LAGRANGE MULTIPLIERS IN MULTIOBJECTIVE OPTIMIZATION UNDER MIXED ASSUMPTIONS OF FRECHET AND DIRECTIONAL DIFFERENTIABILITY

Vicente Novo[1] and Bienvenido Jiménez[2], Departamento de Matemática Aplicada,
Universidad Nacional de Educación a Distancia, Madrid, Spain

ABSTRACT
We study a multiobjective optimization problem in finite-dimensional spaces with a feasible set defined by directionally differentiable (or quasiconvex) inequality constraints and Fréchet differentiable equality constraints. Under a suitable constraint qualification (of the Mangasarian-Fromovitz type) an expression for the contingent cone to the feasible set is obtained. As application, necessary conditions of Pareto optimality both Fritz John type and Kuhn-Tucker type are obtained by means of Lagrange multipliers rules.

Key words: Lipschitz continuity, nonlinear optimization, optimality conditions.

MSC: 90C29

RESUMEN
Estudiamos un problema multiobjetivo de optimización en espacios de dimensión finita con un conjunto de factibilidad definido por restricciones en desigualdades que son direccionalmente diferenciables (o cuasiconvexas) y restricciones de igualdad que son Fréchet diferenciables. Bajo una adecuada calificación (del tipo Mangasarian-Fromovitz) una expresión para el cono contingente de las restricciones es obtenida. Como una aplicación, las condiciones necesarias para la Pareto optimalidad del tipo de Fritz John y del tipo Kuhn-Tucker son obtenidas por medio de las reglas de los multiplicadores de Lagrange.

1. INTRODUCTION

Over the last three decades, the classical multiplier rule was extended in the direction of eliminating the smoothness assumptions. These extensions were given under assumptions of directional differentiability and Lipschitz continuity. In this work we are going on along this way using mixed assumptions of Fréchet and directional differentiability in multiobjective optimization problems with inequality and equality constraints.

Di (1996) obtains some first and second order multiplier rules for nonlinear optimization problems with equality, inequality and abstract constraints and considers all functions Fréchet differentiable. Ye (2001) gives multipliers rules for a scalar nonlinear optimization problem under assumptions of Fréchet differentiability and Lipschitz continuity.

On the other hand, constraint qualifications have a significant role in optimization problems, since they allow us to guarantee the effective intervention of the objective functions in the Fritz John type necessary conditions for a point to be an optimum. Since the decade of the 50's, the study of these qualifications has been the aim of several researchers with different approaches, proposing various regularity conditions. In this work we use a constraint qualification of Mangasarian-Fromovitz type.

This paper is structured as follows: Section 2 is devoted to the definitions, notations and preliminaries, in Section 3 we study explicit expressions for the tangent cone to the feasible set when it is defined by Fréchet differentiable equality constraints and or well Hadamard differentiable inequality or well mixed inequalities functions.

To be more precise, if we consider a feasible set S defined by equality and inequality constraints, then if the equality constraints are continuous on a neighborhood of the point and Fréchet differentiable at this point with

E-mail: [1]vnovo@ind.uned.es
[2]bjimen1@encina.pntic.mec.es

Jacobian of maximal rank, and if the inequality constraints are Dini differentiable with convex derivative, it can be shown that, under a suitable Mangasarian-Fromovitz constraint qualification, the tangent cone is equal to the linearized closed cone in the following cases:

1. When the inequality constraints are Hadamard differentiable (Theorem 3.6).

2. When each inequality constraint is either continuous on a neighborhood of the point and quasiconvex on a neighborhood of the point or Fréchet differentiable at the point (Theorem 3.7).

If there is no equality constraints, it is enough that the inequality constraints are either quasiconvex at the point or Hadamard differentiable (Theorem 3.5).

As an application, necessary optimality conditions of Fritz John type and Kuhn-Tucker type for Pareto optimum are obtained (Theorem 3.10).

2. NOTATIONS AND PRELIMINARIES

Let $S \subset \mathbb{R}^n$ a nonempty set. As usual, cl S, int S, ri S, co S, cone S and Lin S will denote closure, interior, relative interior, convex hull, generated cone and generated subspace by S, respectively.

Give a function f: $\mathbb{R}^n \to \mathbb{R}^p$, the following multiobjective optimization problem is considered

$$\text{(MP) Min}\{f(x) : x \in S\}.$$

A point $x_0 \in S$ is said to be a local weak Pareto minimum, if there exists a neighborhood $B(x_0, \delta)$ of x_0 such that

$$S_f \cap S \cap B(x_0, \delta) = \emptyset \qquad (1)$$

where $S_f = \{x \in \mathbb{R}^n : f(x) < f(x_0)\}$.

Because of the difficulties in verifying condition (1), local approximations of the sets S and S_f through different tangent cones are used.

Definition 2.1. Let S be a subset of $\mathbb{R}^n$ and $x_0 \in \text{cl } S$.

(a) The tangent (contingent) cone to S at x_0 is

$$T(S, x_0) = \{v \in \mathbb{R}^n : \exists t_k \to 0^+, \exists x_k \in S \text{ such that } (x_k - x_0)/t_k \to v\}.$$

(b) The cone of attainable directions is

$$A(S,x_0) = \{v \in \mathbb{R}^n : \exists \delta > 0, \exists \gamma : [0,\delta] \to \mathbb{R}^n / \gamma(0) = x_0, \gamma(t) \in S\ \forall t \in (0, \delta], \gamma'(0) = v\}.$$

We have the following inclusion

$$A(S, x_0) \subset T(S, x_0) \qquad (2)$$

A complete and rigorous analysis of different tangent cones can be found in Bazaraa and Shetty (1976) or in Aubin and Frankowska (1990).

