RECURRENCE AND DIRECT FORMULAS FOR THE HAL & LAH NUMBERS

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ABSTRACT

In this paper we study some properties concerning the Hal (H_n^k) and Lah numbers (L_n^k), which are those that arise in the theory of combinatorics in association with the "upward and downward factorial polynomials" $[x]^n$ and $[x]_n$. By using generating function techniques and simple binomial expansions we establish some nive results for the Hal number (H_n^k), like the recurrence formula, the generating funcion and the direct (closed) formula, among others properties. Finally we compare these results with the corresponding formulas for the Lah numbers (L_n^k). We establish the corresponding recurrence formulas, the generating functions and the closed formulas for this family of numbers, among others properties.

Key words: Hal numbers, Lah numbers, combinatorics, generating functions.

MSC: 05A10.

RESUMEN

En este trabajo se estudian algunas propiedades relativas a los números de Hal (H_n^k) y de Lah (L_n^k), que son aquellos números que surgen en combinatoria asociados con los polinomios factoriales hacia arriba y hacia abajo. Se establecen las fórmulas por recurrencia, las funciones generadoras y las fórmulas cerradas para esta familia de números, entre otras propiedades.

1. THE HAL NUMBERS

For any real number x and any natural number n we define the *upward factorial polynomial* $[x]^n$ to be

$$[x]^n := x(x+1)\dots(x+n-1).$$

In association with these polynomials we have the *Hal numbers* (H_n^k), whose definition are those numbers that satisfy the relation

$$[-x]^n := H_n^1[x]^1 + H_n^2[x]^2 + \cdots + H_n^n[x]^n, \tag{1}$$

and we extend this definition stipulating that $H_n^k = 0$ when $k > n$, or $k = 0$, or $n = 0$.

The terminology employed here (´Hal´ numbers) has been chosen to emphasize the analogy between this numerical family with the more familiar Lah numbers (L_n^k) for the downward factorial polynomials $[x]_n$ (´hal´ is ´lah´ backwards), as we see later. See also [2, 3].

Notice that by substitution of x by – x in the relation (1) we obtain the equivalent relation

$$[x]^n := H_n^1[-x]^1 + H_n^2[-x]^2 + \cdots + H_n^n[-x]^n. \tag{2}$$

As a consequence from the First Inversion Formula (see appendix, or [1]) we inmediately deduce that

$$\sum_{k=1}^{\infty} H_n^k H_k^m = \delta_{nm} = \begin{cases} 1, & \text{si} \quad n = m \\ 0, & \text{si} \quad n \neq m. \end{cases}$$

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Then, the square matrix $H = (H_n^k)$, with $n,k \in \{1,\ldots,N\}$ is inverse of itself, for all $N \in \mathbb{N}^*$. Furthermore, also as a consequence of the same First Inversion Formula we deduce that any one of the equations

$$a_n = \sum_{k=1}^{n} H_n^k b_k, \qquad b_n = \sum_{k=1}^{n} H_n^k a_k$$

implies the other. Let´s see how to get a recurrence formula and a direct (closed) formula for these Hal numbers (H_n^k).

2. RECURRENCE FOR THE HAL NUMBERS

Theorem 1

$$H_{n+1}^k = (n+k)H_n^k - H_n^{k-1}. \tag{3}$$

Proof: Consider the following polynomials of degree n + 1:

$$A(x) = \sum_{k=1}^{n+1} \{(n+k)H_n^k - H_n^{k-1}\}[x]^k,$$

$$B(x) = \sum_{k=1}^{n+1} H_{n+1}^k [x]^k.$$

We will prove that these polynomials are equal, obtaining the result by comparing the corresponding coefficients. In fact, we have:

$$A(x) = \sum_{k=1}^{n+1} \{(n+k)H_n^k - H_n^{k-1}\}[x]^k$$

$$= n\sum_{k=1}^{n+1} H_n^k[x]^k + \sum_{k=1}^{n+1} kH_n^k[x]^k - \sum_{k=1}^{n+1} H_n^{k-1}[x]^k$$

$$= n\sum_{k=1}^{n} H_n^k[x]^k + \sum_{k=1}^{n} kH_n^k[x]^k - \sum_{k=1}^{n} H_n^k[x]^{k+1}.$$

