ANALYSIS OF A TWO PHASES BATCH ARRIVAL QUEUEING MODEL WITH BERNOULLI VACATION SCHEDULE*

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ABSTRACT
We consider a single server bulk arrival queueing system with two phases of heterogeneous service under Bernoulli schedule vacation, where the customers arrive in batches of the random variable 'X'. Using the imbedded Markov chain technique, we first derive the queue size distribution at a stationary point of time. Next, we obtain a recursive solution of the stationary queue size distribution of this model. Finally, we obtain the Laplace Stieltjes Transform of the waiting time distribution and some related performance measures. The method proposed here is not only easily amenable to computation but can be applied to solve more complicated problems of similar nature.

Key words: $M^X/(G_1,G_2)/1$ queue, Queue size, Heterogeneous service, Bernoulli schedule vacation, Imbedded Markov Chain.

RESUMEN
Consideramos un solo servidor con un sistema de colas con grandes arribos y servicio heterogéneo de arribo bajo una política de vacaciones Bernoulli. Los clientes arriban en lotes de una variable aleatoria ´X´. Usando la técnica de la Cadena de Markov empotrada, primero derivamos la distribución del tamaño de la cola en un momento puntual estacionario. Después, obtenemos una solución recursiva de la distribución de tamaño de la cola estacionaria de este modelo. Finalmente obtenemos la Transformada de Laplace Stieltjes del tiempo de la distribución del tiempo de espera y algunas medidas de comportamiento. El método propuesto aquí no es solo responsable de una fácil computación sino que puede ser utilizada para resolver problemas más complicados de una naturaleza similar.

MSC: 60K25, 90B22.

1. INTRODUCTION

The queueing model with vacation under Bernoulli schedule has received attention from many authors due to its applications in many real life situations. Considerable efforts have been devoted to study these models by Keilson and Servi (1986, 1987 1989), Servi (1986), Ramaswamy and Servi (1988), Doshi (1986) and Takagi (1991) among others. Further, Ghafir and Silio (1993) recognized its applications in a Multiple Access Ring Network.

Recently, Madan (2000, 2001) studied two similar types of vacation models for M/G/1 queueing system. In both the models he introduced the concept of two stages heterogeneous models with two phases of heterogeneous service and Bernoulli schedule along with a single vacation policy. However, the two phase queueing system with generalized service times have been classified by Doshi (1991). Although some aspects of these types of models studied by these authors, it seems that batch arrival queues of these types will give us much more information on the number of batches instead of total number of individual units in deciding whether the server is activated or not. Thus in this paper we propose to study such a batch arrival queue, where the concept of Bernoulli schedule along with a vacation time is introduced for a two phase heterogeneous service queueing system .

At present however, most of the studies are devoted to batch arrival vacation models under different vacation policies because of its interdisciplinary character. Numerous researchers including Baba (1986), Choudhury (2000, 2002(a, b)), Choudhury and Borthakur (2000), Lee and Srinivasan (1989), Lee **et al**. (1994, 1995),

*This work is supported by the Department of Atomic Energy, Govt. of India, through NBHM project: 48/2(2001/R&DII/ 1211
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Rosenberg and Yechiali (1993) and Teghem (1990) and many others have studied batch arrival vacation queues under different vacation policies. However, recent progress of $M^x/G/1$ type of queues with vacations have been served by Dishalalow (1997) and Medhi (1997).

In this paper we first obtain the condition for the existence of steady state solution of this model. Next we obtain the queue size distribution at a stationary point of time. The method proposed in this paper not only conduct to the use of a numerical algorithm of easy computation but can be applied to solve much more complicated problems of similar nature. Finally we derive the Laplace Stieltjes transform (LST) of the waiting time distribution and some performance measures of this model.

2. MODEL DESCRIPTION

We consider here a single server batch arrival queueing model with two phases of heterogeneous service under Bernoulli schedule vacation, in which the units arrive in batches of random size 'X' and according to a compound Poisson process with rate λ (> 0). The service times of two phases are assumed to follow a general law of distribution with distribution function (d.f) $S_i(x)$, LST $S_i^*(\theta)$ with finite moment $E(S_i^k)$ for i =1,2. In this model, after completing the first phase service (FPS), the server must provide a second phase service (SPS) to all customers. However, after completion of SPS the server may decide to take a vacation with probability p ($0 \leq p \leq 1$) or may continue to stay in the system with probability (1 - p). Assuming that the vacation time random variable 'V' follows a general law of distribution with d.f V(x), LST $V^*(\theta)$ which is independent of the service times. After returning from the vacation if the server does not find any units in the system it remains in the system till a batch of new customers arrive. Thus the time required by an unit to complete the service cycle which we will call it modified service time is given by

$$B = S_1 + S_2 + V, \text{ with probability 'p'}$$

$$B' = S_1 + S_2, \text{ with probability '1 - p'}$$

For convenience we designate our model as $M^x/(G_1,G_2)/V_S/1(BS)$ queue, where V_S represents the vacation time with single vacation and BS represents Bernoulli schedule .

