NUMERICAL APPROXIMATION OF OPTIMIZATION PROBLEMS WITH L^∞ FUNCTIONALS

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ABSTRACT
We consider a minimization problem which generalizes the problems of optimal lipschitz (extensión to the domain Ω of the functions tha verify the restrictions u = g on $\partial\Omega$). This work deals with the numerical approximations of the problem. Our work includes a numerical procedure for solving the discrete problem, the proof of convergence of the discrete solutions to the solution of the continuous problem and a numerical example that shows the efficiency of the procedure.

Key words: lipschitz extensions, optimization problems, minimax problems, numerical solution, convergence.

RESUMEN
Consideramos un problema de minimización que generaliza el problema de la extensión lipschitziana óptima (extensión a todo el dominio Ω de una función que verifica la restricción u = g en $\partial\Omega$). En este trabajo tratamos la aproximación numérica del problema. Desarrollamos un procedimiento de discretización y un algoritmo para resolver discreto. Probamos la convergencia de las soluciones discretas hacia la solución del problema continuo y presentamos un ejemplo numérico que muestra la eficiencia del procedimiento desarrollado.

MSC: 49K20.

1. INTRODUCTION: THE CONTINUOUS PROBLEMS

We consider the minimization of a functional of the form

$$J(u) = \operatorname*{ess\,sup}_{z\in\Omega} f(x, u, D(u)) \tag{1}$$

where $u \in W^{1,\infty}_g(\Omega) = \{u \in W^{1,\infty}(\Omega)\text{: } u = g \text{ on } \partial\Omega\}$. We assume that Ω is a bounded open domain in $\mathbb{R}^n$. This optimization genealizes the problem of finding an extension to the domain Ω of the functions that verify the restrictions u = g on $\partial\Omega$. In particular, this generalization comprises the problem of finding the lipschitzian extension with minimum lipschitz constant. This problem has been recently studied from different points of view in [Barron 2001, Crandall 1997, Crandall 2001 and Kirjner-Neto 1998].

This work deals with the numerical approximations of the problem. We analyze here the unidimensional case (n = 1). We present a discretization procedure which uses linear finite elements. Our work includes:

- the proof of the convergence of the discrete solutions to the solution of the continuous problem, obtaining different orders of convergence in terms of the regularity or convexity of the function f.
- a numerical procedure, based on techniques of penalization, for solving the discrete problem (which is essentially a minimax optimization problem in $\mathbb{R}^p$).
- a numerical example that the shows the efficiency of the procedure.

Let J be the functional (1), we define

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$$V = \inf_{u \in W_g^{1,\infty}(\Omega)} J(u) = \inf_{u \in W_g^{1,\infty}(\Omega)} \operatorname{ess\,sup}_{u \in \Omega} f(x, u, D(u)). \quad (2)$$

In [Barron 2001], Barron has proved the following theorems concerning issues of existence and necessary conditions of optimally:

Theorem 1.

Lef f: $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy:

(i) For any (x, s) $\in \mathbb{R} \times \mathbb{R}$, f(x, s, $\cdot$): $\mathbb{R} \to \mathbb{R}$ is Morrey quasiconvex, i.e. for any $p \neq q \in \mathbb{R}$ and $t \in (0,1)$

$$f(x, s, tp + (1 - t)q) \leq \max\{f(x, s, p), f(x, s, q)\},$$

(ii) there exists a function ω: $\mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, which continuous in its first variable and non-decreasing in its second variable, such that

$$|f(x_1, s_1, A) - f(x_2, s_2, A)| \leq \omega(|x_1 - x_2)| + s_1 - s_2|, |A|)$$

for any $(x_1, s_1), (x_2, s_2) \in \mathbb{R} \times \mathbb{R}$ and $A \in \mathbb{R}$.

Then for any bounded domain $\Omega \subset \mathbb{R}$ the functional

$$J(u) = \operatorname{ess\,sup}_{z \in \Omega} f(x, u(x), Du(x))$$

is sequentially weak* lower semicontinuous on $W^{1,\infty}(\Omega, \mathbb{R})$.