But $[x]^{k+1} = [x]^k(x + k)$, and so we can simplify the last expression:

$$A(x) = n\sum_{k=1}^{n} H_n^k[x]^k + \sum_{k=1}^{n} kH_n^k[x]^k - \sum_{k=1}^{n} H_n^k[x]^k(x+k)$$

$$= (-x+n)\sum_{k=1}^{n} H_n^k[x]^k = (-x + n)[-x]^n$$

$$= [-x]^{n+1} = \sum_{k=1}^{n+1} H_{n+1}^k[x]^k$$

$= B(x)$. ■

The last recurrence makes it easy to do fast calculations of Hal numbers, whose first terms are illustrated in the Table 1. It can be observed that even columns have only positive numbers, while odd columns have only negative numbers (below the principal diagonal). Besides, all the rows have alternating numbers, beginning by – n!. These facts will be easily proved with the help of the direct formula H_n^k which we will deduce later.

Table 1. First terms of the Hal numbers (H_n^k).

n\k	1	2	3	4	5	…
1	- 1					
2	- 2	1				
3	- 6	6	- 1			
4	- 24	36	- 12	1		
5	- 120	240	- 120	20	- 1	
⋮						

3. GENERATING FUNCTION FOR HAL NUMBER

Following a similar approach to that used by Riordan in (1958), we deduct the generating function for the Hal number.

Theorem 2. *For all* $k \in \mathbb{N}$, *the exponential generating function* $H_k(t)$ of the Hal numbers $(H_n^k)_{n\in\mathbb{N}}$ is

$$H_k(t) := \sum_{n=0}^{\infty} H_n^k \frac{t^n}{n!} = \frac{1}{k!}\left(\frac{-t}{1-t}\right)^k. \qquad (4)$$

Proof: From the identity

$$[-x]^n = (-x)(-x+1)\ldots(-x+n-1)$$

$$= (-1)^n x(x-1)\ldots(x-n+1) = (-1)^n \binom{x}{n}$$

and using the binomial expansion $(1+u)^\beta = \sum_{k=0}^{\infty} \binom{\beta}{k} u^k$, valid for $|u| < 1$, we deduce that:

$$\sum_{n=0}^{\infty} [-x]^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \binom{x}{n} t^n = (1-t)^x = \left(1+\frac{t}{1-t}\right)^{-x}$$

$$= \sum_{k=0}^{\infty} \binom{-x}{k}\left(\frac{t}{1-t}\right)^k = \sum_{k=0}^{\infty} (-1)^k \frac{[x]^k}{k!}\left(\frac{t}{1-t}\right)^k$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{-t}{1-t}\right)^k [x]^k.$$

Then,

$$\sum_{k=0}^{\infty} H_k(t)[x]^k = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} H_n^k \frac{t^n}{n!}[x]^k = \sum_{n=0}^{\infty} \frac{t^n}{n!}\sum_{k=0}^{\infty} H_n^k \frac{t^n}{n!}[x]^k$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!}[-x]^n = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{-t}{1-t}\right)^k [x]^k.$$

The exchange between sum symbols is justified just because all the sums over k are finite, due to $H_n^k = 0$ for $k > n$. The result is obtained by comparing the corresponding coefficients. ■

4. CLOSED FORMULA FOR THE HAL NUMBERS

Although the recurrence (3) for the Hal numbers is a linear and first degree formula in both n and k, this recurrence is not easy to resolve in order to derive a direct formula for the Hal numbers (H_n^k). In fact, we use a different approach to reach the closed formula.

Theorem 3. For $n \geq k$:

$$H_n^k = (-1)^k \binom{n-1}{k-1} \frac{n!}{k!}. \tag{5}$$

Proof: Making the binomial expansion of $\{-t/(1-t)\}^k$, we obtain:

$$\begin{aligned}
H_k(t) &= \frac{1}{k!}(-1)^k t^k (1-t)^{-k} \\
&= \frac{1}{k!}(-1)^k t^k \sum_{i=0}^{\infty} \binom{-k}{i} (-1)^i t^i \\
&= \frac{1}{k!}(-1)^k \sum_{i=0}^{\infty} (-1)^i \binom{k+i-1}{i} (-1)^i t^{k+i} \\
&= \frac{1}{k!}(-1)^k \sum_{n=k}^{\infty} \binom{n-1}{n-k} t^n \\
&= \sum_{n=k}^{\infty} (-1)^k \frac{n!}{k!} \binom{n-1}{k-1} \frac{t^n}{n!}.
\end{aligned} \tag{6}$$

