PSEUDOCONVEXITY IN SET-VALUED OPTIMIZATION

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ABSTRACT
The notion of contingent epiderivative and its properties are studied. A Lagrange multiplier rule is obtained for K-pseudoconvex and generalized K-convexlike multifunctions with contingent epiderivative. Convexity of the domain is replaced with conditions on a certain cone which will be asymptotically compact.

Key words: Set-valued functions, optimality conditions, convex and set-valued analysis, vector optimization.

RESUMEN
En este trabajo se obtiene una regla de multiplicadores de Lagrange para multifunciones K-pseudo-convexas y K-convexlike generalizadas que poseen epiderivada contingente. Por otra parte se estudia esta noción de epiderivada contingente y sus propiedades. La convexidad del dominio se reemplaza por condiciones en un cierto cono que deberá ser asintóticamente compacto.

MSC: 49K27

1. INTRODUCTION AND NOTATION

Set valued optimization problems have been investigated by many authors in recent years. In some cases: Aubin (1981), Aubin-Frankowska (1990), Burwein (1977), Corley (1987) and (1988), Jahn-Rauh (1997), Jahn-Götz (2000) and Luc (1991), they have established necessary and sufficient conditions under certain hypothesis.

In this paper we consider the following standard assumptions:

Let X be a real normed space. Let Y, Z be real normed spaces partially ordered by convex pointed cones $K_Y \subset Y$ and $K_Z \subset Z$ respectively. Let $F: X \to 2^Y$, $G: X \to 2^Z$ be set-valued maps and let M be a nonempty subset of X, $M \subset \mathrm{Dom}(F)$, $M \subset \mathrm{Dom}(G)$.

Under these assumptions we consider the constrained set-valued optimization problem

$$(1) \qquad \begin{cases} \min F(x) \\ \text{subject to the constraints:} \\ G(x) \cap (-K_Z) \neq \phi \\ x \in M \end{cases}$$

The aim of this paper is to study some optimization conditions for the problem (1).

In section 2 we introduce the concepts of contingent epiderative and K-pseudoconvexity of a set-valued map at a point. Several properties of K-pseudoconvex multifunctions are provided. Section 3 deals with a necessary condition. We establish a mutiplier rule for the problem (1), in the case of a set-valued map F x G, K-pseudoconvex and with contingent epiderivative. The notion of minimizer used is the one of weak minimizer. In set-valued optimization there are different optimality concepts in use. We recall two standard optimality notions (see for example Jahn-Rauh (1997) and Luc (1991).

Definition 1.1. Let $F(M) = \cup_{x \in M} F(x)$ denote the image set of M by F.

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a) A point $(x_0,y_0) \in$ graph(F) is called a minimizer of the problem (1), if y_0 is a minimal element of the set F(M),
i.e.:

$$y \in F(M),\ y_0 - y \in K_Y \Rightarrow y = y_0$$

or in other words

$$(\{y_0\} - K_Y) \cap F(M) = \{y_0\}$$

b) Let K_Y have a nonempty interior $\mathrm{int}(K_Y)$. A point $(x_0, y_0) \in$ graph(F), is called a weak minimizer of the problem (1) if y_0 is a weakly minimal element of the set F(M), i.e.:

$$(\{y_0\} - \mathrm{int}(K_Y)) \cap F(M) = \varnothing$$

The following notions of set-valued maps will be used throughout this work. Let E_1, E_2 be real normed spaces. Let E_2 be partially ordered by a pointed convex cone $K \subset E_2$. Let $F:M \to 2^{E_2}$ be a set-valued map and let $M \neq \varnothing$ be a nonempty subset of E_1.

The epigraph of F is the set

$$\mathrm{epi}(F) = \{(x,y) \in E_1 \times E_2 : x \in M,\ y \in F(x) + K\}$$

the epirange of F is the set

$$\mathrm{epiran}(F) = \{y \in E_2 : \text{there exists } x \in M,\ y \in F(x) + K\}$$

Let $(x_0, y_0) \in$ graph(F). The contingent cone T(graph(F); (x_0,y_0)) consists of all tangent vectors $(h,k) = \lim\limits_{n\to\infty} \mu_n(x_n - x_0, y_n - y_0)$ with $\lim\limits_{n\to\infty} (x_n,y_n) = (x_0, y_0)$, $(x_n, y_n) \subset$ graph (F) and $(\mu_n) \subset \mathrm{IR}$, $\mu_n > 0$ for all $n \in N$. Or equivalently, there exists a sequence of real numbers $(t_n) \to 0$, $t_n > 0$, and a sequence of vectors $(h_n, k_n) \to (h, k)$ such that $(x_0 + t_nh_n, y_0 + t_nk_n) \in$ epi(F).