Let $D \subset \mathbb{R}^n$, the polar cone to D is $D^* = \{v \in \mathbb{R}^n : \langle v,d \rangle \le 0\ \forall d \in D\}$, which is a closed and convex set. The normal cone to S at x_0 is defined by $N(S, x_0) = T(S, x_0)^*$.

Let us remark that the normal cone is often used in order to discuss properties of the constraint set. Moreover, if the sets are defined by functions constraints, their approximation is realized through the cones defined by different directional derivatives.

Definition 2.2. Let f: $\mathbb{R}^n \to \mathbb{R}^p$, x_0, $v \in \mathbb{R}^n$.

(a) The Dini derivative (or directional derivative) of f at x_0 in the direction v is

$$Df(x_0, v) = \lim_{t\to 0^+} \frac{f(x_0 + tv) - f(x_0)}{t}.$$

(b) The Hadamard derivative of f at x_0 in the direction v is

$$df(x_0, v) = \lim_{(t,u)\to(0^+,v)} \frac{f(x_0 + tu) - f(x_0)}{t}.$$

(c) f is Dini differentiable (respectively Hadamard differentiable) at x_0 if its Dini derivative (resp. Hadamard derivative) exists in all directions.

The next definition of Dini subdifferential for a function f: $\mathbb{R}^n \to \mathbb{R}$ is well known (see, for example, Penot 1978).

Definition 2.3. The Dini subdifferential of f at x_0 is

$$\partial_D f(x_0) = \{\xi \in \mathbb{R}^n : \langle \xi, v\rangle \le Df(x_0, v)\ \forall v \in \mathbb{R}^n\}.$$

If $Df(x_0,\cdot)$ is a convex function, then there exists the subdifferential (in the sense of Convex Analysis), $\partial Df(x_0,\cdot)(0)$, is a nonempty, compact and convex set in $\mathbb{R}^n$ and we have that:

$$\partial_D f(x_0) = \partial Df(x_0,\cdot)(0) \text{ and } Df(x_0, v) = \mathrm{Max}\{\langle \xi, v\rangle : \xi \in \partial_D f(x_0)\}.$$

If $Df(x_0,\cdot)$ is nonconvex, then $\partial_D f(x_0)$ can be empty.

We also consider the following extensions of convexity.

Definition 2.4. Let $\Gamma \subset \mathbb{R}^n$ be a convex set, $f : \Gamma \to \mathbb{R}$ and $x_0 \in \Gamma$.

(a) f is quasiconvex at x_0 if

$$\forall x \in \Gamma, f(x) \le f(x_0) \Rightarrow f(\lambda x + (1-\lambda)x_0) \le f(x_0)\ \forall \lambda \in (0,1).$$

f is quasiconvex on Γ if f is quasiconvex at each point of Γ.

(b) f is Dini-quasiconvex at x_0 if $\forall x \in \Gamma, f(x) \le f(x_0) \Rightarrow Df(x_0, x - x_0) \le 0$.

The most valuable properties for our purposes related to these concepts are summarized in the next proposition.

Proposition 2.5.

(a) (Bazaraa, Shetty 1979). f is quasiconvex on Γ if and only if the level sets $\Gamma_\alpha = \{x \in \Gamma : f(x) \le \alpha\}$ are convex for all $\alpha \in \mathbb{R}$.

(b) Let f be Dini differentiable at x_0. If f is quasiconvex at x_0, then f is Dini-quasiconvex at x_0.

(c) (Giorgi, Komlósi 1992, Theorem 3.2). If f is continuous and Dini-quasiconvex on Γ, then f is quasiconvex on Γ.

The next proposition, that will be often used, is deduced from (Ref. 3, Theorem 3.9) with $Q = \mathbb{R}^n$.

Proposition 2.6. Let m, r be nonnegative integers with $m + r \geq 1$, $g_1, g_2, ..., g_m$ be sublinear functions (positively homogeneous and convex) from $\mathbb{R}^n$ to $\mathbb{R}$, and $h_1, h_2, ..., h_r$ be linear functions from $\mathbb{R}^n$ to $\mathbb{R}$ given by $h_k(v) = \langle c_k, v\rangle$, $k \in K = \{1,2,...,r\}$. We denote $J = \{1,2,...,m\}$.

Then the following statements are equivalent:

(a) For each set $\{\xi_j$: j = 1,2,…,m$\}$, with $\xi_j \in \partial g_j(0)$, we have that:

$$\sum_{j\in J}\lambda_j\xi_j + \sum_{k\in K}\mu_k c_k = 0,\ \lambda \geq 0, \mu \in \mathbb{R}^n \text{ implies } \lambda = 0, \mu = 0.$$

(b) For each $\lambda \geq 0$, $\mu \in \mathbb{R}^r$ $\sum_{j\in J}\lambda_j g_j(v) + \sum_{k\in K}\mu_k h_k(v) \geq 0, \forall v \in \mathbb{R}^r$, implies $\lambda = 0, \mu = 0$.