Let us define the following probabilities for our model

$a_k = \textit{Prob}\ [\ X=k\ ]\ ;\ k \geq 1$

$a_k^{(j)} = \text{Prob}\ [\ Y_j=k\ ]$ is the k-fold convolution of $\{a_k\}$ with itself and $a_k^{(0)} = 1$

$Y_j = X_1 + X_2 + \ldots + X_j$

g_k = Prob [a batch of k units arrive during the FPS time 'S_1']

$$= \sum_{i=0}^{k} \int_0^\infty \frac{(\lambda x)^i e^{-\lambda x}}{i!} a_k^{(i)} dS_1(x)\ ;\ k \geq 0$$

h_k = Prob [a batch of k units arrive during the SPS time 'S_2']

$$= \sum_{i=0}^{k} \int_0^\infty \frac{(\lambda x)^i e^{-\lambda x}}{i!} a_k^{(i)} dS_2(x)\ ;\ k \geq 0$$

m_k= Prob [a batch of k units arrive during the vacation time '*V*']

$$= \sum_{i=0}^{k} \int_0^\infty \frac{(\lambda x)^i e^{-\lambda x}}{i!} a_k^{(i)} dV(x)\ ;\ k \geq 0$$

Let $X(z)$, $G(z)$, $H(z)$ and $M(z)$ be the probability generating functions (PGF) of $\{a_k;k\geq 1\}$, $\{g_k;k\geq 0\},\{h_k;k\geq 0\}$ and $\{m_i;i\geq 0\}$,respectively, then

$$X(z)=\sum_{k=1}^{\infty}a_k z^k$$

$$G(z)=\sum_{j=0}^{\infty}z^k g_k = S_1^*(\lambda-\lambda X(z))$$

$$H(z)=\sum_{j=0}^{\infty}z^k h_k = S_2^*(\lambda-\lambda X(z))$$

and $$M(z)=\sum_{j=0}^{\infty}z^k m_k = V^*(\lambda-\lambda X(z)).$$

3. EXISTENCE OF STEADY STATE SOLUTION

This section will provide a discussion on the existence of the steady state solution based on the Lyapounov function in the following Theorem.1.

Theorem 1: Let $E(S_i)$ and $E(V)$ be finite for $i = 1, 2$; where S_i is the service time random variable for i-th phase of service and V is the vacation time random variable, then the system is ergodic if and only if $\rho^* = \rho+\lambda pE(X)E(V) < 1$, where $\rho=\lambda E(X)[E(S_1)+E(S_2)]$.

Proof:

Let, t_n be the n–th departure epoch and $N(t_n)$ be the number of units in the system at the time instant "t_n", then $L_n = N(t_n + 0)$; $n\geq 0$ is the number in the system immediately after the n-th departure has a denumerable state space Ω ={0,1,2,…}. Clearly the Markov chain $\{L_n;\ n\geq 0\}$ is irreducible and aperiodic; since it is denumerable with transition probability matrix .

$$P = (P_{ij})$$
$$= pP_1 + (1 - p)P_2;$$

where

$$P_{\mathbf{1}}=\begin{bmatrix} b_0 & b_1 & b_2 & b_3 & . & . & . & . \\ b_0 & b_1 & b_2 & b_3 & . & . & . & . \\ 0 & b_0 & b_1 & b_2 & . & . & . & . \\ 0 & 0 & b_0 & b_1 & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix} \quad \text{and} \quad P_{\mathbf{2}}=\begin{bmatrix} c_0 & c_1 & c_2 & c_3 & . & . & . & . \\ c_0 & c_1 & c_2 & c_3 & . & . & . & . \\ 0 & c_0 & c_1 & c_2 & . & . & . & . \\ 0 & 0 & c_0 & c_1 & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix}$$

c_i = Prob ['i'units arrive during the service time 'S_1+S_2']