The contingent derivative of F at (x_0, y_0) is the set-valued map $D_CF(x_0, y_0): X \to 2^Y$ whose graph equals the contingent cone to the graph of F at (x_0,y_0), i.e.:

$$\mathrm{graph}\big(D_CF(x_0,y_0)\big) = T(\mathrm{graph}(F); (x_0, y_0))$$

The dual cone of K is the set

$$K^* = \{y^* \in E_2^* : y^*(y) \geq 0 \text{ for all } y \in K\}$$

The cone generated by a nonempty subset L of E_2 is the set

$$\mathrm{cone}(L) = \{\lambda y \in E_2 : \lambda \geq 0,\ y \in S\}$$

Definition 1.2 a) The set-valued map F is called K-convexlike if the set epiran(F) = {F(M) + K} is convex.

b) F is called generalized K-convexlike if the set {cone (F(M)) + K} is convex.

2. K-PSEUDOCONVEXITY AND CONTINGENT EPIDERIVATIVE

From the basic idea of J. Aubin and H. Frankowska (1990), J. Jahn and R. Rauh (1997) develope the concept of contingent epiderivative for a multifunction defined between two arbitrary real normed spaces. Next we recall the definition of this epiderivative. Let Π_1 denote the projection map of $E_1 \times E_2$ onto E_1.

Definition 2.1. Let $(x_0, y_0) \in$ graph(F). Let $A = \Pi_1(T(\mathrm{epi}(F); (x_0,y_0)))$. The contingent epiderivative of F at (x_0, y_0) is the single-valued map $DF(x_0, y_0)$, from A to E_2 which verifies

$$\text{epi}(DF(x_0,y_0)) = T(\text{epi}(F); (x_0,y_0))$$

Compare the previous definition with the one given in Janh-Rauh (1997). In that definition it is supposed that $\text{Dom}(DF(x_0, y_0)) = E_1$. It considerably limits the class of set-valued maps which have contingent epiderivative. It is obvious that

$$\text{cl}(\text{Dom}(DF(x_0, y_0)) \subset T(\text{Dom}(F); x_0)$$

because

$$\Pi_1(T(\text{epi}(F);(x_0, y_0)) \subset \Pi_1(T(\text{Dom}(F) \times (\text{epiran}(F) + K); (x_0,y_0))$$

$$\subset \Pi_1 (T(\text{Dom}(F); x_0) \times T(\text{epirang}(F) + K; y_0))$$

$$= T(\text{Dom}(F);x_0)$$

If the contingent epiderivative exists, it is an unique and positively homogeneous function. It has been proved in Janh-Rauh (1997). Moreover in that work it has been noted about the difficulty of the calculus of this epiderivative. In this sense we establish the following result.

Theorem 2.2. $DF(x_0, y_0)$ exists if and only if for every $\bar{x} \in A$, the set

$$L(\bar{x}) = \{y \in E_2 \mid (\bar{x}, y) \in T(\text{epi}(F); (x_0,y_0))\}$$

has a minimum. Then $DF(x_0, y_0)(\bar{x}) = \min L(\bar{x})$.

Proof. Let us suppose that $DF(x_0, y_0)$ exists. Let $\bar{y} = DF(x_0, y_0)(\bar{x})$. Since

$$\text{epi } (DF(x_0,y_0)) = T(\text{epi}(F); (x_0, y_0))$$

we have

$$(\bar{x},\bar{y}) + (\{0\} \times K) = \{\bar{x}\} \times L(\bar{x})$$

Therefore $y \in \bar{y} + K$ for all $y \in L(\bar{x})$ and $\bar{y}$ is the minimum of $L(\bar{x})$.

Let us now assume that, for every $\bar{x} \in A$, $\min L(\bar{x})$ exist. We define the function $f(\bar{x}) = \min L(\bar{x})$. We will show that f verifies

$$\text{epi } (f) = T(\text{epi } (F); (x_0,y_0))$$

and then $f = DF (x_0,y_0)$.