(c) $\{c_k$: $k \in K\}$ is linearly independent and there exists $v \in \mathbb{R}^n$ such that

$$g_j(v) < 0, j = 1,2,...,m, h_k(v) = 0, k = 1,2,....,r.$$

(d) $\{c_k$: $k \in K\}$ is linearly independent,

$$0 \notin \text{co}\left(\bigcup_{j\in J}\partial g_j(0)\right) \text{ and } \left[\text{cone co}\left(\bigcup_{j\in J}\partial g_j(0)\right)\right] \bigcap \text{Lin}\{c_k : k \in K\} = \{0\}.$$

3. SOME EXPRESSIONS FOR THE CONTINGENT CONE AND APPLICATIONS TO OPTIMALITY CONDITIONS

In this section we consider a set S defined by equality and inequality constraints, a point of S at which we will obtain the tangent cone and we suppose that h is Fréchet differentiable at x_0 and continuous on a neighborhood of x_0. Under different conditions, we provide an expression for the tangent cone (Theorems 3.5, 3.6 and 3.7).

When the feasible set of the problem (MP) is defined by:

$$S = \{x \in \mathbb{R}^n : g(x) \leq 0,\ h(x) = 0\} \tag{5}$$

with g: $\mathbb{R}^n \to \mathbb{R}^m$ and h: $\mathbb{R}^n \to \mathbb{R}^r$, whose component functions are, respectively, $g_j, j \in J = \{1,2,...,m\}$, $h_k, k \in K = \{1,2,...,r\}$, we shall adopt the following notation. Given $x_0 \in S$, the active index set at x_0 is $J_0 = \{j \in J: g_j(x_0) = 0\}$. The sets defined by the constraints g and h are denoted, respectively, by $G = \{x \in \mathbb{R}^n: g(x) \leq 0\}$, $H = \{x \in \mathbb{R}^n: h(x) = 0\}$, following that $S = G \cap H$.

Assuming that the constraint functions are Dini differentiable at x_0, the cones that we shall use in order to approximate S at x_0 are (linearized cones):

$$C_0(S) = \{v \in \mathbb{R}^n: Dg_j(x_0, v) < 0\ \forall j \in J_0, \nabla h(x_0)v = 0\},$$

$$C(S) = \{v \in \mathbb{R}^n: Dg_j(x_0, v) \leq 0\ \forall j \in J_0, \nabla h(x_0)v = 0\}.$$

$C_0(G)$ and $C(G)$ are defined analogously and we denote $K(H) = \text{Ker } \nabla h(x_0)$. Consequently, $C_0(S) = C_0(G) \cap K(H)$ and $C(S) = C(G) \cap K(H)$.

If g is differentiable at x_0, the relationship between these cones and the ones given in Definition 2.1 is (Bazaraa, Shetty 1976, Theorems 6.1.2 and 6.2.2):

(a) If there is no equality constraints $(S = G)$ then

$$C_0(G) \subset A(G,x_0) \subset T(G,x_0) \subset C(G).$$

(b) (1) $T(S,x_0) \subset C(S)$.

(2) If $\{\nabla h_k(x_0),\ k \in K\}$ is linearly independent, then $C_0(S) \subset A(S,x_0)$.

Let us begin by studying these inclusions for Dini or Hadamard differentiable functions. We will suppose that all the functions are continuous at x_0 and the active ones Dini differentiable at x_0.

Our aim is to obtain the inclusions

$$C_0(S) \subset T(S,x_0) \subset C(S), \tag{6}$$

such that if the first set is nonempty and its closure is C(S), we have that the two last sets are the same.

In the differentiable case it is enough that $C_0(S)$ is nonempty in order to have $\mathrm{cl}C_0(S) \subset C(S)$. But, in general, this result is not true. It is still possible to obtain this result if the inequality constraints have convex Dini derivative and the equality constraints have linear Dini derivative. The following lemma gives us the basic properties of the linearized cones and the aforementioned sufficient condition.

Lemma 3.1. Let $Dg_j(x_0,\cdot), j \in J_0$ be convex and $C_0(S) \neq \varnothing$. Then:

(i) $C_0(S)$ is a convex cone, relative open in K(H), and C(S) is a closed convex cone.

(ii) $\mathrm{cl}C_0(S) = C(S)$.

Proof.

Part (i) follows immediately from the convexity of $Dg_j(x_0,\cdot), j \in J_0$, the linearity of $\nabla h(x_0)$ and continuity of all of them.

(ii) The following is a classical reasoning. Let $u \in C_0(S)$, $v \in C(S)$, $u_\lambda = \lambda u + (1-\lambda)v$ with $\lambda \in (0,1)$. Then $\lim_{\lambda \to 0^+} u_\lambda = v$ and the proof will be completed if we show that $u_\lambda \in C_0(S)$. But this follows because of the convexity of $Dg_j(x_0,\cdot), j \in J_0$ and the linearity of $\nabla h(x_0)$.

In the following proposition sufficient conditions for the second inclusion in the sequence (6) are stated.

Proposition 3.2. Let suppose that for each $j \in J_0$, either g_j is Hadamard differentiable at x_0 or g_j is Dini-quasiconvex at x_0 whose derivative $Dg_j(x_0,\cdot)$ is continuous on $\mathbb{R}^n$. Then $T(S,x_0) \subset C(S)$.

Proof.

Let us consider $v \in T(S,x_0)$. Then, there exist $t_n \to 0^+, v_n \to v$ such that $x_n = x_0 + t_n v_n \in S$, $\forall n \in \mathbb{N}$.