$$=\sum_{j=0}^{i} g_j h_{j-i} \tag{3.1}$$

b_i = Prob ['i'units arrive during the service time 'S_1+S_2+V']

$$=\sum_{j=0}^{i} c_j m_{j-i} \tag{3.2}$$

and $$P_{ij} = \Pr ob[L_{n+1} = j / L_n = i]$$

Clearly, the transition probabilities are given by

$$P_{ij} = \begin{cases} = pb_j + (1-p)c_j & \text{if} \quad i = 0, j \ge 0 \\ = pb_{j-i+1} + (1-p)c_{j-i+1} & \text{if} \quad i \ge 1, j \ge i-1 \\ = 0 & \text{if} \quad i \ge 1,\ 0 \le j < i-1 \end{cases} \tag{3.3}$$

Then the drift also known as Lyapounov function [see Pakes (1969)] is defined as $\gamma(i) = E\{L_{n+1} - L_n \mid L_n = i\}$ for $i \in \Omega$, where $\Omega = \{0,1,2,\ldots..\}$ is the state space (i.e. the number of units in the system) .

Now, for this model the drift is given by

$$\gamma(0) = [\rho + \lambda pE(X)E(V)] = \rho^*$$

$$\gamma(i) = [\rho + \lambda pE(X)E(V)] - 1 = \rho^* - 1;\ \text{for } i \ge 1\ .$$

Then there exist an $\in > 0$, such that $\gamma(i) < -\in$ for all $i \neq 0$. Hence by Fosters criterion the condition $[\rho + \lambda pE(X)E(V)] = \rho^*$, (say) < 1, is sufficient for ergodicity. Hence the proof is complete.

4. THE QUEUE SIZE DISTRIBUTION AT A STATIONARY POINT OF TIME

The derivation of the PGF of the queue size distribution at a stationary point of time under the steady state condition of this $M^X/(G_1,G_2)/V_S/1(BS)$ queue is done in this section.

Let ψ_j $(j \ge 0)$ be the steady state probability that 'j' customers are left behind by a departing customer, then

$$\psi_j = \lim_{n\to\infty} \Pr ob[L_n = j]\ ; \quad j \ge 0$$

Now, $\{\psi_j; j \ge 0\}$ can be obtained by solving the system of equations

$$\Psi' P = \Psi\ ; \tag{4.1}$$

where, $\Psi = ((\psi_0, \psi_1, \psi_2, \ldots\ldots..)'$ is a column vector.

Utilizing equations (3.3) in (4.1), we observe that ψ_j satisfies the following Kolmogorov equation

$$\psi_j = \psi_0\{pb_j + (1-p)c_j\} + \sum_{i=1}^{j+1} \psi_i\{pb_{j-i+1} + (1-p)c_{j-i+1}\}\ ;\ j \ge 0 \tag{4.2}$$

and the normalizing condition $\sum_{j=0}^{\infty} \psi_j = 1.$

Let us now assume $\Psi(z)$ be the PGF of $\{\psi_j; j \ge 0\}$, then multiplying equation (4.2) by appropriate powers of 'z^j' and taking summation over all possible values of 'j' $(j \ge 0)$ and after simplification we get $\Psi(z)$ as

$$\psi(z) = \sum_{j=0}^{\infty} \psi_j z^j$$

$$= \psi_0[pB(z) + (1-p)C(z)] + [\psi(z) - \psi_0][pB(z) + (1-p)C(z)]z^{-1}; \tag{4.3}$$

where B(z) and C(z) are the PGFs of $\{b_j\,;\ j \ge 0\}$ and $\{c_j;\ j \ge 0\}$ which can be obtained from equations (3.1) and (3.2) as follows :

$$B(z)=\sum_{j=0}^{\infty} z^j b_j = S_1^*(\lambda-\lambda X(z))S_2^*(\lambda-\lambda X(z)) \tag{4.4}$$

$$C(z)=\sum_{j=0}^{\infty} z^j c_j = S_1^*(\lambda-\lambda X(z))S_2^*(\lambda-\lambda X(z))V^*(\lambda-\lambda X(z)) \tag{4.5}$$

Now, utilizing (4.4) and (4.5) in (4.3) and simplifying we get

$$\psi(z)=\frac{\psi_0(1-z)[S_1^*(\lambda-\lambda X(z))S_2^*(\lambda-\lambda X(z))\{pV^*(\lambda-\lambda X(z))+(1-p)\}]}{[S_1^*(\lambda-\lambda X(z))S_2^*(\lambda-\lambda X(z))\{pV^*(\lambda-\lambda X(z))+(1-p)\}-z]} \quad \ldots \quad (4.6)$$

Now, using the normalizing condition $\lim_{z\to 1}\psi(z)=1$ we get ψ_0 after differentiation as

$$\psi_0 = 1-\lambda E(X)[E(S_1)+E(S_2)+pE(V)] = (1-\rho^*).$$

where, $\rho^* = \rho+\lambda pE(X)E(V)$ is the utilization factor of the system under which the steady state solution exist.