In fact, if $(\bar{x}, y) \in T(\text{epi}(F); (x_0, y_0))$, we get that $y \in L(\bar{x})$, so $y \in f(\bar{x}) + K$ and $(\bar{x}, y) \in \text{epi}(f)$. On the other hand, if $(\bar{x}, \bar{y}) \in \text{epi}(f)$, then there exists $k \in K$ such that $y = f(\bar{x}) + k$. Since $f(\bar{x}) \in L(\bar{x})$ we obtain that $(\bar{x}, f(\bar{x})) \in T(\text{epi}(F); (x_0, y_0))$. Therefore there exist sequences (x_n), (y_n), with $y_n \in F(x_n) + K$, that converge to x_0 and y_0 respectively and a sequence (l_n) of real positive numbers such that $\lim_{n\to\infty} l_n(x_n - x_0, y_n - y) = (\bar{x}, f(\bar{x}))$. If $(x'_n) = (x_n), (y'_n) = \left(y_n + \dfrac{k}{l_n}\right)$, as $(l_n) \to \infty$ it follows that $\lim_{n\to\infty} l_n(x'_n - x_0, y'_n - y_0) = (\bar{x}, f(\bar{x}) + k) \in T(\text{epi}(F);(x_0, y_0))$.

□

The K-pseudoconvexity concept which will be used here is the generalization of the one of J.P. Aubin and H. Frankowska (1990). The contingent epiderivative of K-pseudoconvex set-valued maps verifies some interesting properties. We prove them in propositions 2.4-2.6.

Definition 2.3. F is said K-pseudoconvex at $(x_0, y_0) \in \text{graph}(F)$ if epi(F) is a pseudoconvex set at (x_0, y_0), i.e.

$$\mathrm{Epi}(F) \subset \{(x_0, y_0)\} + T(\mathrm{epi}(F); (x_0, y_0))$$

Proposition 2.4. If F is K-pseudoconvex at (x_0,y_0) then

$$\mathrm{cl}(\mathrm{Dom}(DF(x_0, y_0))) = T(\mathrm{Dom}(F); x_0)$$

Proof. Let $\tilde{F}:M \to 2^{E_2}$ be defined by $\tilde{F}(x) = F(x) + K$ and let $D_C\tilde{F}(x_0, y_0)$ be its contingent derivative. Since $\tilde{F}$ is K-pseudoconvex at (x_0,y_0) and

$$\mathrm{Dom}(DF(x_0, y_0)) = \mathrm{Dom}(D_c\tilde{F}(x_0,y_0))$$

we obtain

$$\begin{aligned} \mathrm{cl}(\mathrm{Dom}(DF(x_0, y_0))) &= \mathrm{cl}(\mathrm{Dom}(D_C\tilde{F}(x_0,y_0))) && = \mathrm{cl}(\Pi_1(\mathrm{graph}(D_C\tilde{F}(x_0,y_0)))) = \\ &= \mathrm{cl}(\Pi_1(T(\mathrm{graph}(\tilde{F}); (x_0,y_0)))) && = \mathrm{cl}(T(\Pi_1(\mathrm{graph}\,\tilde{F})); x_0)) = \\ &= T(\mathrm{Dom}(\tilde{F}); x_0) && = T(\mathrm{Dom}(F); x_0) \end{aligned}$$

□

Proposition 2.5. Let E_1, E_2 be real normed semi-reflexive spaces. Let $F:M \to 2^{E_2}$ be K-pseudoconvex at (x_0, y_0). If $T(\mathrm{epi}(F); (x_0, y_0))$ is a convex set and

$$(T(\mathrm{Dom}(F); x_0) \times \{0_{E_2}\}) \cap T(\mathrm{epi}(F); (x_0,y_0)) = \{(0_{E_1}, 0_{E_2})\}$$

then

$$\mathrm{Dom}(DF(x_0, y_0)) = T(\mathrm{Dom}(F); x_0)$$

Proof. Since F is K-pseudoconvex at (x_0, y_0), by proposition 2.4 we have $\mathrm{cl}(\mathrm{Dom}(DF(x_0, y_0))) = T(\mathrm{Dom}(F); x_0)$.

Let $u \in \mathrm{cl}(\mathrm{Dom}(DF(x_0,y_0))) \setminus \mathrm{Dom}(DF(x_0,y_0))$. There exists a sequence $(u_n) \subset \Pi_1(T(\mathrm{epi}(F); (x_0,y_0)))$, such that $\lim\limits_{n\to\infty} u_n = u$. Therefore there exists a sequence (v_n) such that $(u_n, v_n) \subset T(\mathrm{epi}(F);(x_0,y_0))$. Since $u \neq 0$ because
$0 \in \mathrm{Dom}(DF(x_0, y_0))$, we can suppose that $\|u_n\| > k_1$. On the other hand, (v_n) doesn't converge to 0 because in other case

$$(u,0) \in T(\mathrm{epi}(F); (x_0, y_0)) \cap (T(\mathrm{Dom}(F); x_0) \times \{0_{E_2}\})$$

which contradicts the hypothesis. So we can assume that $\|v_n\| > k_2$. Let $k = \min\{k_1,k_2\} > 0$. Let us consider

$$\mu_n = \inf\{\mu > 0 \mid \mu u_n \notin B(0,\tfrac{k}{2}), \mu v_n \notin B(0, \tfrac{k}{2}), \mu u_n \in B(0,k), \mu v_n \in B(0,k)\}$$