(a) Let $j \in J_0$. 1) If g_j is Hadamard differentiable at x_0, then

$$dg_j(x_0,v) = \lim_{n\to\infty} \frac{g_j(x_n) - g_j(x_0)}{t_n}.$$

As $x_n \in S$ and $j \in J_0$ we obtain that $g_j(x_n) \le 0, g_j(x_0) = 0$ and, therefore, $g_j(x_n) - g_j(x_0) \le 0$. Thus $Dg_j(x_0, v) = dg_j(x_0, v) \le 0$.

2) Now let g_j be Dini-quasiconvex with continuous Dini derivative. Then, as $x_n \in S$, it follows that $g_j(x_n) \le 0 = g_j(x_0)$. Because of Dini-quasiconvexity we have that $Dg_j(x_0, x_n - x_0) \le 0$, i.e. $Dg_j(x_0, t_n v_n) \le 0$, and consequently, $Dg_j(x_0, v_n) \le 0$. By continuity we obtain $Dg_j(x_0, v) \le 0$.

(b) As $x_n \in S = G \cap H$, we have that $h(x_n) = h(x_0) = 0$ for all n. Hence,

$$\lim_{n\to\infty} \frac{h(x_n) - h(x_0)}{t_n} = \nabla h(x_0) v = 0 .$$

So, $\forall j \in J_0$, $Dg_j(x_0, v) \le 0$ and $\nabla h(x_0) v = 0$, which means that $v \in C(S)$.

If it is only required that $Dg_j(x_0, \cdot)$ is continuous on $\mathbb{R}^n$, it is not true in general that $T(G, x_0) \subset C(G)$, even if $C_0(G)$ is nonempty and $Dg_j(x_0, \cdot)$ is convex, as the next example shows.

Example 3.3. In $\mathbb{R}^2$, let D be the convex hull of the arc of the parabola $y^2 = -4(x+1)$ between the points (-2,2) and (-2,-2), that is, $D = \text{co}\{(-t^2 - 1, 2t) : -1 \le t \le 1\}$. Let σ be the support function of D, i.e., $\sigma(v) = \text{Max}\{\langle d, v\rangle : d \in D\}$. As D is a compact convex set, σ is positively homogeneous, convex and finite on $\mathbb{R}^2$. We have that

$$\sigma(x,y) = \begin{cases} (y^2 - x^2)/x & \text{if} \quad -x < y < x \\ -2x + 2y & \text{if} \quad y \ge x, y \ge 0 \\ -2x - 2y & \text{if} \quad y \le -x, y < 0 \end{cases}$$

Let φ: $\mathbb{R} \to \mathbb{R}$ given by $\varphi(\alpha) = \begin{cases} |\alpha - 2| & \text{if} \quad 1 \le \alpha \le 3 \\ 1 & \text{otherwise,} \end{cases}$

$W = \{(x,y) \in \mathbb{R}^2 : x^2 \le y \le 3x^2\} \setminus \{(0,0)\}$

and g: $\mathbb{R}^2 \to \mathbb{R}$ given by $g(x,y) = \begin{cases} \sigma(x,y) & \text{if} \quad (x,y) \notin W \\ \varphi(y/x^2)\sigma(x,y) & \text{if} \quad (x,y) \in W \end{cases}$

Clearly $Dg(x_0, v) = \sigma(v)$, for all $v \in \mathbb{R}^2$, being $x_0 = (0,0)$.

The feasible set is $G = \{(x, y): g(x,y) \le 0\} = A \cup \{(x, y): y = 2x^2, x < 0\}$, where $A = \{(x, y): -x \le y \le x\}$, $C_0(G) = \{v: Dg(x_0, v) < 0\}$ = int A and $C(G) = \{v: Dg(x_0, v) \le 0\}$ = A.

But $T(G, x_0) = A \cup \{(x, 0): x \le 0\}$ and, consequently, $T(G, x_0) \not\subset C(G)$.

Next we are going to analyze under what conditions it is verified that $\text{cl}\, C_0(S) = T(S, x_0) = C(S)$. Of course it depends on the kind of the involved functions. If there are some equality constraints, these functions should be Fréchet differentiable. Proof of the next previous result is immediate.

Proposition 3.4. If there is no equality constraints, S = G, and the functions $g_j, j \in J_0$, are Dini differentiable at x_0, then $C_0(G) \subset A(G, x_0) \subset T(G, x_0)$.

Theorem 3.5. Assume that each $g_j, j \in J_0$, is Dini-quasiconvex or Hadamard differentiable at x_0, in both cases with convex derivative at x_0. If $C_0(G) \neq \emptyset$ then

(i) $\mathrm{cl}\, C_0(G) = T(G, x_0) = C(G)$.

(ii) $N(G, x_0) = \mathrm{cone\ co}\left(\bigcup_{j \in J_0} \partial_D g_j(x_0) \right)$.

Proof.

(i) From Propositions 3.4 and 3.2, we have that $C_0(G) \subset T(G, x_0) \subset C(G)$ and, as the derivatives are convex, the conclusion follows from Lemma 3.1.