Theorem 2.

Let $\Omega \subset \mathbb{R}$ be a bounded domain and leg $g \in C^2(\Omega, \mathbb{R})$. Let f: $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be nonnegative and satisfy:

(f1) $f \in C^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$

(f2) For each (x, s) $\in \mathbb{R} \times \mathbb{R}$, f(x,s,$\cdot$): $\mathbb{R} \to \mathbb{R}$ is strictly quasi-convex, i.e.

$$f(x,s,tp + (1 - t)q) < \max\{f(x,s,p), f(x,s,q)\}$$

for any $p \neq q \in \mathbb{R}$ and $t \in (0,1)$.

(f3) For each (x,s) $\in \mathbb{R} \times \mathbb{R}$, f(x,s,0) = 0, f_p(x, s, 0) = 0 and f_p(x,s,q) $\neq 0$ some $0 \neq q \in \mathbb{R}$.

(f4) Coercivity condition:

$$f(x,s,A) \geq C_1 |A|^p - C_2, \forall (x,s,A) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \quad (3)$$

for some $C_1 > 0$, $C_2 > 0$, and $p > 0$.

Then, there exists at least one function u* an absolute minimizer for the functional J over $W_g^{1,\infty}(\Omega)$, moreover u* is a viscosity solution of the Aronson-Euler equation (see [Barron 2001]).

In the following sections we present different issues concerning the direct numerical solution of problem 2.

2. DISCRETIZATION PROCEDURE

We present here a discretization procedure based in the finite difference. We deal only with the one dimensional case, so, without loss of generality we suppose that

$$\Omega = (0,1) \text{ and } g(0) = 0, g(1) = 1.$$

Let $p \in N$, we define $k = 1(p + 1)$ and we consider the discretization Ω_k of Ω, where $\Omega_k = \{x_l\text{: } l = 0,\dots,p+1\}$ and $x_l = lk$. We also define a discretization W_k of $W = W_g^{1\infty}(\Omega)$, i.e.

$$W_k = \{v_k\text{: } \overline{\Omega} \to R,\ \text{continuous and piecewise linear in any } (x_l, x_{l+1}),\ l = 0,1,\dots,p;\ l = v_k = g \text{ in } \partial\Omega\}. \quad (4)$$

Function v_k will be considered as a vector with p + 2 components, i.e. the components $\{v_i\}_1^p, v_i \in \mathbb{R}$, are free while always $v_0 = 0$ and $v_{p+1} = 1$. As $W_k \subset W$, we have

$$V = \inf_{u \in W} J(u) \leq \inf_{u_k \in W_k} J(u_k) = V_k. \quad (5)$$

We will prove below that $V_k \to V$ when $k \to 0$.

We define

$$(D^+v)_l = \frac{v_{l+1} - v_l}{k}, \qquad l = 0,1,\dots,p,$$

$$(D^-v)_l = \frac{v_l - v_{l-1}}{k}, \qquad l = 1,\dots,p+1.$$

Once the variables are discretized, we define F^l, the discretization of function f(x,v,Dv) at the nodes x_l in the following form:

$$F^l(v) = f(x_1, v_l (D^+v)_l) \qquad l = 0,1,\dots,p. \quad (6)$$

Also, we define the discrete functional in W_k:

$$J_k(u_k) = \max_{l=0,1,\dots,p} F^l(u_k),$$

and we denote

$$\hat{V}_k = \inf_{u_k \in W_k} J_k(u_k).$$

Remark 1. In a similar way, we can define

$$\hat{F}^l(v) = f(x_l, v_l, (D^- v)_l) \qquad l = 1,\dots,p+1 \quad (7)$$

and

$$J_k(u_k) = \max_{l=1,\dots,p+1} \hat{F}^l u_k).$$

All the convergence results presented below are also valid for this discretization.

