CHARACTERIZING COMPLETE WEAK EFFICIENCY IN MULTIPLE OBJECTIVE LINEAR PROGRAMMING

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ABSTRACT
The phenomenon consisting in all the feasible region of a multiple objective linear programming problem being weakly efficient, may be less unlikely than has been believed, particularly for problems whose feasible regions have no interior. To deal with such possibility of complete weak efficiency, we would want to have at our disposal proficient methods to check this situation. So, in this paper we give a few characterizations of that kind. The provided tests are easily put into practice and can lead us to important procedural simplifications and computational savings, specially if vector maximum or interactive approaches are going to be used in the resolution of a multiple objective linear program.

Key words: Multicriteria decision making, multiple objective linear programming, weakly efficient solution, complete weak efficiency.

MSC: 90C29, 90C99

RESUMEN
El fenómeno consistente en que toda la región factible de un problema de programación lineal multiobjetivo sea débilmente eficiente quizás el estimado más probable de lo que podría pensarse a priori, especialmente en aquellos problemas cuyas regiones factibles tienen interior vacío. Para tratar la eventualidad de la eficiencia débil completa sería deseable la disposición de herramientas adecuadas que comprueben esta situación. Por ello, en este artículo damos algunas caracterizaciones de este tipo. Los tests obtenidos son fácilmente aplicables y pueden proporcionar importantes simplificaciones procedimentales y ahorros de cómputo, especialmente cuando se utilizan los enfoques de maximización vectorial o interactivos en la resolución de un programa lineal multiobjetivo.

1. INTRODUCTION

One of the more practical models used in multiple criteria decision making is that which involves the simultaneous maximization of $k \geq 2$ objective functions over a feasible region given in an implicit way (see, for instance, Steuer (1986).

Let $X \subseteq R^n$ be a nonempty set of feasible solutions given in an implicit way and let $z: R^n \to R^k$ be the vector-valued criterion function defined for each $x \in X$. Then, the multiple objective programming problem (MOP) may be written as:

$$\max\{z(x) \,/\, x \in X\} \tag{1}$$

When all the objective functions that appear in (1) are linear and X is a polyhedron, we have a multiple objective linear programming problem (MOLP). Without loss of generality, we can assume that the formulation of a MOLP is as follows:

$$\max\{Cx \,/\, x \in X\} \tag{2}$$

where $X = \{z \in R^n \,/\, Ax = b,\ x \geq 0\}$, being $A \in R^{m\times n}$, $b \in R^{m\times 1}$ and $C \in R^{k\times n}$, fixed real matrices.

Throughout the paper we will employ the following notation: Let $x, y \in R^n$, then:

1. $x \leq y \Leftrightarrow x_j \leq y_j,\ \forall\, j \in \{1,\ldots,n\}$.
2. $x \leq y \Leftrightarrow x \leq y,\ x \neq y$.
3. $x < y \Leftrightarrow x_j < y_j,\ \forall\, j \in \{1,\ldots,n\}$.
4. $R^n_+ \Leftrightarrow \{x \in R^n \,/\, x \geq 0\}$.
5. $R^n_{++} \Leftrightarrow \{x \in R^n \,/\, x > 0\}$.
6. $e \Leftrightarrow$ Vector whose components are each equal to one.

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Let P be a MOP. Several strategies for solving problem (1) have been suggested. Many of these involve generating efficient or weakly efficient solutions which are defined in the following sense:

Definition 1.1. $\overline{x} \in X$ is said to be an efficient solution of P if, and only if, $\nexists\, x \in X$ such that $z(x) \geq z(\overline{x})$.

Definition 1.2. We say that $\overline{x} \in X$ is a weakly efficient solution of P if, and only if, $\nexists\, x \in X$ such that $z(x) \geq z(\overline{x})$.

Let WE^P denote the set of all weakly efficient solutions for problem P.

The following definition plays a key role in this paper.

Definition 1.3. P is said to be completely (weakly) efficient if, and only if, every feasible solution is also (weakly) efficient, i.e., $WE^P = X$.

It is clear that if X = ∅, or z(x) is a null vector objective function, or z(X) contains only one point, then P is, trivially, completely weak efficient. Therefore, without loss of generality, we can assume that X = ∅ and that z(x) is a not null function.

The research made in this work provides us with a few tests to check the complete weak efficiency in a MOLP and seeks a better understanding of the mathematical underlying structure of this problem.

To the best of our knowledge, the issue of complete weak efficiency has not been previously treated in the literature. The only studies found closely to the subject of which we are aware are certain communications related with complete efficiency.