The sequence $\mu_n(x_n, y_n) \in T(\mathrm{epi}(F); (x_0, y_0)) \cap B((0_{E_1}, 0_{E_2}), k)$. This set is weakly relativelly compact because E_1 and E_2 are semi-reflexive sets. And from $0 \le \mu_n \le 1$, by the compacity of [0, 1], we deduce that there exist subsequences of $(\mu_n u_n, \mu_n v_n)$ and (μ_n), (which we will denote in the same way) such that $(\mu_n u_n, \mu_n v_n)$ converges to $(u',v') \in E_1 \times E_2$ weakly and $(\mu_n) \to \mu_0 \ge 0$. As $T(\mathrm{epi}(F); (x_0, y_0))$ is a closed and convex set, it is weakly closed and $(u',v') \in T(\mathrm{epi}(F); (x_0,y_0))$. On the other hand $\mu_0 \neq 0$. In other case $(\mu_n u_n) \to 0_{E_1}$ but it is impossible because the definition of μ_n implies that $u' \notin B(0,\tfrac{k}{2})$. So $\mu_0 > 0$ and we can define $u = \frac{1}{\mu_0}u'$ and $v = \frac{1}{\mu_0}v'$ which verify $(u,v) = \frac{1}{\mu_0}(u',v') \in T(\mathrm{epi}(F); (x_0,y_0))$. In consequence $u \in \mathrm{Dom}(DF(x_0,y_0))$.

□

The next proposition relates the contingent epiderivative to the contingent derivative.

Proposition 2.6. Let us suppose that there exist $D_CF(x_0, y_0)$ and $DF(x_0, y_0)$. It verifies

a) $\mathrm{epi}(D_CF(x_0,y_0)) \subset \mathrm{epi}(DF(x_0,y_0))$

b) If F is pseudoconvex or K-pseudoconvex at (x_0, y_0) then $\mathrm{cl}(\mathrm{epi}(D_CF(x_0,y_0))) = \mathrm{epi}(DF(x_0,y_0))$.

Proof. a) Let $(x,y) \in \mathrm{epi}(D_CF(x_0,y_0))$. There exists $\bar{y} \in D_CF(x_0, y_0)(x)$, $k \in K$ such that $y = \bar{y} + k$. From $(x,\bar{y}) \in T(\mathrm{graph}(F);(x_0, y_0))$ we have

$$(x,y) = (x, \bar{y}) + (0, k) \in T(\mathrm{graph}(F); (x_0,y_0)) + (\{0_{E_1}\} \times K) \subset T(\mathrm{epi}(F); (x_0,y_0)) + (\{0_{E_1}\} \times K)$$

$$= T(\mathrm{epi}(F); (x_0,y_0))$$

$$= \mathrm{epi}(DF(x_0,y_0))$$

b) From a) we obtain that $\mathrm{cl}(\mathrm{epi}(D_CF(x_0,y_0))) \subset \mathrm{epi}(DF(x_0,y_0))$. Let us show that $\mathrm{epi}(DF(x_0, y_0)) \subset \mathrm{cl}(\mathrm{epi}(D_CF(x_0,y_0)))$.

Let $(x,y + k) \in \mathrm{epi}(DF(x_0, y_0)) = T(\mathrm{epi}(F); (x_0,y_0))$. There exists $(x_n) \to x_0$, $(y_n) \to y_0$, with $y_n = \bar{y}_n + k_n$, $\bar{y}_n \in F(x_n)$, and $(\lambda_n) \subset \mathbb{R}$, with $\lambda_n > 0$ such that $(\lambda_n(x_n - x_0, \bar{y}_n + k_n - y_0)) \to (x, y + k)$. As F is pseudoconvex

$$\lambda_n(x_n - x_0, \bar{y}_n + k_n - y_0) = \lambda(x_n - x_0,y_n - y_0) + (\{0_{E_1}\} \times K) \in T(\mathrm{graph}(F); (x_n ,y_0)) + (\{0_{E_1}\} \times K)$$

and in consequence

$$(x, y + k) = \lim_{n\to\infty} (\lambda_n(x_n - x_0, \bar{y}_n + k_n - y_0)) \in \mathrm{cl}(T(\mathrm{graph}(F); (x_0, y_0)) + (\{0_{E_1}\} \times K)$$

$$= \mathrm{cl}(\mathrm{epi}(D_CF(x_0,y_0)))$$

3. A NECESSARY CONDITION FOR WEAK MINIMIZER

We will establish a Lagrange multiplier rule for the problem (1) at a point $(x_0, y_0) \in \mathrm{graph}(F)$, weak minimizer of F. With this purpose we will first prove some properties about the images of the contingent epiderivative. We will consider the set $L = \mathrm{cone}(M - \{x_0\})$ and we will use the concept of asymptotically compact set. Let us recall this concept.