Hence we can summarize our result in the following Theorem 2.

Theorem 2: Let $\psi(z)$ be the PGF of the queue size distribution at a stationary point of time of this $M^x/(G_1,G_2)/V_S/1$(BS) queue, then

$$\psi(z)=\frac{(1-\rho^*)(1-z)[S_1^*(\lambda-\lambda X(z))S_2^*(\lambda-\lambda X(z))\{pV^*(\lambda-\lambda X(z))+(1-p)\}]}{[S_1^*(\lambda-\lambda X(z))S_2^*(\lambda-\lambda X(z))\{pV^*(\lambda-\lambda X(z))+(1-p)\}-z]};$$

where $\rho^* = [\rho+\lambda pE(X)E(V)]$.

Remark 1:

Taking p = 0 in the above Theorem 2, we get

$$\psi(z)=\frac{(1-\rho)(1-z)S_1^*(\lambda-\lambda X(z))S_2^*(\lambda-\lambda X(z))}{[S_1^*(\lambda-\lambda X(z))S_2^*(\lambda-\lambda X(z))-z]};$$

which is the PGF of the queue size distribution at a departure epoch of an $M^x/(G_1,G_2)/1$ queue without server vacation. In this model, the total service time required by a customer to complete both the phases of service is $B=S_1+S_2$. Thus the LST of B is given by $B^*(\theta)=S_1^*(\theta)S_2^*(\theta)$ with utilization factor $\rho=\lambda E(X)[E(S_1)+E(S_2)]$.

Similarly by putting p = 1 in the above Theorem 2, we get

$$\psi(z)=\frac{(1-\rho^*)(1-z)V^*(\lambda-\lambda X(z))S_1^*(\lambda-\lambda X(z))S_2^*(\lambda-\lambda X(z))}{[V^*(\lambda-\lambda X(z))S_1^*(\lambda-\lambda X(z))S_2^*(\lambda-\lambda X(z))-z]};$$

where $\rho^* = \lambda E(X)[E(S_1)+E(S_2)+E(V)]$ (<1) is the utilization factor of this system.

Note that the above expression is nothing but PGF of the queue size distribution at a stationary point of time for the $M^x/(G_1,G_2)/V_S/1$ queue with limited service. In limited service model, the server takes a vacation each time after completing the service of an unit. However, after returning from the vacation, he serves the remaining units in the queue, if any, otherwise the system becomes idle until a new batch of customers arrive.

5. MEAN QUEUE SIZE

This section will provide the mean queue size of the $M^x/(G_1,G_2)/V_S/1(BS)$ queuing model. This is obtained by differentiating $\psi(z)$ w.r.t z and taking limit as $z \to 1$. i.e.

$$L = \frac{d\psi(z)}{dz}\Big]_{z=1}$$

$$= \rho^* + \frac{\lambda^2 E^2(X)[pE(V)\{E(S_1)+E(S_2)\}+E(S_1)E(S_2)]}{(1-\rho^*)}$$

$$+ \frac{[\lambda^2 E^2(X)\{E(S_1^2)+E(S_2^2)+pE(V)\}]}{2(1-\rho^*)} + \frac{E\{X(X-1)\}}{E(X)}\rho^*; \qquad (5.1)$$

where L is the mean queue size of this $M^x/(G_1,G_2)/V_S/1(BS)$ queueing model, $E(S_i^2)$, j =1,2 and $E(V^2)$ are the second moments of the service time and vacation time random variables respectively .

