INCLUDING DEPENDENCE OF THE COSTS ON TIME IN THE TRAVELING SALESMAN PROBLEM WITH TIME WINDOWS(*)

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ABSTRACT

We present in this paper a generalization of the Traveling Salesman Problem with Time Windows (TSPTW) in which arc costs depend on the period of time in which the cycle starts to traverse the arcs. This new problem fits more accurately some real routing problems in big cities than the TSPTW, because the time, and then the cost, of traversing certain avenues depends on the instant we start to do it. For instance, at peak hours this time is bigger than in other moment of the day. We prove that this new problem can be transformed in pseudo-polynomial time into an Asymmetric Generalized Traveling Salesman Problem and then, into an Asymmetric Traveling Salesman Problem. Thus, we can solve the problem with known techniques. We also present computational results on this transformation in a set of 140 instances with up to 30 vertices with an exact algorithm. We consider our results satisfactory according to the complexity of the new problem.

Key words: Traveling salesman problem, time windows, dependence on time.

RESUMEN

En este artículo presentamos una generalización del Problema del Agente Viajero con Ventanas de Tiempo (PAVVT), en la que el coste de los arcos depende del periodo de tiempo en el cual el ciclo comience a atravesar los arcos. Este nuevo problema se ajusta mejor a las situaciones reales de vehículos en grandes ciudades que el PAVVT, porque el tiempo y por tanto el coste, de circular por ciertas avenidas, depende del instante en que se realice. Por ejemplo, en las horas punta este tiempo es mucho mayor que en cualquier otro instante del día. Nosotros probamos que este nuevo problema puede ser transformado en un tiempo pseudo-polinomial a un Problema del Agente Viajero Generalizado y Asimétrico y después, en un Problema del Agente Viajero Asimétrico. En consecuencia, podemos resolver este problema con técnicas conocidas. Presentamos también resultados computacionales sobre esta transformación en un conjunto de 140 instancias con hasta 30 vértices con un algoritmo exacto. Consideramos los resultados satisfactorios acorde a la complejidad del nuevo problema.

MSC: 90B06.

1. INTRODUCTION

Given a directed graph G = (V, A) with nonnegative costs associated with its arcs, the well-known Asymmetric Traveling Salesman Problem (ATSP) consists of finding a minimum cost Hamiltonian circuit in G, that is, a minimum cost circuit passing through each vertex exactly once.

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An interesting generalization of the ATSP is the Traveling Salesman Problem with Time Windows (TSPTW). In this problem, each vertex i has associated a time window $[a_i, b_i]$, one of the vertices, say i_0, is considered as a depot and traversing arc $(i, j) \in A$ implies a travel time $t_{i,j} > 0$. The aim of the TSPTW is then to find a minimum cost circuit in G starting in i_0 at time a_{i_0} and passing through each vertex exactly once, such that the circuit must live each vertex in its associated time window, and ending at i_0 not latter than b_{i_0}. Note that it is allowed to arrive at vertex i before a_i (waiting time), but in this case, the circuit will leave i at time a_i. For simplicity, if in a vertex *i* it is necessary a service time, this time is included in the travel times $t_{j,i}$ $j \neq i$.

The TSPTW has important applications, especially in sequencing and distribution problems. Because of that, many studies have been written about it in the last decade. See for example Dumas, Desrosiers, Genlinas and Solomon (1995) and Gendreau, Hertz, Laporte and Stan (1998) or the more recent papers of Wolfer (2000) and Ascheuer, Fischetti and Grötchel (2000).

In some real vehicle routing problems, as for example the distribution of goods inside a big city, in addition to the time windows required by the customers, the time (which normally corresponds with the cost) of traversing some streets, like main avenues, depends on the moment in which we start to traverse them. For example at peak hours such as going to or leaving the school or work. If we take into account this issue, the costs of the arcs in some routing problem must depend on time. In this case, probably nearly all the easy problems used as subroutines to solve routing problems (shortest path, shortest spanning tree, matching, minimum cost flow…) will not be useful.