Definition 3.1. A subset L of X is called asymptotically compact if there exist $\varepsilon_0 > 0$ and an open ball $B(0, r)$ such that $([0, \varepsilon_0]) \cap B(0, r)$ is a relativelly compact set.

To simplify the notation, let us consider in the problem (1): $E = Y \times Z$, $H = F \times G$, $K = K_Y \times K_z$. Let us suppose that the next conditions are satisfied

$$\begin{cases} \text{Let } (x_0,u_0) \in \mathrm{graph}(H) \\ \text{Let the set } L = \mathrm{cone}(M - \{x_0\}) \text{ be closed and asymptotically compact} \\ (L \times \{O_E\}) \cap (\mathrm{epi}(H); (x_0,u_0)) = \{O_X \times O_E\} \end{cases} \qquad (2)$$

Proposition 3.2. Let us assume conditions (2). Let us suppose that H is a K-pseudoconvex at (x_0, u_0) function. Let the contingent epiderivative $DH(x_0, u_0)$ exists. Then the set

$$\mathrm{cone}\{\cup_{x\in M} DH(x_0, u_0)(x - x_0)\} + K$$

is closed.

Proof. Let $\psi: X \to 2^E$ be defined by $\psi(x) = DH(x_0, u_0)(x) + K$. We will first prove that $\psi(L)$ is a closed set. Let (u_n) be a sequence of elements of $\psi(L)$, $(u_n) \to b \in E$. Let us prove that $b \in \psi(L)$. We will consider a sequence $(x_n) \subset L$ such that $u_n \in \psi(x_n)$ for all $n \in N$. We can suppose that there exists $\gamma > 0$ such that $\|x_n\| > \gamma$ for all $n \in N$. In other case there exists a subsequence $(x_{n_i}) \to 0$ such that $(x_{n_i}, u_{n_i}) \in (\psi)$. But graph($\psi$) is a closed set because

$$\text{graph}(\Psi) = \{(x,y) \in X \times Y: x \in X, y \in DH(x_o, u_0)(x) + K\}$$

$$= T(\text{epi}(H),(x_0,u_0))$$

So $(0, b) \in \text{graph}(\Psi)$ and $b \in \psi(0) \subset \psi(L)$. In this way, given γ such that $||x_n|| > \gamma$, we consider $B(0, \gamma)$. For every n let

$$\mu_n = \inf\{\mu > 0: \mu x_n \in B(0, \gamma), \mu x_n \notin B(0, \gamma/2)\}$$

The sequence $(\mu_n x_n) \in L \cap B(0, \gamma)$ which is a relatively compact set. From this fact and from the compacity of [0,1] we deduce that there exist subsequences of $(\mu_n x_n)$ and (μ_n), (which we will denote in the same way) such that $(\mu_n x_n\})$ converges to $a' \in X$ and $(\mu_n) \to \mu_0 \geq 0$. Furthermore $a' \in L$, because L is a closed set.

If $\mu_0 > 0$, then $(x_n) \to a = 1/\mu_0 a' \in L$. As we showed before, $\text{graph}(\Psi)$ is a closed set, then from $(x_n, u_n) \in \text{graph}(\psi)$ we deduce that $(a,b) \in \text{graph}(\Psi)$ and $b \in \Psi(a) \subset \Psi(L)$. Moreover $\mu_0 \neq 0$. In other case on the one hand there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have $\mu_n x_n \notin B(0, \gamma/2)$. On the other hand from $a' \in L$ we get that the sequence $(\mu_n x_n, \mu_n u_n) \to (a', 0) \in L \times \{0_E\}$. Furthermore

$$(\mu_n x_n, \mu_n u_n) = \mu_n(x_n,u_n) \in \mu_n \text{graph}(\Psi) = \text{graph}(\Psi)$$