2.1. The discrete problem

The problem to be solved numerically, i.e.; compute $\hat{V}_k = \inf_{u_k \in W_k} J_k(u_k)$, becomes:

$$P_p\text{: compute } \min_{u \in R^p} \max_{l=0,1,\dots p} F^l(u). \quad (8)$$

This problem is equivalent to solve the following linear optimization problem with non linear constraints:

$$P_p : \begin{cases} \text{minimize } C \\ \text{subject to } C \geq F^l(u), \forall l = 0,1,\ldots,p. \end{cases} \tag{9}$$

Then the problem can be stated:

$$P_p : \text{Find}(\overline{C}, \overline{u}) \in \mathbb{R}^{p+1} \text{ such that}$$

$$\overline{C} = \max_{l=0,\ldots,p} F^l(\overline{u}) = \min_{u \in R^p} \max_{l=0,\ldots,p} F^l(\overline{u}) = \min_{u \in \mathbb{R}^p} \{C : C \geq F^l(u),\ \forall l = 0,1,\ldots,p\}. \tag{10}$$

Remark 2. *Condition (f1) and (f4) imply that P_p has at least a solution.*

Using simple tools from comvex analysis, it is easy to get the necessary conditions of optimality for the previous problem:

Theorem 3

If $(\overline{C} : \overline{u})$ is an optimal element then there exists $\lambda \in \mathbb{R}^{p+1}$, $\lambda_l \geq 0$, $\sum_{l=0}^{p} \lambda_l = 1$ such that

$$\sum_{l=0}^{p} \lambda_l \nabla F^l(\overline{u}) = 0 \tag{11}$$

and

$$\lambda_l(\overline{C} - F^l(u)) = 0, \qquad \forall l = 0,1,\ldots,p. \tag{12}$$

2.2. Convergence of the discrete problem

We consider the discretizations Ω_k and W_k defined above. We have

$$V = \inf_{u \in W} J(u) \leq \inf_{v_k \in W_k} J(v_k) = V_k. \tag{13}$$

As $J_k(v_k) = J(v_k)$, then

$$\hat{V}_k = V_k. \tag{14}$$

Condition (f_1) and (f_4) imply that f is a lipschitz function on the first two variables, i.e.

$$|f(x_1,s_1,A) - f(x_2,s_2,A)| \leq L_f(|x_1 - x_2| + |s_1 - s_2|),$$

then for $v_k \in W_k$ we have

$$|J_k(v_k) - J(v_k)| \leq L_f|k + Lv_k k). \tag{15}$$

Finally, from (13), (14) and (15) we obtain

$$V_k - L_f(k + \hat{L}k) \leq \hat{V}_k \leq V_k. \tag{16}$$

Let u be a solution of (2). To prove that $Vk \to V$ when $k \to 0$, we have to show that the minimizer of the original problem can be approximated by elements of W_k which verifies $J_k(u_k) \cong J(u)$. This approximation is more or less tighter depending on the structure of f.

2.3. Rates of convergence

Approximation of $u \in W_g^{1,\infty}(\Omega)$: **Case**: f = f(Du)

We can suppose w.l.g. that

$$f(Du) \le 1 = \operatorname*{ess\,sup}_{x\in\Omega} f(D(u)) = J(u). \tag{17}$$

We define the following set

$$\mathbf{D} = \{\xi \in [0,M]: f(\xi) \le 1\}.$$

By the continuity of f, **D** is a closed set. Moreover Du(s) $\in$ **D** a.e. s $\in$ [0,1]. As

$$\int_0^1 Du(s)ds \in \overline{C_0\mathbf{D}} = C_0\mathbf{D}$$

and it verifies

$$\int_0^1 Du(s)ds = 1,$$

then there exist $\lambda \in [0,1]$, and $\xi_1, \xi_2 \in$ **D** such that

$$1 = \lambda\xi_1 + (1 - \lambda)\xi_2.$$

Approximation using uniformly spaced sub-intervals

Let $\bar{\lambda} = [\lambda(p + 1)] / (p + 1)$. We define w_k by the following form

$$\begin{cases} Dw_k = \xi_1 & \left[0, \bar{\lambda}\right] \\ Dw_k = \xi_2 & \left[\bar{\lambda}, \bar{\lambda} + \frac{1}{(p+1)}\right] \\ Dw_k = (1-\rho)\xi_2 & \left[\bar{\lambda} + \frac{1}{(p+1)}, 1\right], \end{cases}$$

where ρ is such that $w_k(1) = 1$, hence

$$\rho = \frac{(\xi_2 - \xi_1)(\lambda - \bar{\lambda})}{\xi_2\left(1 - \bar{\lambda} - \frac{1}{(p+1)}\right)}.$$