Despite the traffic jams we have to stand at certain times and in certain areas of a big city, routing problems with time dependent costs have hardly been considered in the literature. In fact, as far as we know, the more recent paper on this topic is by Haouari and Dejax (1997), who solved the shortest path problem with time windows and time dependent costs in pseudo-polynomial time.

In this paper we present a generalization of the TSPTW in which the cost and the travel time of each arc are time dependent and some waiting times are allowed. For instance, if the instant a_{i_0} (the lower bound of the depot time window) corresponds to a peak hour, we can minimize the cost of the tour by waiting for a short period of time (working inside the warehouse) instead of starting the route at a_{i_0}.

In Section 2 we define this new problem which we have called the Traveling Salesman Problem with Time Dependent Costs (TSPTDC). In order to solve this problem with known techniques, we must mention another combinatorial optimization problem studied in the OR literature: the Asymmetric Generalized Traveling Salesman Problem (AGTSP). Given a directed graph G = (V, A) with nonnegative costs associated with its arcs, such that V is partitioned into k nonempty subsets $\{S_i\}_{i=1}^{k}$, the AGTSP consists of finding a minimum cost circuit passing through exactly one vertex of each subset S_i $\forall_i = 1,\ldots,k$.

To solve the AGTSP, several polynomial time transformations of this problem into an ATSP have been described. It seems that the most efficient transformation is the one given by Noon and Bean (1993). As we will use this transformation, we basically describe it: construct a new directed graph with the same vertex set but order the vertices of each subset S_i consecutively in an arbitrary fashion $\{v_1,\ldots,v_r\}$; then, for $j = 1,\ldots,r - 1$ define the cost $c_{j,j+1}$ of an arc (v_j, v_{j+1}) as - M, where M is an arbitrary large positive constant; also define $c_{r,1}$ as – M. Then, for each $v_j \in S_t$ and for each $v_l \in S_k$ with $t \neq k$, $c_{j,l}$ **is equal to the cost** in G of the arc from vertex $v_{j+1 (\mathrm{mod}\ r)}$, to vertex v_l; any other arc has cost infinite. Solve the AGTSP in *G* is equivalent to solve the ATSP in the new graph.

In Section 3 we present a pseudo-polynomial time transformation of the TSPTDC into an AGTSP and a little example of this transformation. Finally, in Section 4 we show some computational results with the exact procedure by Fischetti and Toth (1992) on a set of 140 instances obtained from the instances used by Gendreau, Hertz, Laporte and Stan (1998) for the TSPTW. Our results show that in TSPTDC instances with up to 20-25 customers at day it is possible to obtain the optimal solution in a reasonable time.

2. DEFINITION OF THE TSPTDC

The Traveling Salesman Problem with Time Dependent Cost (TSPTDC) can be defined as follows:

Let G = (V, A) be a directed graph, being $V = \{v_i\}_{i=0}^{n}$ its vertex set, where v_0 is the depot vertex. Each vertex $v_i \in V$ has associated a time window $[a_i, b_i]$ such that $a_i, b_i \in Z^+ \cup \{0\}$ and $[a_i,b_i] \subseteq [a_0,b_0]$ $\forall i \in \{1,...,n\}$ .. Consider for each time window $[a_i,b_i]$, $p_i = b_i - a_i + 1$ periods of time $\{ [a_i + k - 1, a_i + k[\}_{k=1}^{p_i}$. For simplicity we will denote $T_i^k = [a_i + k - 1, a_i + k[$ and in order to discretize time, we identify period T_i^k with the instant of time $a_i + k - 1$.

On the other hand, the time and the cost of traversing an arc $(v_i, v_j) \in A$ depend on the period of time $T_i^k (k \in \{1,...,p_i\})$ in which we start to traverse it. Denote $T_{i,j}^k \in Z^+$ and $c_{i,j}^k \geq 0$ the time and the cost respectively of traversing arc (v_i, v_j) starting at period T_i^k.

The TSPTDC consists of finding a Hamiltonian circuit in G, starting and ending at v_0 in its time window $[a_0, b_0]$ such that the circuit leaves each vertex $v_i \in V$ with $i > 0$ in its associated time window, the cost of the circuit be minimum and in order to minimize this cost, it is allowed a zero cost waiting time in each vertex v_i if it is reached before time a_i, but in this case the circuit will leave the vertex at instant a_i.