So $(a',0) \in \text{graph}(\psi)$. By virtue of the hypothesis $(L \times \{0_E\}) \cap \text{graph}(\Psi) = \{0_X \times 0_E\}$ and we obtain $a' = 0$. But this is impossible because $\mu_n x_n \notin B(0, \gamma/2)$. In consequence the set $\Psi(L)$ is closed. Taking into account that the set-valued map $DH(x_0, u_0)$ is positively homogeneus, if $\lambda > 0$ it follows that

$$\Psi(L) = \cup_{x\in M} DH(x_0, u_0)(\lambda(x - x_0)) + K$$

$$= \text{cone}\{\cup_{x\in M} DH(x_0,u_0)(x-x_0)\} + K$$

□

Proposition 3.3. Let us suppose that the assumption (2) is satisfied. Let the multifunction H be generalized K-convexlike and K-pseudoconvex at (x_0, u_0) and let us assume that the contingent epiderivative $DH(x_0, u_0)$ exists. Then the set

$$C = \text{cone}\{\cup_{x\in M} DH(x_0, u_0)(x - x_0)\} + K$$

is convex.

Proof. a) We will first suppose that $u_o = (0_Y,0_Z)$. Let us prove that if $\alpha_i h_i \in C$, $i = 1,2$, with $\alpha_i \geq 0$, $h_i \in \cup_{x\in M} DH(x_0,u_0)(x - x_0)$ and $\lambda \in [0,1]$, then $\lambda\alpha_1 h_1 (1 - \lambda)\alpha_2 h_2 \in C$.

Given the previous elements, there exist $x_1, x_2 \in M$ such that

$$((x_i - x_0\}),h_i) \in T(\text{epi}(H),(x_0, u_0))$$

therefore

$$\alpha_i h_i \in T(\text{epirang}(H), u_o) \subset T(\text{cone}(H(M)) + K, u_0)$$

Since the multifunction H is generalized K-convexlike, then the sets $\{\text{cone}(H(M)) + K\}$ and $T(\text{cone}(H(M)) + K,u_0)$ are convex. So $\lambda\alpha_1 h_1 (1 - \lambda)\alpha_2 h_2 \in T(\text{cone}(H(M)) + K,u_0)$ and there exist sequences $(t_n)_{n\in N}, (w_n)_{n\in N}$, such that $t_n > 0$, $\lim_{n\to\infty} t_n = 0$, $\lim_{n\to\infty} w_n = \lambda\alpha_1 h_1 + (1-\lambda)\alpha_2 h_2$, with $t_n w_n \in \text{cone}(H(M)) + K$.

We will suppose that $t_n w_n \notin K$. In other case it is obvious that $w_n \in C$ and $\lim_{n\to\infty} w_n \in C$ because the set C is closed. In this way there exist sequences $(\alpha_n) \subset \mathbb{R}$ with $\alpha_n > 0$ and $(x_n) \subset M$ such that $\left(x_n, \frac{t_n w_n}{\alpha_n}\right) \in \text{epi}(H)$.

Since the multivalued function H is K-pseudoconvex at (x_0, u_0) we get that

$$\left(x_n - x_0, \frac{t_n w_n}{\alpha_n} - u_0\right) \in T(\mathrm{epi}(H),(x_o,u_0))$$

hence $\frac{t_n w_n}{\alpha_n} - u_0 = \frac{t_n w_n}{\alpha_n} \in DH(x_0,u_0)(x_n - x_0) + K$ and $w_n \in C$. By proposition 3.2, C is a closed set and it follows that

$$\lambda\alpha_1 h_1 + (1 - \lambda)\alpha_2 h_2 = \lim_{n\to\infty} w_n \in C$$

b) If $u_0 \neq (0_Y, 0_Z)$ we define $G = H - u_0$. It verifies that $DG(x_0, 0) = DH(x_0, u_0)$ and the result is consequence of a).

From now on we will consider the assumptions (2). We will suppose that $u_0 = (y_0, z_0)$ where $y_0 \in F(x_0)$ and $z_0 \in G(x_0) \cap (-K_Z)$. □

Theorem 3.4. Let us assume conditions (2). Let the cones K_Y, K_Z have nonempty interiors $\mathrm{int}(K_Y)$, $\mathrm{int}(K_Z)$. Assume that $(x_0, y_0) \in \mathrm{graph}(F)$ is a weak minimizer of the problem (1). Let the contingent epiderivative $D(F \times G)(x_0,(y_0, z_0))$ exists. If the set-valued map $F \times G$ is generalized $K_Y \times K_Z$-convexlike and $K_Y \times K_Z$-pseudo-convex at $(x_0,(y_0,z_0))$ then there exist $u \in K_{Y^*}$ and $v \in K_{Z^*}$, $(u,v) \neq (0,0)$ such that $v(z_0) = 0$ and

$$u(y) + v(z) \geq 0$$

for all $(y, z) = D(F \times G)(x_0,(y_0, z_0))(x-x_0)$ with $x \in M$.