(ii) As

$$C(G) = \{v \in \mathbb{R}^n: Dg_j(x_0, v) \leq 0\ \forall j \in J_0\} =$$

$$= \{v \in \mathbb{R}^n: \langle \xi_j, v \rangle \leq 0,\ \forall \xi_j \in \partial_D g_j(x_0),\ \forall j \in J_0\}$$

using (Rockafellar 1970, page 122), the polar of C(G) is the closure of the convex cone generated by all the ξ_j, i.e.,

$$C(G)^* = \mathrm{cl\ cone\ co}\left(\bigcup_{j \in J_0} \partial_D g_j(x_0) \right).$$

By hypothesis, there exists $v \in \mathbb{R}^n$ such that $Dg_j(x_0, v) < 0, \forall j \in J_0$. By Proposition 2.6 (with r = 0), $0 \notin \mathrm{co}\left(\bigcup_{j \in J_0} \partial_D g_j(x_0) \right)$ and, as $\mathrm{co}\left(\bigcup_{j \in J_0} \partial_D g_j(x_0) \right)$ is compact (Hiriart-Urruty, Lemaréchal 1996, Theorem 1.4.3, Chap. 3), it follows that the set $\mathrm{cone\ co}\left(\bigcup_{j \in J_0} \partial_D g_j(x_0) \right)$ is closed, (Hiriart-Urruty, Lemaréchal 1996, Proposition 1.4.7, Chap. 3), consequently $C(G)^* = \mathrm{cone\ co}\left(\bigcup_{j \in J_0} \partial_D g_j(x_0) \right)$.

The result follows taking into account that $N(G, x_0) = T(G, x_0)^* = C(G)^*$.

This theorem generalizes some results of the Convex Analysis, for example, Corollary 23.7.1 of Rockafellar (1970) and Theorems 1.3.4 and 1.3.5 (Hiriat-Urruty, Lemaréchal 1996, Chap. 6).

Theorem 3.6. Suppose that the following conditions are true.

(a) For each $j \in J_0$, g_j is Hadamard differentiable at x_0 with convex derivative.

(b) $\nabla h(x_0)$ has maximal rank.

(c) $C_0(S) \neq \emptyset$.

Then:

(i) $\text{cl}\, C_0(S) = A(S,x_0) = T(S,x_0) = C(S)$.

(ii) $N(S,x_0) = N(G,x_0) + N(H,x_0) = \text{cone co}\left(\bigcup_{j\in J_0} \partial_D g_j(x_0)\right) + \text{Lin}\{\nabla h_k(x_0) : k \in K\}$.

Proof.

(i) If we prove that $C_0(S) \subset A(S,x_0) \subset T(S,x_0)$, by Proposition 3.2 and Lemma 3.1 we have the conclusion. The second inclusion is a well known result. Let us proof the first one.

By hypothesis, $\nabla h_k(x_0), k = 1,2,...,r$ are lin. indep. (we suppose $r < n$).

Given a vector v, by (Hestenes 1981, Theorem 5.3, Chap. 3), the system

$$h_k(x) = h_k(x_0) + t\nabla h_k(x_0)v,\ k = 1,2,...,r$$

has a solution $x = \gamma(t), -\delta \le t \le \delta$ such that $\gamma(0) = x_0$ and $\gamma'(0) = v$.

Now, let $v \in C_0(S)$. We have that $\nabla h_k(x_0)v = 0,\ k = 1,2,...,r$ and, as $x_0 \in S$, it follows that $h_k(\gamma(t)) = 0$, $\forall t \in [-\delta,\delta],\ k = 1,2,...,r$ and therefore, $\gamma(t) \in H\ \ \forall t \in [-\delta,\delta]$.

Let us see that $\gamma(t) \in G$ for $t > 0$ small enough.

Since γ is differentiable at t = 0, γ is Hadamard differentiable at t = 0, and we have that $d\gamma(0,1) = \nabla\gamma(0)1 = v$.

On the other hand, since the Hadamard derivative verifies the chain rule (Demyanov, Rubinov 1995, Theorem 3.3, Chap. 1), if we put $\psi = g_j \circ \gamma$, it follows that

$$d\psi(0,1) = dg_j(\gamma(0), d\gamma(0,1)) = dg_j(x_0, v).$$

Consequently, $d\psi(0,1) = D\psi(0,1) = \lim\limits_{t\to 0^+} \dfrac{\psi(0+t)-\psi(0)}{t} = \lim\limits_{t\to 0^+} \dfrac{g_j(\gamma(t))}{t} < 0.$

Therefore, for *t* small enough $g_j(\gamma(t)) < 0, \forall j \in J_0$. For $j \in J \setminus J_0$, by the continuity of g_j at x_0 and the continuity of γ at t = 0, we have $g_j(\gamma(t)) < 0$ for *t* small enough.

So, $\gamma(t) \in S$ for *t* small enough and, consequently, $v \in A(S,x_0)$, completing the proof.

(ii) $N(S,x_0) = C(S)^* = [C(G) \cap K(H)]^* = N[C(G) \cap K(H), 0] = N[C(G),0] + N[K(H),0]$.

The last equality holds by (Rockafellar 1970, Corollary 23.8.1) because K(H) is polyhedral and $(\text{ri}\, C(G)) \cap K(H) \ne \emptyset$ is verified. To see the latter, we note that, by hypothesis, $C_0(S) \ne \emptyset$, so there exists v such that $Dg_j(x_0,v) < 0, \forall j \in J_0$ and $\nabla h(x_0)v = 0$. By continuity of $Dg_j(x_0,.)$, it follows that $v \in (\text{int}\, C(G)) \cap K(H)$. Finally $N[C(G),0] = C(G)^* = \text{cone co}\left(\bigcup_{j\in J_0} \partial_D g_j(x_0)\right)$ by Theorem 3.5 (ii).

In this theorem it is only required h to be differentiable at x_0 and not on a neighborhood of the point and, far from it, to be C^1. That are the conditions in which Hestenes's Theorem is valid. On the other hand, this theorem generalizes and improves Theorem 3.3 of Di (1996) who suppose that *g* is Fréchet differentiable at x_0 and obtains that $T(S, x_0) = C(S)$.