As in the TSPTW, for simplicity we assume that (if necessary) the time of traversing arc (v_i, v_j) with $j > 0$ includes the service time at v_j. In the particular case of the TSPTDC in which $T_{i,j}^k = T_{i,j}^s = c_{i,j}^k = c_{i,j}^s$ $\forall k$, $s \in \{1,\ldots,p_i\}$ and $\forall (v_i, v_j) \in A$, we have a TSPTW with the objective function equal to the total time of the circuit. Then the TSPTDC is an NP-hard problem.

3. TRANSFORMATION OF THE TSPTDC INTO AN ATSP

In this section we show a way to solve the TSPTDC by transforming it in pseudo-polynomial time into an ATSP. We have seen in Section 1 that the AGTSP can be transformed into an ATSP in polynomial time (Noon and Bean), then, our aim is to prove that the TSPTDC can be transformed in pseudo-polynomial time into an AGTSP.

To see this, we need to construct an auxiliary directed graph G´= (V´, A´) from the graph *G* given in the definition of the TSPTDC, in the following way:

1. For each v_i and for each period of time T_i^k $\forall k \in \{1,\ldots,p_i\}$ and $\forall i \in \{0,1,\ldots,n\}$ create a vertex $v_i^k \in V'$.

2. For each pair of vertices $v_i^k, v_j^l \in V'$ with $i \neq j$ and such that $a_j + l - 1 = \max\{a_j, a_i + k - 1 + T_{i,j}^k\}$, add to G' an arc (v_i^k, v_j^l) with cost equal to $c_{i,j}^k$. Note that $a_i + k - 1 + T_{i,j}^k < a_j$ implies a waiting time at vertex $v_j \in G$ if the circuit traverses arc (v_i, v_j) at period T_i^k.

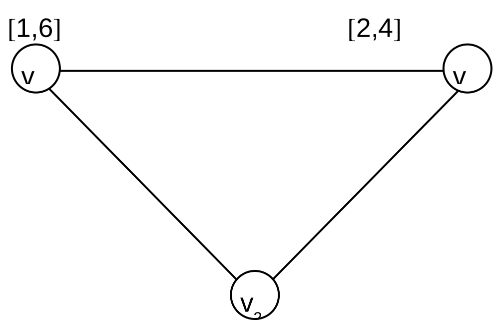
3. Partition $\{1, 2, ..., p_0\}$ into four subsets I_1, I_2, I_3, I_4, such that:

   - If $k \in I_1, v_0^k$ only has leaving arcs in G'. In these case, change also name v_0^k by v_{0s}^k (starting vertex).
   - If $k \in I_2, v_0^k$ only has entering arcs in G'. In these cases, change also name v_0^k by v_{0e}^k (ending vertex).
   - If $k \in I_3, v_0^k$ has both entering and leaving arcs in G'. In this case, split vertex v_0^k into two new vertices v_{0s}^k and v_{0e}^k such that v_{0s}^k is only incident with the leaving arcs from vertex v_0^k in G' and v_{0e}^k is only incident with the entering arcs to vertex v_0^k in G'.
   - If $k \in I_4, v_0^k$ has not neither leaving arcs nor entering arcs. Delete from G' v_0^k for all $k \in I_4$.

4. Add to G' two new vertices v_s and v_e and the following arcs, all of them with cost zero:

   - For each $k \in I_1 \cup I_3$, an arc (v_s, v_{0s}^k).
   - For each $k \in I_2 \cup I_3$, an arc (v_{0e}^k, v_e).
   - Arc (v_e, v_s).

In Figure 1 we show an example of a TSPTDC with n = 3. Time windows are given in the figure and the time dependent costs are given in Table 1. From Table 1 we can easily obtain the travel time depending on the instant we start to traverse the arcs, because each T_i^k has in brackets its corresponding instant of time. For example, the element (T_0^1, T_1^2) implies that if we traverse arc (v_0, v_1) starting at period of time T_0^1, which corresponds with instant 1, $c_{0,1}^1 = 70$ and $T_{0,1}^1 = 2$ because we arrive to v_1 to T_1^2, which corresponds with instant 3. A dash inside a cell (T_i^k, T_j^l) means that if we traverse arc (v_i, v_j) starting at period T_i^k we will not arrive to v_j at period T_j^l, neither before in the case T_j^1.