Proof. a) By the proposition 3.3, the set

$$S = \mathrm{cone}\{\cup_{x\in M} D(F \times G)(x_0,(y_0, z_0))(x - x_0)\} + (K_Y \times (K_Z + \{z_0\}))$$

is convex.

Let us show that

(1) $S \cap [(-\mathrm{int}(K_Y)) \times (-\mathrm{int}(K_Z))] = \phi$

In fact, let us suppose that there exists $(y, z) \in Y \times Z$ such that

(2) $(y,z + z_0) \in S \cap [(-\mathrm{int}(K_Y)) \times (-\mathrm{int}(K_Z))]$

therefore there exist

(3) $x \in M$; $\lambda > 0$; $y^1 \in Y$, $y^2 \in K_Y$; $z^1 \in Z$, $z^2 \in K_Z$;

such that $y = \lambda y^1 + y^2$, $z = \lambda z^1 + z^2$ and it verifies

(4) $(x - x_0,(y^1, z^1)) \in T(\mathrm{epi}(F \times G),(x_0,(y_0,z_0)))$

hence there exists a sequence $(x_n,(y_n, z_n)) \in \mathrm{epi}(F \times G)$ and a sequence (μ_n) of real positive numbers such that $(x - x_0,(y_0,z_0)) = \lim_{n\to\infty} (x_n,(y_n,z_n))$ and

(5) $(x-x_0,(y^1,z^1)) = \mu_n \lim_{n\to\infty} (x_n - x_0,(y_n - y_0, z_n-z_0))$

From (2) and (3) we deduce that for a sufficiently large n

$$\lambda\mu_n(y_n - y_0) + y^2 \in -\mathrm{int}(K_Y)$$

$$\lambda\mu_n(z_n - z_0) + z^2 + z_0 \in -\mathrm{int}(K_Z)$$

therefore

$$\lambda\mu_n(y_n - y_0) \in -\text{int}(K_Y)$$

$$\lambda\mu_n(z_n - z_0) + z_0 \in -\text{int}(K_Z)$$

and hence

(6) $\quad y_n \in y_0 - \text{int}(K_Y)$

(7) $\quad z_n \in z_0 \left(1 - \frac{1}{\lambda\mu_n}\right) - \text{int}(K_Z)$

On the other hand, there exist sequences $(y_n^\bullet)_{n\in N}$, $(z_n^\bullet)_{n\in N}$, with $y_n^\bullet \in F(x_n)$, $z_n^\bullet \in G(x_n)$, such that

$$y_n \in y_n^\bullet + K_Y,\ z_n \in z_n^\bullet + K_Z$$

and from this, taking into account (6) and (7) we get

(8) $\quad y_n^\bullet \in y_n - K_Y \subset y_0 - \text{int}(K_Y) - K_Y = y_0 - \text{int}(K_Y)$

(9) $\quad z_n^\bullet \in z_n - K_Z \subset z_0\left(1 - \frac{1}{\lambda\mu_n}\right) - \text{int}(K_Z) - K_Z = z_0 \left(1 - \frac{1}{\lambda\mu_n}\right) - \text{int}(K_Z)$

Then on the one hand from (8) we have that

(10) $\quad F(x_n) \cap (y_0 - \text{int}(K_Y)) \neq \phi$

On the other hand from (2) we deduce that $y = \lambda y^1 + y^2 \neq 0$. Furthermore $y^1 \neq 0$, because in other case $y = y^2 \in K_Y \cap (-\text{int}(K_Y))$. So $\mu_n \to \infty$ and for a sufficiently large n we obtain that $\lambda\mu_n > 1$. As by hypothesis $z_0 \in -K_Z$, then $z_0\left(1 - \frac{1}{\lambda\mu_n}\right) \in -K_Z$. From (9) we deduce that $z_n^\bullet \in (-\text{int}(K_Z))$, and in consequence

(11) $\quad z_n^\bullet \in G(x_n) \cap (-\text{int}(K_Z))$

From (10) and (11) we conclude that (x_0, y_0) isn't a weak minimizer so (1) is proved.