In consequence we have:

$$\sup_{x\in\Omega} f(Dw_k) \le \sup_{x\in\Omega} f(Du) + L_f \rho M \le 1 + L_f M \frac{1}{(p+1)}.$$

From (13) we obtain

$$V_k \le V + L_f M \frac{1}{(p+1)},$$

then, we have

$$0 \le V_k - V \le \frac{L_f M}{(p+1)} \le L_f M k. \tag{18}$$

Approximation of $u \in W_g^{1\infty}(\Omega)$ **: Case**: f = f(x,u,Du). We assume that p + 1 = q^2 and

$$\Omega^k = \{x_0\} \cup \{x_{iq+j}: i = 0,\ldots,q-1;\ j = 1,\ldots,q\}.$$

Essentially, in each interval $[x_{iq}, x_{(i+1)q}]$ we use the results obtained in the case f = f(Du).

We define

$$\mathbf{D}_i = \left\{ \xi \in [0,M]: f(x_{iq}, u(x_{iq}), \xi) \le \frac{L_f(1+M)}{q} + \operatorname*{ess\,sup}_{x \in [x_{iq}, x_{(i+1)q}]} f(x,u,Du) \right\}$$

$\mathbf{D}_i$ is closed by the continuity of f. Moreover Du(s) $\in \mathbf{D}_i$ a.e. s $\in [x_{iq}, x_{(i+1)q}]$.

Let $d_i = u(x_{(i+1)q}) - u(x_{iq})$. We have

$$q \int_{x_{iq}}^{x_{(i+1)q}} Du(s)ds = qd_i \in \overline{C_0 \mathbf{D}_i} = C_0 \overline{\mathbf{D}_i} = C_0 \mathbf{D}_i.$$

Then there exist $\lambda \in [0,1]$, and $\xi_1, \xi_2 \in \mathbf{D}_i$ such that

$$qd_i = \lambda\xi_1 + (1 - \lambda)\xi_2.$$

(without loss of generality we suppose $\xi_2 > \xi_1$ and $\lambda \le ½$).

Approximation using uniformly spaced sub-intervals

For the i[th] subinterval, we define: $\bar{\lambda} = [\lambda q] / q$ and Dw_k in $(x_{iq}, x_{(i+1)q})$ by the following form

$$\begin{cases} Dw_k = \xi_1 - \rho_i & \left(x_{iq}, x_{(i+1)q} + \bar{\lambda}\frac{1}{q}\right), \\ Dw_k = \xi_2 - \rho_i & \left(x_{iq} + \bar{\lambda}\frac{1}{q}, x_{iq} + \bar{\lambda}\frac{1}{q} + \frac{1}{q^2}\right), \\ Dw_k = \xi_2 - \rho_i & \left(x_{iq} + \bar{\lambda}\frac{1}{q} + \frac{1}{q^2}, x_{(i+1)q}\right), \end{cases}$$

When $\rho_i = \frac{(\xi_2 - \xi_1)(\lambda - \bar{\lambda})\frac{1}{q}}{(\frac{1}{q})} \le \frac{M}{q}$, we have $w_k(1) = 1$ and $d(Dw_k, \mathbf{D}_i) \le \frac{M}{q}$. Moreover

$$\operatorname*{ess\,sup}_{x \in [x_{iq}, x_{(i+1)q}]} \{f(x, u_k(x), Du_k(x))\} \le \frac{L_f(2M+1)}{q} + \operatorname*{ess\,sup}_{x \in [x_{iq}, x_{(i+1)q}]} \{f(x,u,Du)\}. \tag{19}$$

Then,

$$J(u_k) \le J(u) + \frac{L_f(2M+1)}{q},$$

which implies that

$$0 \le V_k - V \le \frac{L_f(2M+1)}{q} \le L_f(2M+1)\sqrt{k}. \qquad (20)$$

3. SOLUTION BY PENALIZATION

Instead of solving directly Problem Pp, we solve a penalized problem that approximates Pp. Some related approaches can be seen in [Kirjner-Neto 1998, Polak 1996 and Polak 2001]. This choice enables us to apply methods of Newton type to solve the penalized problem.