If the functions g_j are only Dini differentiable, then the proof of Theorem 3.6 is not valid, because the chain rule is not verified. However, we can still have the same conclusions with extra hypothesis.

Theorem 3.7. Suppose the following:

(a) For each $j \in J_0, g_j$ is either Dini-quasiconvex and continuous on a neighborhood of x_0 with convex derivative or Fréchet differentiable at x_0.

(b) $\nabla h(x_0)$ has maximal rank.

(c) $C_0(S) \neq \varnothing$.

Then

(i) $\mathrm{cl}\, C_0(S) = T(S, x_0) = C(S)$.

(ii) $N(S, x_0) = \sum_{j \in J_0} \mathrm{cone}\, \partial_D g_j(x_0) + \mathrm{Lin}\{\nabla h_k(x_0) : k \in K\}$.

Firstly, we need next Lemma 3.8. We consider the following sets:

$J_{01} = \{j \in J_0 : g_j$ is Dini-quasiconvex on a neighborhood of $x_0\}$,

$J_{02} = \{j \in J_0 \setminus J_{01} : g_j$ is Fréchet differentiable at $x_0\}$,

$G_1 = \{x \in \mathbb{R}^n : g_j(x) \leq 0,\ \forall j \in J_{01}\}$ and $S_2 = \{x \in \mathbb{R}^n : g_j(x) \leq 0,\ \forall j \in J \setminus J_{01},\ h(x) = 0\}$.

Notice that $S = G_1 \cap S_2$.

Lemma 3.8. $\nabla h(x_0)$ has maximal rank and $C_0(S) \neq \varnothing$ if and only if $C_0(G_1) \neq \varnothing$ and the following constraint qualification (CQ) holds at x_0:

(CQ) there exists no $(\lambda, \mu) \in \mathbb{R}^{J_{02}} \times \mathbb{R}^r$ such that

1) $\lambda \geq 0$, 2) $(\lambda, \mu) \neq 0$ and 3) $\sum_{j \in J_{02}} \lambda_j \nabla g_j(x_0) + \sum_{k \in K} \mu_k \nabla h_k(x_0) \in -N(G_1, x_0)$

Proof.

($\Rightarrow$) It is obvious that $C_0(G_1) \neq \varnothing$. Conditions $C_0(S) \neq \varnothing$ and $\{\nabla h_k(x_0) : k \in K\}$ lin. indep. are just statement (*c*) in Proposition 2.6 applied to the convex functions $Dg_j(x_0, \cdot), j \in J_{01}$, $\nabla g_j(x_0)(\cdot), j \in J_{02}$ and the linear functions $\nabla h_k(x_0)(\cdot), k \in K$. Such statement is equivalent to (*a*) of that proposition.

If we suppose that (CQ) is not true, then there exist $\lambda_j \geq 0, j \in J_{02}, \mu_k \in \mathbb{R}$, $k \in K$ not all zero such that $\sum_{j \in J_{02}} \lambda_j \nabla g_j(x_0) + \sum_{k \in K} \mu_k \nabla h_k(x_0) \in -N(G_1, x_0)$.

Now, by Theorem 3.5(*ii*), $N(G_1, x_0) = \mathrm{cone}\, \mathrm{co}\left(\bigcup_{j \in J_{01}} \partial_D g_j(x_0)\right) = \sum_{j \in J_{01}} \mathrm{cone}\, \partial_D g_j(x_0)$, thus there exist $\lambda_j \geq 0, \xi_j \in \partial g_j(0), \forall j \in J_{01}$ such that

$$\sum_{j\in J_{01}}\lambda_j\xi_j+\sum_{j\in J_{02}}\lambda_j\nabla g_j(x_0)+\sum_{k\in K}\mu_k\nabla h_k(x_0)=0\ ,$$

contradicting statement (*a*) in Proposition 2.6.

($\Leftarrow$) We shall prove that (*a*) of Proposition 2.6 is verified.

Let $\sum_{j\in J_0}\lambda_j\xi_j+\sum_{k\in K}\mu_k\nabla h_k(x_0)=0$ with $\lambda_j\geq 0,\xi_j\in\partial_D g_j(x_0),\forall j\in J_0$.

For each $j\in J_{02}, g_j$ is Fréchet differentiable at x_0, and so $\partial_D g_j(x_0)=\{\nabla g_j(x_0)\}$. Thus:

$$\sum_{j\in J_{01}}\lambda_j\xi_j+\sum_{j\in J_{02}}\lambda_j\nabla g_j(x_0)+\sum_{k\in K}\mu_k\nabla h_k(x_0)=0\ ,$$

and because $N(G_1,x_0)=\sum_{j\in J_{01}}\partial_D g_j(x_0)$ we have that

$$\sum_{j\in J_{02}}\lambda_j\nabla g_j(x_0)+\sum_{k\in K}\mu_k\nabla h_k(x_0)\in -N(G_1,x_0)\ .$$

By hypothesis (CQ), $\lambda_j=0,\forall j\in J_{02},\mu=0$, and consequently, $\sum_{j\in J_{01}}\lambda_j\xi_j=0$.

Now, since $C_0(G_1)\neq\varnothing$, by Proposition 2.6 (with r = 0), we deduce that $\lambda_j=0,\ \ \forall j\in J_{01}$, and the proof of the lemma is completed.

Proof of Theorem 3.7.