	$T_0^3(3)$	$T_0^4(4)$	$T_0^5(5)$	$T_2^2(3)$
$T_1^1(2)$	43	-	-	40
$T_1^2(3)$	-	40	-	-
$T_1^3(4)$	-	-	31	-

$T_2^2(3)$
-
39
-
-
-

	$T_0^3(3)$	$T_0^4(4)$	$T_1^2(3)$	$T_1^3(4)$
$T_2^1(2)$	35	-	37	-
$T_2^2(3)$	-	36	-	38

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Table 1. Time dependent costs from g

In Figure 2 we show the transformed graph G´ from G, where in order to give a clarifying drawing, vertex k inside ellipse v_i corresponds with vertex v_i^k defined above.

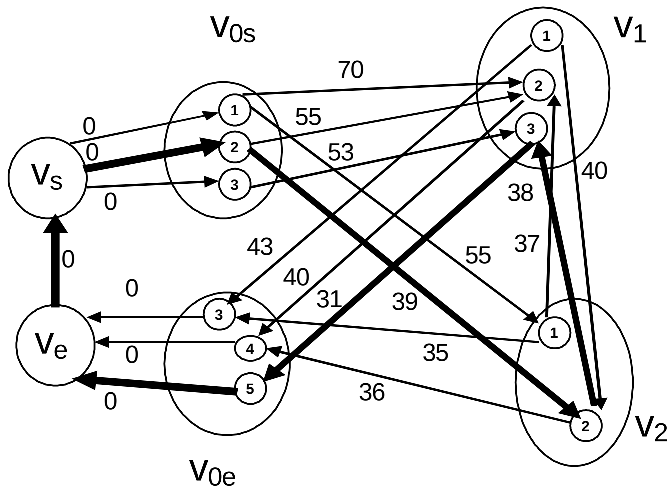

Figure 2. Graph G´.

Theorem. The TSPTDC can be transformed in pseudo-polynomial time into an AGTSP.

Proof:

Let G = (V, A) be a directed graph where the TSPTDC is defined. We will consider all its elements as they are given in our definition of the TSPTDC $(v_0, [a_i, b_i],\ T_i^k$, etc.).

Let G´= (V´, A´) be the transformed graph constructed from G, as it has been explained above. Consider in G´ an AGTSP corresponding to the partition of V´ into the following subsets: $S_i = \{v_i^k\}_{k=1}^{p_i}$ for all $i \in \{1,\ldots,n\}$, $S_0 = \{v_{0s}^k\}_{k \in I_1 \cup I_3}$, $S_{n+1} = \{v_{0e}^k\}_{k \in I_2 \cup I_3}$, $S_{n+2} = \{v_s\}$ and $S_{n+3} = \{v_e\}$.

By construction of G´, there is a one-to-one correspondence between the set of circuits solution to the AGTSP in G´ and the set of feasible solutions to the TSPTDC in *G.* To see this it is enough to identify the circuit solution to the AGTSP in G´. T´ = $\{v_s, v_{0s}^{k_0}, v_{i_1}^{k_1}, v_{i_2}^{k_2}, ..., v_{i_n}^{k_n}, v_{0e}^{k_{n+1}}, v_e, v_s\}$ with the TSPTDC feasible solution H in G consisting of the Hamiltonian circuit in G $\{v_0, v_{i_1}, v_{i_2}, ..., v_{i_n}, v_0\}$ starting in v_0 at time $k_0 \in$ [a_0, b_0], leaving each vertex v_{i_r} at time $a_{i_r} + k_r - 1 \in [a_{i_r}, b_{i_r}]$ for all $r \in \{1,..., n\}$ and ending at v_0 at time $a_0 + k_{n+1} - 1 \in [a_0, b_0]$ (note that two TSPTDC feasible solutions in *G* with the same Hamiltonian circuit but with at least one different leaving period of time T_i^k are considered as distinct solutions). Both T´ and H have the same cost so the optimal solution of the AGTSP in G´ leads to the optimal solution of the TSPTDC in G.