The first one was presented by M. Benveniste in 1977 (see Benveniste (1977)). In that work, she examines the case in which the objective functions are linear and the feasible region is a convex set with a nonempty interior. Later, in 1991, H. Benson (Benson (1991)) carried out a new study, with special attention to the linear case, achieving a useful test (based on the resolution of a linear scalar program) that does not require X to have a nonempty interior. Recently, in 2001, J. Jorge (Jorge (2001)) has analyzed again the linear case finding new tests nearly related to, but different from, that proposed by Benson.

This paper was thought as a natural extension to the previous work of Jorge on complete efficiency and is organized as follows:

Section 2 illustrates the notion of complete weak efficiency and states a theorem of the alternative that will be used latter.

In Section 3 we give the mathematical framework that allows us to obtain some necessary and sufficient conditions that characterize the complete weak efficiency of a linear problem.

Some conclusions are given in the final section.

2. PRELIMINARIES FOR THE MOLP

At this point we will discuss about how graphical detection of weakly efficient solutions in the linear case is achieved. This will allow us to illustrate the notion of completely weak efficient problem. At last, an useful theorem of the alternative is stated.

Definition 2.1. The weak preference cone associated with matrix C is the positive polar cone of the cone generated by the rows of C, that is, the set $C^{>} = \{d \in R^n \,/\, Cd > 0\} \cup \{0\}$.

The next result is an extension of another one well-known (see Steuer (1986), Theorem 6.16).

Theorem 2.2. $\overline{x} \in WE^P$ if, and only if, $(\overline{x} + C^{>}) \bigcap X = \{\overline{x}\}$.

The following example shows us a completely weak efficient MOLP.

Example 2.3**. Let us consider the problem P given by**

$$\left.\begin{array}{l} \max Cx = \max \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} x \\ \text{s.t. } x \in X = \{x \in R^3 \,/\, 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, x_3 = 0\} \end{array}\right\}$$

whose feasible region and weak preference cone are represented in Figure 1. Note that the graphical representation of the weak preference cone, analytically

described by $\left\{ x \in R^n / \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} x > 0 \right\} \cup \{0\}$, has been truncated by the plane $x_3 = 1$ for the sake of clarity. Now, it can be verified that $WE^P = X$.

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Now, a theorem of the alternative, which will be proved useful in the next section.

Let us consider an arbitrary constant $\alpha \in R^k$.

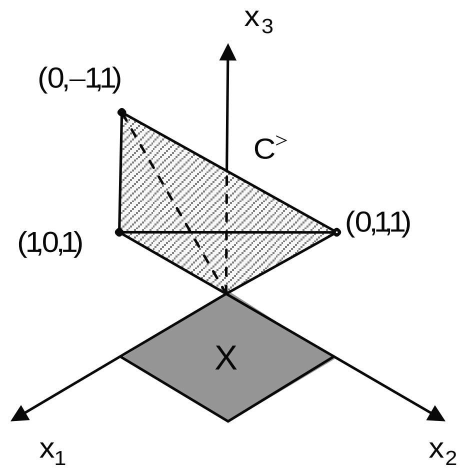

Figure 1.

Theorem 2.4. (Jorge (2002), Corollary 2.4) If $X = \{x \in R^n_+ / Ax = b\} \neq \emptyset$ the the system $Ax = b$, $x \geq 0$, $Cx > \alpha$, has no solution if, and only if, $u'A - \lambda'C \geq 0'$, $\lambda'\alpha \geq u'b$, $\lambda \geq 0$, has a solution.

3. TESTS FOR DETECTING COMPLETE WEAK EFFICIENCY IN A MOLP

Our goal in this section shall be concerned with establishing some necessary and sufficient conditions, easily put into practice, which allow us to detect the possibility of complete weak efficiency in a MOLP before its resolution.

First we start with a key result that can be applied to the general case.

Theorem 3.1. If P is a MOP, then the following are equivalent:

(i) P is completely weakly efficient.

(ii) The system

$$z(x) - z(y) > 0, \ x, y \in X, \tag{3}$$

has no solution. (See an analogous result for the efficient case in Benveniste (1977), Theorem 1).

(iii) The scalar program

$$\max\{s / z(x) - z(y) - es \geq 0, \ x, y \in X, \ s \geq 0\} \tag{4}$$

has optimal objective value equal to 0.

Proof. Straightforward ■

In spite of its simplicity, the above theorem is decisive in the development that we have done below.