Given $q > 0$, we define

$$G_q(C,u) = C + \sum_{l=0}^{p} \exp(-q(C - F^l(u))) \qquad (21)$$

and we try to find (C_q, u_q) in R^{p+1} such that

$$G_q(C_q,u_q) = \min_{(C,u)\in R^{p+1}} G_q(C,u) \qquad (22)$$

The optimality conditions for this problem are

$$\frac{\partial G_q}{\partial C} = 1 - q\sum_{l=0}^{p} \exp(-q(C_q - F^l(u_q))) = 0, \qquad (23)$$

$$\frac{\partial G_q}{\partial u} = q\sum_{l=0}^{p} \nabla F^l(u_q)\exp(-q(C_q - F^l(u_q)) = 0. \qquad (24)$$

In order to find a solution of (22) we find a solution of the non-linear equation system (23-24), using fast algorithms of Newton type. From condition (f1) $G \in C^2(R^{p+1})$ and so it is possible to apply Newton's method.

3.1. Convergence of the penalized problem

In this section we are interested in finding the asymptotic properties of C_q when $q \to \infty$ and the quasi-optimality of the cluster points of the sequence $\{u_q\}$.

Theorem 4

If $(\overline{C},\overline{u})$ is an optimal element then

$$\lim_{q\to\infty} C_q = \overline{C}.$$

Proof: From definition (22) we know that $\forall(C,u) \in R \times R^p$,

$$G_q(C_q, u_q) \le G_q(C, u).$$

Hence, by (23) and (24) we have

$$C_q + \frac{1}{q} = G_q(C_q,u_q) \le G_q(\overline{C}+\delta,\overline{u}) \quad \forall\delta,$$

and then

$$C_q + \frac{1}{q} \le \min_{\delta\in R} G_q(\overline{C}+\delta,\overline{u}). \qquad (25)$$

Since

$$\min_{\delta\in R} G_q(\overline{C}+\delta,\overline{u}) = \min_{\delta\in R}\left(\overline{C}+\delta+\exp(-q\delta)\sum_{l=0}^{p}\exp(-q(\overline{C}-F^l(\overline{u})))\right).$$

we obtain , for any optimal $\overline{\delta}$, the condition

$$\frac{1}{q}\exp(-q\overline{\delta})\sum_{l=0}^{p}\exp(-q(\overline{C}-F^l(\overline{u}))). \tag{26}$$

Hence, as $\overline{C} = \max_{l=0,\ldots,p} F^l(\overline{u})$ we have $\sum_{l=0}^{p}\exp(-q(\overline{C}-F^l(u)))\le p$ and by (26)

$$\exp(q\overline{\delta}) = q\left(\sum_{l=0}^{p}\exp(-q(\overline{C}-F^l(\overline{u})))\right)\le q\,p;$$

therefore

$$\overline{\delta}\le\frac{\log(q\,p)}{q}. \tag{27}$$

Finally, from (25), (26) and (27) we obtain

$$Cq+\frac{1}{q}\le G_q(\overline{C}+\overline{\delta},\overline{u})$$

$$=\overline{C}+\overline{\delta}+\exp(-q\overline{\delta})\sum_{l=0}^{p}\exp(-q(\overline{C}-F^l(\overline{u})))\le\overline{C}+\frac{\log(q\,p)}{q}+\frac{1}{q}$$