As $|V'|$ depends on the width of the time windows besides $|V|$ ($|V'|$ is O((n + 1)p*) where p* = $\max_{0 \le i \le n}\{b_i - a_i + 1\}$), we conclude that this transformation is pseudo-polynomial.

According to the transformation of G´ given by Noon and Bean (1993), we obtain the optimal solution to the ATSP corresponding to our example, from which we obtain the optimal solution to the AGTSP in G´, which is given in Figure 2 (the bold arcs), and then the optimal solution to the TSPTDC in graph G $\{v_0, v_2, v_1, v_0\}$ with the time sequence {2, 3, 4, 5} and with cost 39 + 38 + 31 = 108. Note that there are not waiting times in this tour solution.

Proposition. The size of graph G´ can be reduced in the following way:

i. Condense all vertices $v_{0_s}^k$ together with v_s into a single vertex v_s.

ii. $\forall v$ vertex of G´ not condensed before, calculate $\min_k\{cost(v_{0_s}^k, v)\}$ and give this value to $cost(v_s, v)$.

iii. Condense all vertices v_{0e}^k together with v_e into a single vertex v_e.

iv. For each arc (v, v_{0e}^k) with finite cost in G´, being v vertex of G´ not condensed in point iii, do cost (v, v_e) = cost(v, v_{0e}^k).

Proof:

It is easy to see that solving the AGTSP in the condensed graph is equivalent to solve the AGTSP in G´, because the only difference in the condensed graph is the substitution of an initial section $(v_s, v_{0_s}^k, v)$ of an AGTSP solution in G´ for a single arc (v_s, v) which in the optimal solution will necessary have cost $\min_k\{cost(v_{0_s}^k, v)\}$, and the substitution of a final section (v, v_{0e}^k, v_e) of an AGTSP solution in G´ for the arc (v, v_e) with the same cost.

In Figure 3 we show the condensed graph of G´ corresponding to our example, passing from 13 vertices in G´ to 7 vertices in the condensed graph.

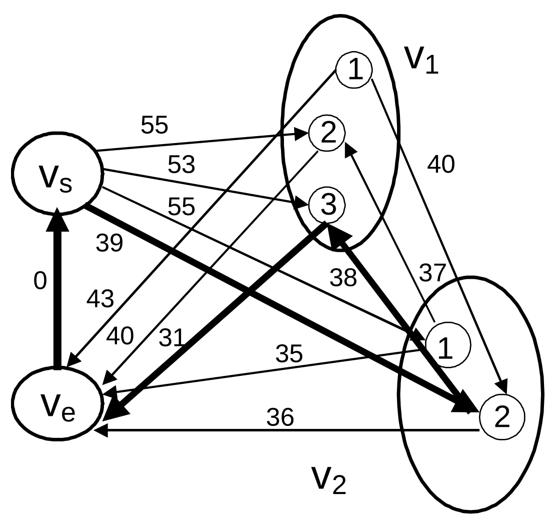

Figure 3. Graph G´ condensed. Bold arcs represent the optimal solution to the AGTSP, with cost 108.

4. COMPUTATIONAL EXPERIMENTS

In order to test the efficiency of this transformation, we have make some computational experiments on a set of 140 instances obtained from the instances used by Gendreau, Hertz, Laporte and Stan (1998) for the TSPTW. The main difference is that the number of vertices of each one of our instances is always smaller than the number of vertices of the one from which it has been obtained. This is a consequence of the definition of the TSPTDC, because it implies that the travel time of each arc (v_i^k, v_j^l) is greater or equal than a period of time, which we have considered as 10 minutes in our instances. The original instances have many vertices with the same or similar tight time windows, so many of these instances have not solution to the TSPTDC because it is impossible to visit all vertices in their time windows consuming at least a period of time each time that an arc in traversed. Under this restriction, we have decided to eliminate some vertices of each original instance and to weight the time windows according to a working day from 8:00h. untill 20:00h. in a department store. We show in Table 2 a small example of this weighting of the time windows, with a depot and two customers, corresponding to an instance of Gendreau, Hertz, Laporte and Stan.