Note that the scalar program specified in (4) constitutes a very practical tool in the analysis of complete weak efficiency since, generally, it is easier to solve (4) than to check the infeasibility of (3).

However, when we deal with a MOLP, we also have at our disposal the theorem of the alternative stated in the previous section. So, if we suppose that P is a MOLP.

Theorem 3.2. Let $u, v \in R^n$ and $\lambda \in R^k$. P is completely weakly efficient if, and only if, the system

$$\left.\begin{array}{c} u^t A - \lambda^t C \geq 0 \\ v^t A + \lambda^t C \geq 0 \\ u'b + v^t b \leq 0 \\ \lambda \geq 0 \end{array}\right\} \tag{5}$$

has a solution.

Proof. By hypothesis P is completely weak efficient. Applying Theorem 3.1, system (5) has a solution if, and only if, $\left.\begin{array}{r} Cx - Cy > 0 \\ Ax = b \\ Ay = b \\ x \geq 0, y \geq 0 \end{array}\right\}$ has no solution. Since this system can be rewritten as: $\left.\begin{array}{c} (C,-C)\begin{pmatrix} x \\ y \end{pmatrix} > 0 \\ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ b \end{pmatrix} \\ (x^t, y^t) \geq 0 \end{array}\right\}$ and applying Theorem 2.4 it is equivalent to the following system $\left.\begin{array}{c} (u^t, v^t)\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} - \lambda^t(C,-C) \geq 0^t \\ (u^t, v^t)\begin{pmatrix} b \\ b \end{pmatrix} \leq 0, \lambda \geq 0 \end{array}\right\}$ having a solution. Now the proof is done. ■

Theorem 3.2 has two corollaries. First, due to the homogeneity of system (5) it is possible to write:

Corollary 3.3*.* P is completely weakly efficient if, and only if, the system

$$\left.\begin{array}{r} u^t A - \lambda^t C \geq 0 \\ v^t A + \lambda^t C \geq 0 \\ u^t b + v^t b \leq 0 \\ e^t \lambda \geq 1, \lambda \geq 0 \end{array}\right\} \tag{6}$$

has a solution.

Note that the inequality $u^t b + v^t b \leq 0$ is actually an implicit equality. Indeed, multiplying both sides of $u^t A + v^t A \geq 0$ by any $x \in X$ yields $u^t b + v^t b \geq 0$. Thus we obtain:

Corollary 3.4. P is completely weakly efficient if, and only if, the system

$$\left.\begin{array}{r} u^t A - \lambda^t C \geq 0 \\ v^t A + \lambda^t C \geq 0 \\ u^t b + v^t b = 0 \\ e^t \lambda \geq 1, \lambda \geq 0 \end{array}\right\} \tag{7}$$

has a solution.

Here is another important result:

Theorem 3.5. P is completely weakly efficient if, and only if, the problem

$$\left.\begin{array}{r} \min u^t b + v^t b \\ \text{s.t.} \, u^t A - \lambda^t C \geq 0 \\ v^t A + \lambda^t C \geq 0 \\ e^t \lambda \geq 1, \lambda \geq 0 \end{array}\right\} \tag{8}$$

has optimal objective value equal to 0.

The utility of the above test is due to the fact that it allows us to analyze the complete weak efficiency in a MOLP through the resolution of an easy scalar linear program.

Actually, Theorem 3.1 gives us another similar possibility by means of problem (4). Since, the linear formulation of (4) is written as:

$$\left.\begin{array}{l}\max s\\ \text{s.t.}\\ (-C \quad C \quad e)\begin{pmatrix}x\\y\\s\end{pmatrix}\le 0\\ \begin{pmatrix}A & 0 & 0\\ 0 & A & 0\end{pmatrix}\begin{pmatrix}x\\y\\s\end{pmatrix}=\begin{pmatrix}b\\b\end{pmatrix}\\ \begin{pmatrix}x\\y\\s\end{pmatrix}\ge 0\end{array}\right\} \tag{9}$$

we have:

Theorem 3.6. P is completely weakly efficient if, and only if, the scalar linear program (9) has optimal objective value 0.

A question might be asked about what relationship is established, if any, between programs (8) and (9). The answer comes immediately.

Proposition 3.7. The problems (8) and (9) are a primal-dual pair of linear scalar programs.

Proof.