When comparing the corresponding coefficients in the formulas (4) and (6) we obtain the result. ■

From this closed formula for Hal numbers we can deduce, among other things, that the numbers $(-1)^k H_n^k$ are always positive (for $k \in \{1,\dots,n\}$): the rows of the table of Hal numbers have alternating entries, beginning by the negative factorials $H_n^1 = n!$. Also note that the parity of Hal number H_n^k depends only on index k and does not depend on index n.

5. THE LAH NUMBERS

For any real number x and for any natural number n we define the *downward factorial polynomial* $[x]_n$ to be

$$[x]_n := x(x-1)\dots(x-n+1).$$

The Lah numbers are defined as the coefficients L_n^k that satisfy the identity

$$[-x]_n = L_n^1[x]_1 + L_n^2[x]_2 + \dots + L_n^n[x]_n. \tag{7}$$

We also extend the definition for other indexes, stipulating that $L_n^k = 0$ when $k > n$, or $n = 0$, or $k = 0$. By similar methods used for the Hal numbers, we can deduce that the Lah numbers also form square matrices which are inverse of itself:

$$\sum_{k=1}^{\infty} L_n^k L_k^m = \delta_{nm} = \begin{cases} 1, & \text{si } n = m \\ 0, & \text{si } n \neq m. \end{cases}$$

The complete analogy between Lah and Hal numbers will be stated in the following result.

Theorem 4. *The Lah numbers* (L_n^k) *satisfy the following recurrence and direct formula:*

$$L_{n+1}^k = -(n+k)L_n^k - L_n^{k-1} \tag{8}$$

$$L_n^k = (-1)^n \frac{n!}{k!}\binom{n-1}{k-1} \tag{9}$$

Proof: First we find the connection between the polynomials $[-x]_n$ and $[x]^n$:

$$\begin{aligned}[-x]_k &= (-x)(-x-1)\ldots(-x-k+1)\\ &= (-1)^k x(x+1)\ldots(x+k-1)\\ &= (-1)^k[x]^k,\end{aligned}$$

that is, $[x]^k = (-1)^k[-x]_k$. Applying this in formula (1) we obtain

$$(-1)^n[-x]_n = (-1)^1 H_n^1 [x]_1 + \ldots + (-1)^n H_n^n [x]_n,$$

that also implies that

$$[-x]_n = (-1)^{n+1} H_n^1 [x]_1 + \ldots + (-1)^{n+n} H_n^n [x]_n.$$

The identity (7) then implies that

$$L_n^k = (-1)^{n+k} H_n^k. \tag{10}$$

Therefore, the recurrence (8) and the direct formula (9) are a consequence of theorems (1) and (3) and the last identity (10). ■

APPENDIX: First Inversion Formula

The following is a well known result in combinatorics.

Theorem 5. *Let* $\varphi_n(x)$ *and* $\psi_n(x)$ *be families of polynomials of degree n and let* α_n^k, β_n^k *(with* $0 \le k \le n$*) be any collection of real numbers. Suppose that the following relations are satisfied* :

$$\varphi_n(x) = \sum_{k=0}^{n} \alpha_n^k \psi_k(x), \qquad (n = 0,1,\ldots n_0)$$

$$\psi_n(x) = \sum_{k=0}^{n} \beta_n^k \varphi_k(x), \qquad (n = 0,1,\ldots n_0).$$

If $a_0, a_1, \ldots, a_{n_0}, b_0, b_1, \ldots, b_{n_0}$ *are numbers that satisfy the relations* $a_n = \sum_{k=0}^{n} \alpha_n^k b_k$, *for* $n = 0,1,\ldots,n_0$, *then* $b_n = \sum_{k=0}^{n} \beta_n^k a_k$, *for* $n = 0, 1, \ldots n_0$. *Furthermore*,

$$\sum_{k=0}^{\infty} \alpha_n^k \beta_k^m = \delta_{nm} = \begin{cases} 1 & \text{si } n = m \\ 0 & \text{si } n \neq m. \end{cases}$$

Prof: May be looked up in the books of Berge (1971) or Piza (2002). ■

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