and then

$$C_q\le\overline{C}+\frac{\log(q\,p)}{q}. \tag{28}$$

From (23), we have $\exp(-q(C_q-F^l(u_q)))\le\frac{1}{q},\ \forall l=0,\ldots,p;$ then

$$C_q\ge\frac{\log q}{q}+F^l(u_q)\quad\forall l=0,\ldots,p \tag{29}$$

and this implies $C_q\ge\frac{\log q}{q}+\max_{l=0,\ldots,p}F^l(u_q)$. As $\overline{C}\le\max_{l=0,\ldots,p}F^l(u)\ \forall u\in R^p$, it results

$$\overline{C}\le\max_{l=0,\ldots,p}F^l(u)\le C_q-\frac{\log q}{q}. \tag{30}$$

From (28) and (30), we have

$$\overline{C}+\frac{\log q}{q}\le C_q\le\overline{C}+\frac{\log(q\,p)}{q}, \tag{31}$$

which implies

$$\lim_{q\to\infty} C_q = \overline{C}.$$

Theorem 5.

If w is a cluster point of the sequence $\{u_q\}$ then w verifies the necessary conditions of optimality (11 –12).

Proof: Let w be an cluster point of the sequence $\{u_q\}$. From (30) and (31)

$$\max_{l=0,\ldots,p} F^l(w) = \lim_{q\to\infty} \max_{l=0,\ldots,p} F^l(u_q) = \overline{C}.$$

Let us define

$$\lambda_l = \lim_{q\to\infty} q\exp(-q(C_q - F^l(u_q))), \tag{32}$$

obviously, $\lambda_l \geq 0$. From (23) it follows that $\sum_{l=0}^{p} \lambda_l = 1$.

Taking limit in (24) we obtain

$$\sum_{l=0}^{p} \lambda_l \nabla F^l(u_q) = 0\ .$$

We will finally prove the orthogonality condition (12). Naturally, we have to prove this only in the case $F^l(w) < \max_{l=0,\ldots,p} F^l(w)$.

In consequence, there are exist $\delta > 0$, $q_\delta > 0$ such that $F^l(u_q) \leq C_q$ - δ $\forall q > q_\delta$.

Then $\lambda_l = \lim_{q\to\infty} q\exp(-q(C_q - F^l)u_q))) \leq \lim_{q\to\infty} q\exp(-q\delta) = 0$ and therefore

$$\lambda_l(\overline{C} - F^l(w)) = 0.$$

3.2. Practical implementation

In practice, it is not convenient to solve directly the penalized problem for large values of q, since that choice leads to the treatment of ill conditioned optimization problems. The implementation that we have used comprises a sequence of approximated optimizations with increasing values q.

Algorithm

Let $\varepsilon_\rho \to 0$, $q_p \to \to \infty$

Step 0. ρ = 0, choose $\hat{u}_{q_\rho}$.

Step 1: Solve in approximated form (with an iterative algorithm from the $\hat{u}_{q_\rho}$) the penalized problem until it verifies the condition

$$\left\| \frac{\partial Gq}{\partial u} \right\| \leq \varepsilon_\rho$$

obtaining the new $\hat{u}_{q_{\rho+1}}$.

Step 2: $\rho = \rho + 1$, and go to step 1

Remark 3. *The algorithm above defined generates a sequence of points* $(\hat{C}_q, u_{q_\rho})$ *that verify*:

$$\lim_{q\to\infty} \hat{C}_q = \overline{C}.$$

If w is a cluster point of the sequence $\{\hat{u}_{q_\rho}\}$ *then w verifies the necessary conditions of optimality* (11 - 12).

