3.5 Performance-Test

For a first test of the performance of the Mean – TSP-vector – portfolio data from the Japanese capital market were used. Out of the 681 biggest Japanese stocks were 50 stocks arbitrarily chosen. On this data-base, the Mean – TSP-vector efficient Portfolio was computed and also the Mean – Variance efficient Portfolio with the restriction to achieve at least the return of the Mean – TSP-vector efficient Portfolio. This procedure was repeated 54 times.

Contrary to the classical performance-tests the achieved returns of the two portfolio selection models were compared in a bear market and in a bull market.

Table 2 shows the observed "return-performance". The Mean - TSP-Portfolios seem not to possess an advantage in the bear market in comparison with the Mean-Variance-Portfolios. The difference between the obtained return of the 54 Mean - TSP-Portfolios and the Mean-Variance-Portfolios were 0,07%. Within the bull market the average difference was 2,05%. This indicates a return advantage of the Mean – TSP model. It must be pointed out, that the results do not have a remarkably significant level. Nevertheless, the results should be mentioned because of their plausibility.

It seems that the TSP-restrictions as well as the minimization of the variance make a limitation of risk possible (cf. bear market). The minimization of the variance however can turn out to be a small disadvantage. The reason could be that the yield is not exactly normal resp. symmetric distributed.

Table 2. Return-Performance.

	Mean–TSP–Portfolios	Mean–Variance-Portfolios
Average Return:		
Bear market:	- 10.85 %	- 10.78 %
Bull market:	+ 34.82 %	+ 32.77 %
total:	+ 9.63 %	+ 8.83 %

4. CONCLUSION

The TSP is a criterion which is controlled in fund management especially in the asset-liability management of retirement funds. Due to the acceptable calculation time it already now offers the possibility not only to control the TSP but to integrate it into the portfolio optimization in the form of a TSP-vector. The development of faster calculators and the improvement of the optimization software. The department for "mixed integer programming" at the University of Darmstadt (Germany) researches the structure of TSP restrictions to find a faster way to solve such optimization problems, will make it possible to optimize bigger sizes of problems in the near future. The flexible utility theoretical qualities of the TSP-vector, the possibility, to combine it with other linear models, the intuitive understanding of TSP by investors, the possibility to determine efficient portfolios independent of the distribution of the returns and maybe a favorable performance show that it is worth using a TSP-vector in portfolio optimization.

An important topic for further research is the influence of the skewness on the observed differences in the performance.

APPENDIX

For the construction of the $E(u(R))$ a discrete return distribution R represented by the probability p and the utility function of a Mean – TSP-vector investor (cf. Figure 3) is used:

$$E(u(R)) = \sum_{\tau_1 \le r_i} r_i p_i + \sum_{\tau_2 \le r_i < \tau_1} (r_i - g_1) p_i + \sum_{\tau_3 \le r_i < \tau_2} +(r_i - g_1 - g_2) p_i + \ldots + \sum_{r_i < \tau_m} \left(r_i - \sum_{1 \le k \le m} g_k \right) p_i$$

$$= \sum_{\tau_1 \le r_i} r_i p_i + \sum_{\tau_2 \le r_i < \tau_1} r_i p_i - \sum_{\tau_2 \le r_i < \tau_1} g_1 p_i + \sum_{\tau_3 \le r_i < \tau_2} r_1 p_i - \sum_{\tau_3 \le r_i < \tau_2} g_1 p_i - \sum_{\tau_3 \le r_i < \tau_2} g_2 p_i + \ldots +$$

$$+ \sum_{r_i \le \tau_m} r_i p_i - \sum_{r_i < \tau_m} g_1 p_i - \sum_{r_i < \tau_m} g_2 p_i - \ldots - \sum_{r_i < \tau_m} g_m p_i$$

$$= \sum_{-\infty \le r_i \le \infty} r_i p_i - \sum_{r_i < \tau_1} g_1 p_i - \sum_{r_i < \tau_2} g_2 p_i - \ldots - \sum_{r_i < \tau_m} g_m p_i$$

$$= \mu - g_1 \alpha_1 - g_2 \alpha_2 - \ldots - g_m \alpha_m$$

$$= \mu - \sum_{k=1}^{m} g_k \alpha_k$$

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