Then S is convex and equality (1) holds. By Hahn-Banach's theorem there exist $u \in Y^*$, $v \in Z^*$ such that

$$u(y') + v(z') \leq u(y) + v(z)$$

for all $(y', z') \in (-\text{int}(K_Y)) \times (-\text{int}(K_Z))$, $(y,z) \in S$. Taking into account the continuity of u and v and that $0_Y \in \text{cl}(-\text{int}(K_Y))$ and $0_Z \in \text{cl}(-\text{int}(K_Z))$ we have that

$$u(y) + v(z) \geq 0 \text{ for all } (y,z) \in S$$

Furthermore since $(0_Y, z_0) \in S$ for all $y' \in (-\text{int}(K_Y))$ and for all $z' \in (-\text{int}(K_Z))$ we get

$$u(y') + v(z') \leq u(0_Y) + v(z_0) = v(z_0)$$

As $0_Y \in \text{cl}(-\text{int}(K_Y))$ and u is continuous it follows that

$$v(z') \leq v(z_0)$$

for all $z' \in (\text{-int}(K_Z))$. Moreover $v(z') \leq 0$, for all $z' \in \text{-int}(K_Z)$, because in other case there exists $z' \in \text{-int}(K_Z)$ such that $v(z')>0$. Then we have $v(\alpha z') = \alpha v(z') \leq v(z_0)$ for all $\alpha > 0$, which does not make sense. From $K_Z \subset cl(int(K_Z))$, we obtain $v \in K_{Z^*}$. With a similar reasoning for 0_Z we deduce that

$$u(y') \leq v(z_0) \text{ for all } y' \in \text{-int}(K_Y)$$

since by hypothesis $z_0 \in \text{-}K_Z$, we have $v(z_0) \leq 0$, and $u(y') \leq 0$. Therefore $u \in K_{Y^*}$. Furthermore from $v(z') \leq v(z_0)$ for $z' = 0_Y$ we obtain that $0 \leq v(z_0)$ and in consequence $v(z_0) = 0$. From this fact we arrive to

$u(y) + v(z) \geq 0$ for all $(y,z) \in \cup_{x\in M} D(F \times G)(x_0,(y_0, z_0))(x - x_0)$ because $(y,z + z_0) \in S$.

□

Example 3.5. Let $f,g:[-1, 1] \rightarrow \mathbb{R}$ be functions where

$$f(x) = \begin{cases} 2/3x + 1/(3\cdot 2^{3n}) & \text{if} \quad 1/2^{2+3n} \leq x \leq 1/2^{3n} \\ 3x - 1/2^{2+3n} & \text{if} \quad 1/2^{3+3n} \leq x \leq 1/2^{2+3n} \\ -2/3x + 1/(3\cdot 2^{3n}) & \text{if} \quad -1/2^{3n} \leq x \leq -1/2^{2+3n} \\ -3x + 1/2^{2+3n} & \text{if} \quad -1/2^{2+3n} \leq x \leq -1/2^{3+3n} \\ 0 & \text{if} \quad x = 0 \\ & \quad n = 0,1,2,\ldots \end{cases}$$

Let $F:[-1,1] \rightarrow 2^{\mathbb{R}^2}$ be a set-valued map defined by

$$F(x) = \{(x,y) \mid f(x) \leq y \leq g(x)\}$$

Let $G:[-1,1] \rightarrow 2^{\mathbb{R}}$ be the multifunction given by $G(x) = \{-|x|\}$. We consider the cones $K_Y = \mathbb{R}^2_+ \subset \mathbb{R}^2$, $K_Z = \mathbb{R}_+ \subset \mathbb{R}$. The contingent epiderivative of $F \times G$ at $(0,((0,0),0))$, is

$$D(F \times G)\,(0,((0,0),0))(x) = \begin{cases} (x,x,-x) & \text{if} \quad x \geq 0 \\ (x,-x,x) & \text{if} \quad x < 0 \end{cases}$$

The problem

$$\begin{cases} \min F(\mathbb{R}) \\ \text{subject to} \\ x \in [-1,1] \end{cases}$$

is a particular case of the problem (1). It is easy to see that $(0,(0,0))$ is a weak minimizer of this problem.

The set-valued map $F \times G$ is not $K_Y \times K_Z$-convex at $(0,(0,0))$, therefore the results of Aubin (1981) can not be applied. Nevertheless it is not difficult to verify that $F \times G$ is generalized $K_Y \times K_Z$-convexlike and $K_Y \times K_Z$-pseudoconvex at $(0,((0,0),0))$. Consequently, by the previous theorem we obtain the functions $u \in K^*_Y$, $v \in K^*_Z$ with $(u,v) \neq (0,0)$.

We note that for instance the functions $u(x,y) = 2x + 2y$, $v(x) = x$ belong to K^*_Y and K^*_Z respectively and comprise a pair of multipliers for this problem.

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