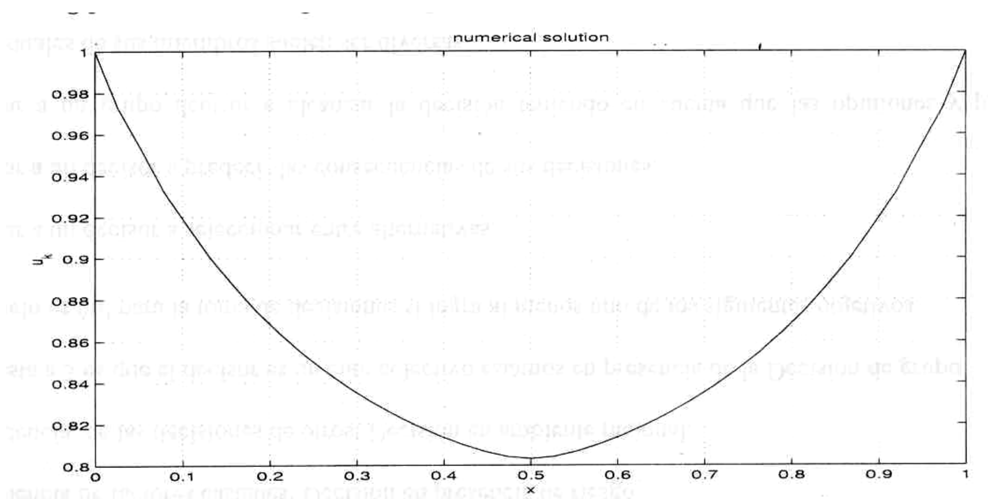
4. EXAMPLE

We present here the numerical results obtained for the following data:

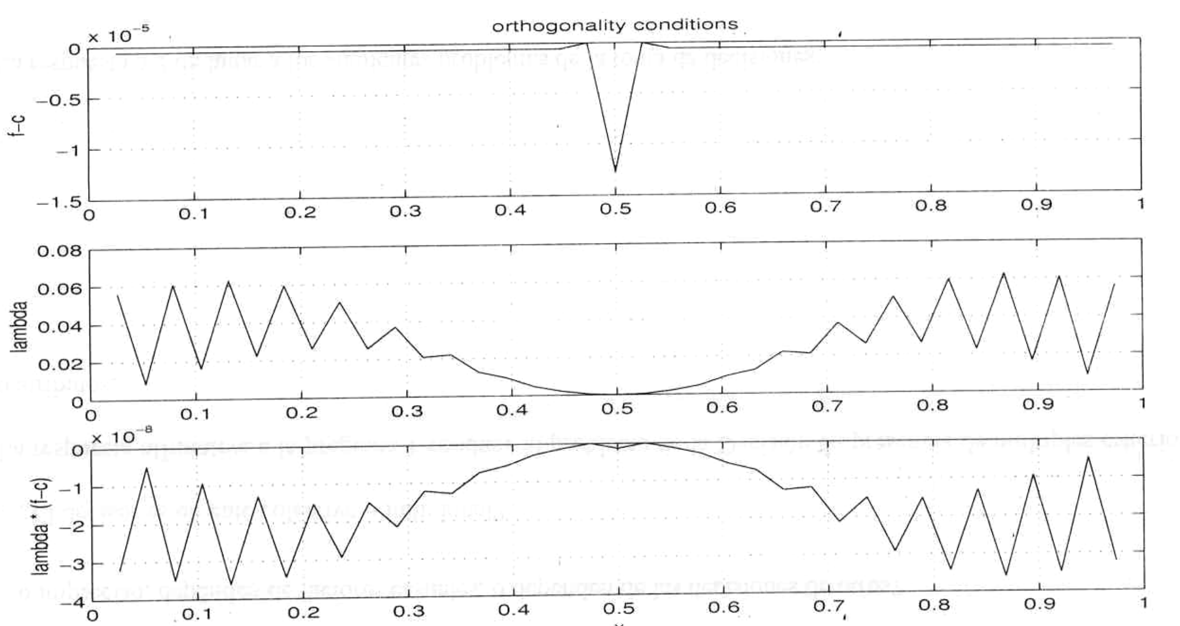
$$f(x,u,Du) = 2\|Du\|^2 + 80x(1 - x)(u - 0.5)^2 u(0) = u(1) = 1.$$

Number of discretization points: $p = 38$.

The following picture shows the general shape of the solution



The following picture shows how the optimality conditions are verifies (up to errors of order 10^{-8}) by the numerical solution obtained.



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