Let us prove that

$$T(G_1\cap S_2,x_0)=T(G_1,x_0)\cap T(S_2,x_0) \tag{7}$$

By hypothesis (a), taking a closed neighborhood of x_0, $\overline{B}(x_0,\delta)$, with δ small enough, we can assume that on this neighborhood all the functions $g_j, j\in J_{01}$, are continuous and quasiconvex (Proposition 2.5(c)). Hence, $G_0=G_1\cap\overline{B}(x_0,\delta)$ is convex (Proposition 2.5(a)) and closed. We have that $N(G_0,x_0)=N(G_1,x_0)$, and therefore, the constraint qualification (CQ) holds at x_0 for G_0 instead of G_1. From Theorem 4.1 of Di (1996), we obtain

$$T(G_0\cap S_2,x_0)=T(G_0,x_0)\cap T(S_2,x_0)$$

and, because the tangent cone is a local concept, $T(G_0,x_0)=T(G_1,x_0)$ and $T(G_0\cap S_2,x_0)=T(G_1\cap S_2,x_0)$, completing the proof of (7).

Now, let us see that

$$C_0(S)\subset T(S,x_0)\subset C(S) \tag{8}$$

The second inclusion follows from Proposition 3.2. By Theorem 3.6, $C_0(S_2)\subset T(S_2,x_0)$, by Proposition 3.4, $C_0(G_1)\subset T(G_1,x_0)$, and using (7) we have that

$$C_0(S)=C_0(G_1)\cap C_0(S_2)\subset T(G_1,x_0)\cap T(S_2,x_0)=T(G_1\cap S_2,x_0)=T(S,x_0)\ .$$

Finally, let us prove parts (i) and (ii). By (8), $C_0(S) \subset T(S, x_0)$, and by Proposition 3.2, $T(S, x_0) \subset C(S)$. Then, by Lemma 3.1, $\mathrm{cl}\, C_0(S) = T(S, x_0) = C(S)$. The proof of part (ii) is analogous to that of Theorem 3.6(ii).

In the next example a problem is proposed in which the results obtained here can be applied. Note that neither the results of the Convex Analysis nor the ones of the Theory of Generalized Gradients of Clarke are applicable though.

Example 3.9. First, we define a real function with real variable, quasiconvex, Dini differentiable and not locally Lipschitzian and from it, we will construct other function with two variables, keeping these properties.

Let δ_n be a decreasing sequence of positive numbers converging to 0 such that $\delta_{n+1}/\delta_n \to 1$ and let c_n be such that $\delta_{n+1} < c_n < \delta_n, \forall n$. We define φ: $\mathbb{R} \to \mathbb{R}$ such that is odd ($\varphi(-t) = -\varphi(t)$), $\varphi(t) = t$ if $t > \delta_1$, $\varphi(0) = 0$ and for $0 < t \le \delta_1$ is given by

$$\varphi(t) = \begin{cases} \delta_{n+1} & \text{if } \delta_{n+1} < t < c_n,\ n = 1, 2, \ldots \\ l_n(t) & \text{if } c_n \le t \le \delta_n,\ n = 1, 2, \ldots \end{cases}$$

where $l_n(t)$ is the affine function through the points $A_n = (\delta_n, \delta_n)$, $C_n = (c_n, \delta_{n+1})$. Its expression is $l_n(t) = m_n(t - \delta_n) + \delta_n$, where $m_n = (\delta_n - \delta_{n+1})/(\delta_n - c_n)$.

Let $\delta_n = 1/n$ and $c_n = 1/n - 1/4^n$. Then the function so defined is continuous on $\mathbb{R}$ and it is not Lipschitz near 0 (the sequence of the slopes of l_n tends to $+\infty$).

It can easily be proved that φ is Dini differentiable at 0. As a matter of fact, it is differentiable with $\varphi'(0) = 1$. In fact,

$$\varphi^+(0) = \limsup_{t \to 0^+} \frac{\varphi(t) - \varphi(0)}{t} = \lim_{\varepsilon \to 0^+} \sup_{t \in (0,\varepsilon)} \frac{\varphi(t)}{t} = 1.$$

$$\varphi^-(0) = \liminf_{t \to 0^+} \frac{\varphi(t) - \varphi(0)}{t} = \lim_{\varepsilon \to 0^+} \inf_{t \in (0,\varepsilon)} \frac{\varphi(t)}{t} = \lim_{n \to \infty} \inf_{t \in (0,\delta_n)} \frac{\varphi(t)}{t} =$$

$$= \lim_{n \to \infty} \inf_{k \ge n+1} \inf_{t \in [\delta_k, \delta_{k-1})} \frac{\varphi(t)}{t} = \lim_{n \to \infty} \inf_{k \ge n+1} \frac{l_k(c_k)}{c_k} = \lim_{n \to \infty} \inf_{k \ge n+1} \frac{\delta_{k+1}}{c_k} = 1.$$

φ is quasiconvex on $\mathbb{R}$, since it is increasing, but it is not convex.