Table 2. Original time windows (left) and their weighted time windows (right).

a_i	b_i
0	408
181	205
306	324

a_i	b_i
0	720
320	360
540	570

In all these TSPTDC instances we have considered the cost of traversing arc (v_i, v_j) in period T_i^k as the integer part of $p(T_i^k)\left\|(v_i, v_j)\right\|$, being $0.5 \le p(T_i^k) \le 1$ weights that depend on the strip of time in which T_i^k is included (see Table 3). We have established the strips of time showed in Table 3, according to the traffic density in the city of Valencia.

Table 3. Assignment of weighs $p(T_i^k)$ according to the strips of time.

Strip of time	$p(T_i^k)$
[8:00, 9:40[	$p(T_i^k) = 1$
[9:40, 11:40[	$p(T_i^k) = 0{,}5$
[11:40, 12:40[	$p(T_i^k) = 0{,}75$
[12:40, 13:30[	$p(T_i^k) = 0{,}65$

Strip of time	$p(T_i^k)$
[13:30, 15:20[	$p(T_i^k) = 1$
[15:20, 16:20[	$p(T_i^k) = 0{,}5$
[16:20, 18:40[	$p(T_i^k) = 0{,}75$
[18:40, 20:00]	$p(T_i^k) = 1$

The 140 TSPTDC instances were transformed into ATSP instances and then, solved with the exact algorithm by Fischetti and Toth (1992), which can be considered one of the best coded exact algorithms for the ATSP. In Table 4 we summarize the results obtained in these computational experiments.

It is easy to see in Table 4 the exponential complexity of the exact algorithm, according to the number of vertices in the condensed graphs, which increases with the number of vertices in the TSPTDC and the size of the time windows. We can conclude that the TSPTDC can be solved optimally in instances with size that we consider reasonable in a working day for a traveling salesman or a delivery-man (up to 20-25 customers), with a running time limit of 12 hours for the code (the code was run in a PC Pentium IV). We expect that if we increase this time limit, TSPTDC instances with more costumers could be solved optimally, especially if the time windows are not very wide. In fact, in our computational experience, all instances with 30 vertices and original amplitude of the time windows not greater than 20 were solved optimally in less than 12 hours.

Table 4. Computational results.

NV: Number of vertices;

TW: Maximum amplitude of the time windows in the original instances;

NI: Number of instances;

ANV: Average number of vertices in the condensed graph;

NISO: Number of instances solved optimally;

WT: Worst time computed in seconds to obtain the optimal solution in the 10 instances (>12 h. means that at least an instance was not solved optimally in 12 hours).

NV	TW	NI	ANV	NISO	WT
10	20	10	36	10	0.00
	40	10	51	10	0.21
	60	10	62	10	1.00
15	20	10	54	10	0.43
	40	10	78	10	10.05
	60	10	97	10	463.79
20	20	10	70	10	11.21
	40	10	101	10	216.59
	60	10	124	10	242268.81
25	20	10	83	10	477.74
	40	10	116	8	> 43200 (12h)
	60	10	128	4	> 43200 (12h)

5. CONCLUSIONS

We have presented here a new problem that generalizes the Traveling Salesman Problem with Time Windows, in such a way that the arc costs depend on the interval of time in which we start to traverse them. We think that this problem fits more accurately than the TSPTW to real problems involving visits, collection or delivery inside a big city, where it is evident that many arc traversing times depend on the moment we traverse the arcs.

We can solve this new problem from a theoretical point of view by transforming it in pseudo-polynomial time into an ATSP. In order to check the efficiency of this transformation, we have applied it to a set of 140 instances obtained from a known set of instances for the TSPTW, that then were solved with an existing exact algorithm for the ATSP. Our conclusion is that we can expect to obtain the optimal solution in a reasonable time for real problems with up to 25 customers in a working day. Problems with more costumers or with wide time windows could be solved optimally with more time consuming or heuristically, but in this last case, we have to note that the traditional heuristics for the ATSP could not obtain a feasible solution; due to the fact that in the transformed graph many arcs have infinite cost (the graph is not complete).

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