Problem (9) can be written as $\left.\begin{array}{l}\max(0^t, \quad 0^t, \quad 1)\begin{pmatrix}x\\y\\s\end{pmatrix}\\ \text{s.t.}\\ (-C \quad C \quad e)\begin{pmatrix}x\\y\\s\end{pmatrix}\le 0\\ \begin{pmatrix}A & 0 & 0\\ 0 & A & 0\end{pmatrix}\begin{pmatrix}x\\y\\s\end{pmatrix}=\begin{pmatrix}b\\b\end{pmatrix}\\ \begin{pmatrix}x\\y\\s\end{pmatrix}\ge 0\end{array}\right\}$. Now, by scalar linear duality, we obtain

$\left.\begin{array}{c}\min(u^t, \quad v^t, \quad \lambda^t)\begin{pmatrix}b\\b\\0\end{pmatrix}\\ \text{s.t.}\\ (u^t, \quad v^t, \quad \lambda^t)\begin{pmatrix}A & 0 & 0\\ 0 & A & 0\\ -C & C & e\end{pmatrix}\ge (0^t, \quad 0^t, \quad 1)\\ \lambda\ge 0\end{array}\right\}$ and the proof is concluded.

■

4. CONCLUSIONS

In this work we provide a few tests for detecting complete weak efficiency in a MOLP before its resolution. Some of them are stated in terms of feasibility of certain linear systems of inequalities and others are formulated through the resolution of some scalar linear programs. Anyway, all of them enjoy the remarkable advantages of their simplicity and ease of application.

The theory developed in section 3 is based on the combination of system (3) under linear hypothesis together with the theorem of the alternative stated in Section 2.

We would like to mention that the possibility of complete weak efficiency is greater than the existing for complete efficiency. Not in vain, the weakly efficient set of solutions contains all the efficient set. In addition, we think, together with Benson (see Benson (1991), p. 482), that complete efficiency may happen with certain frequency for problems whose feasible regions have no interior, although up to the present time this remains to be seen.

Finally, it is worthy of remark that the complete weak efficiency tests also yield practical tools for checking weak efficiency of an arbitrary face in a MOLP. This easily comes from the fact that every face of a polyhedron can be seen also as a polyhedron.

APPENDIX

Let us consider an arbitrary $\alpha \in R^k$. In order to prove Theorem 2.4 we need a more general result.

Theorem 1. The system

$$Ax = b, \quad x \geq 0, \quad Cx > \alpha \tag{1}$$

has a solution if, and only if, the system

$$u^tA - \lambda^tC \geq O^t,\ \lambda^t\alpha - \delta = u^tb,\ (\lambda^t, \delta) \geq 0 \tag{2}$$

has no solution.

Proof. Clearly, system (1) having a solution is equivalent to system – Ax + by = 0, y > 0, x ≥ 0, Cx - αy > 0, having a solution. Since this system can be rewritten as:

$$(-A, b)\begin{pmatrix} x \\ y \end{pmatrix} = 0,\ \begin{pmatrix} C & -\alpha \\ 0^t & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} > 0,\ (I, 0)\begin{pmatrix} x \\ y \end{pmatrix} \geq 0, \tag{3}$$

applying Motzkin´s theorem of the alternative (see the book “Mangasarian, 1969, Nonlinear Programming, McGraw-Hill” p. 28), system (3) has a solution if, and only if, system $(\lambda^t, \delta)\begin{pmatrix} C & -\alpha \\ 0^t & 1 \end{pmatrix} + s^t(I, 0) + u^t(-A, b) = 0$, $s \geq 0$, $(\lambda^t, \delta) \geq 0$, has no solution, which is equivalent to system (2) having no solution. ■

The above result can be improved under the assumption that $X = \{x \in R^n_+ / Ax = b\}$ is not empty. Thus we obtain:

Theorem 2.4. If $X \neq \varnothing$ then system (1) has no solution if, and only if, the system

$$u^tA - \lambda^tC \geq 0^t,\ \lambda^t\alpha \geq u^tb,\ \lambda \geq 0 \tag{4}$$

has a solution.

Proof. By Theorem 1, system (1) has no solution if, and only if, $u^tA - \lambda^tC \geq 0^t$, $\lambda^t \alpha - \delta = u^tb$, $(\lambda^t, \delta) \geq 0$, has a solution. Now, suppose by contradiction that $\lambda = 0$, $\delta > 0$. Thus $u^tA \geq 0^t$ and $u^tb < 0$. On the other hand, taking an arbitrary $x \in X$ we obtain $u^tAx = u^tb \geq 0$, which is a contradiction. Therefore, system (4) has a solution. ■

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