From φ we define ψ: $\mathbb{R} \to \mathbb{R}$, given by

$$\psi(t) = \begin{cases} 2\varphi(t) & \text{if } t \ge 0 \\ \varphi(t) & \text{if } t < 0. \end{cases}$$

ψ has the same properties than φ (continuity, not Lipschitz near 0, quasiconvexity) except that it is not differentiable at 0, but it is Dini differentiable, with convex derivative

$$D\psi(0, t) = \begin{cases} 2t & \text{if } t \ge 0 \\ t & \text{if } t < 0 \end{cases}$$

And, from ψ we define g: $\mathbb{R}^2 \to \mathbb{R}$ given by

$$g(x,y)=\begin{cases}\psi\left(\sqrt{x^2+y^2}\right) & \text{if} \quad x\geq 0, y\geq 0\\ \psi(y) & \text{if} \quad x<0, y\geq x\\ \psi(x) & \text{if} \quad y<0, y<x\end{cases}$$

It is a continuous and quasiconvex function on $\mathbb{R}^2$, but it is neither Lipschitz near x_0 = (0,0) nor convex. Its Dini derivative has the expression:

$$Dg(x_0,(x,y))=\begin{cases}2\sqrt{x^2+y^2} & \text{if} \quad x\geq 0, y\geq 0\\ 2y & \text{if} \quad x<0, y\geq 0\\ y & \text{si} \quad y<0, y\geq x\\ x & \text{if} \quad x\leq 0, y<x\\ 2x & \text{if} \quad x>0, y<0\end{cases}$$

which is convex on $\mathbb{R}^2$. Its Dini subdifferential is

$$\partial_D g(x_0) = \mathrm{co}(\{(1,0),(0,1)\}\cup\{(x,y): x^2+y^2=4, x\geq 0, y\geq 0\}).$$

If we consider *g* as a constraint function, let $G=\{(x,y): g(x,y)\leq 0\}$. In order to obtain $N(G,x_0)$, Corollary 23.7.1 or Theorem 23.7 of Rockafellar (1970) cannot be applied, because *g* is not convex. Neither Theorem 2.4.7 (or Corollary 1) of Clarke (1983) can be applied because *g* is not Lipschitz near x_0. However, from Theorem 3.5, it follows that $N(G,x_0)=\text{cone}\ \partial_D g(x_0)=\{(x,y): x\geq 0, y\geq 0\}$.

We can add an equality constraint. Let

$$h(x,y)=\begin{cases}-x+y+xy & \text{if} \quad y\geq 0\\ -x+y-xy & \text{if} \quad y<0\end{cases}$$

This function is differentiable at x_0, but it is not differentiable on any of its neighborhoods. Let $H=\{(x,y): h(x,y)=0\}$ and $S=G\cap H$. Theorem 3.7 can be used since $C_0(S)=\{(x,y): y=x, x<0\}\neq\emptyset$. Thus,

$$N(S,x_0)=\text{cone}\,\partial_D g(x_0)+\text{Lin}\{\nabla h_k(x_0)\}=\{(x,y): x+y\geq 0\}$$

and $T(S,x_0)=\{t(-1,-1): t\geq 0\}$.

As an application of previous results, we can formulate necessary optimality conditions both Fritz John and Kuhn-Tucker type for a multiobjective optimization problem.

Theorem 3.10. Consider the problem (MP) where S = $\{x\in\mathbb{R}^n: g(x)\leq 0,\ h(x)=0\}$,f is Hadamard differentiable at $x_0\in S$ with convex derivative, and assume that conditions (a) and (b) of Theorem 3.6 or Theorem 3.7 hold. If x_0 is a weak local Pareto minimum for (MP), then there exist $(\lambda,\mu,\nu)\in\mathbb{R}^p\times\mathbb{R}^{J_0}\times\mathbb{R}^r$, $(\lambda,\mu)\geq 0$, $(\lambda,\mu)\neq 0$ such that

$$0\in\sum_{i=1}^{p}\lambda_i\partial_D f_i(x_0)+\sum_{j\in J_0}\mu_j\partial_D g_j(x_0)+\sum_{k=1}^{r}\nu_k\nabla h_k(x_0). \tag{9}$$

If, moreover, $C_0(S)\neq\emptyset$, then $\lambda\neq 0$.

Proof.

It is known that if x_0 is a weak local minimum of a Hadamard differentiable function f on S, then $T(S, x_0) \cap C_0(f) = \varnothing$, where $C_0(f) = \{u \in \mathbb{R}^n : df_i(x_0, u) < 0,\ i = 1,2,\ldots,p\}$.

If $C_0(S) \neq \varnothing$, from Theorem 3.6 or from Theorem 3.7 it follows that $C(S) \cap C_0(f) = \varnothing$.

Hence,

$$C_0(S) \cap C_0(f) = \varnothing \tag{10}$$

If $C_0(S) = \varnothing$, then (10) is obviously satisfied.

Consequently, the following system has no solution $u \in \mathbb{R}^n$:

$$df(x_0, u) < 0, \quad Dg_j(x_0, u) < 0\ \forall j \in J_0, \quad \nabla h(x_0)u = 0 .$$

Using Proposition 2.6, there exist

$(\lambda, \mu, \nu) \in \mathbb{R}^p \times \mathbb{R}^{J_0} \times \mathbb{R}^r$, $(\lambda, \mu) \geq 0$, $(\lambda, \mu) \neq 0$ such that (9) holds and

$$\sum_{i=1}^{p} \lambda_i df_i(x_0, u) + \sum_{j \in J_0} \mu_j Dg_j(x_0, u) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0)u \geq 0 \quad \forall u \in \mathbb{R}^n. \tag{11}$$

For the second part, assume that $C_0(S) \neq \varnothing$. If were $\lambda = 0$, as there exists $w \in C_0(G) \cap K(H)$ and $\mu \neq 0$, we have that

$$\sum_{j \in J_0} \mu_j Dg_j(x_0, w) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0)w < 0 ,$$

which is a contradiction with what is obtained choosing u = w in (11).

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