In particular putting p = 0 in (5.1), we get the mean queue size of the $M^x/(G_1,G_2)/1$ queueing model without server vacation, which is given as

$$L_1 = \rho + \frac{\lambda^2 E^2(X)E(S_1)E(S_2)}{(1-\rho)} + \frac{\lambda^2 E^2(X)\{E(S_1^2)+E(S_2^2)\}}{2(1-\rho)} + \frac{E\{X(X-1)\}}{E(X)}\rho$$

Similarly for p = 1, we get $\rho^* = \lambda E(X)[E(S_1)+E(S_2)+E(V)]$ and therefore (5.1) reduces to

$$L_2 = \rho^* + \frac{\lambda^2 E^2(X)[pE(V)\{E(S_1)+E(S_2)\}+E(S_1)E(S_2)]}{(1-\rho^*)}$$

$$+ \frac{\lambda^2 E^2(X)\{E(S_1^2)+E(S_2^2)+E(V)\}}{2(1-\rho^*)} + \frac{E\{X(X-1)\}}{E(X)}\rho^*;$$

which is the mean queue size of the $M^x/(G_1,G_2)/V_s/1$ queue with limited service.

6. RECURSIVE SOLUTION OF THE MODEL

In this section an attempt has been made to obtain the recursive solution of the steady state queue size distribution of this model. Since our Markov chain is aperiodic and irreducible, therefore the PGF of the steady state queue size distribution can be written as

$$\psi(z) = [1-\{p\sum_{k=0}^{\infty} kb_k + (1-p)\sum_{k=0}^{\infty} kc_k\}][\frac{(1-z)\{pB(z)+(1-p)C(z)\}}{\{pB(z)+(1-p)C(z)\}-z}]$$

we also get

$$\psi_0 = 1-[p\sum_{k=0}^{\infty} kb_k + (1-p)\sum_{k=0}^{\infty} kc_k].$$

Other limiting probabilities can be computed by

$$\psi_n = \frac{1}{n!}\frac{d^n}{dz^n}\psi(z)\Big]_{z=0}$$

Unfortunately this is not a very practical method since taking derivatives is not easy. As an alternative one may use equation (4.2) to compute ψ_j' s recursively.

$$\psi_{j+1} = \frac{\psi_j}{[pb_0 + (1-p)c_0]} - \frac{\psi_0[pb_j + (1-p)c_j]}{\psi_0} - \sum_{i=1}^{j} \frac{\psi_i[pb_{j-i+1} + (1-p)c_{j-i+1}]}{[pb_0 + (1-p)c_0]}, j \geq 0$$

This method runs into numerical trouble because of the substraction involved.

However equation (4.2) can be rearranged as follows

$$\psi_1 = \frac{1-[pb_0 + (1-p)c_0]}{[pb_0 + (1-p)c_0]}\psi_0$$

$$\psi_2 = \frac{1-[pb_0 + (1-p)c_0] - [pb_1 + (1-p)c_1]}{[pb_0 + (1-p)c_0]}(\psi_0 + \psi_1)$$

$$\psi_3 = \frac{1-[pb_0 + (1-p)c_0] - [pb_1 + (1-p)c_1] - [pb_2 + (1-p)c_2]}{[pb_0 + (1-p)c_0]}(\psi_0 + \psi_1 + \psi_2) + \frac{[pb_2 + (1-p)c_2]}{[pb_0 + (1-p)c_0]}\psi_2$$

Now proceeding in a similar manner, we get

$$\psi_{j+1} = \frac{1-[\sum_{i=0}^{j}\{pb_i + (1-p)c_i\}]}{[pb_0 + (1-p)c_0]}[\sum_{i=1}^{j}\psi_i] + \sum_{i=2}^{j}\psi_i \sum_{k=j-i+2}^{j}\frac{[pb_k + (1-p)c_k]}{[pb_0 + (1-p)c_0]};\ j \geq 2$$

These equations, involving only sums of positive numbers are very stable and yield a good numerical method for computing $\{\psi_j;\ j \geq 0\}$.

Note that it even yields a simple method of truncation, for a given $\eta \geq 0$, stop the Computation at j if

$$\sum_{i=0}^{j}\psi_i \geq 1-\eta$$

and set $\psi_k = 0$, for all k > j.

7. WAITING TIME DISTRIBUTION

In this section we derive the LST of the waiting time distribution for an arbitrary (test) customer at a random point of time. In an $M^x/G/1$ queue, the waiting time of the test customer can be obtained by summing the waiting time of the first customer in the test customers group and the additional delay (waiting time) for the service of the group who are served before the test customer under consideration .

For our convenience we consider a test unit and let 'D' be the total waiting time of the unit in queue, that is 'D' is the queueing time of an arbitrary test unit. Then the delay 'D' is seen by the test unit to consist of two independent delays, 'D_1' and 'D_2'. 'D_1' is the delay (or waiting time) of the first member to be served of the batch in which the test unit arrives, and 'D_2' is the delay caused by the service times of the members of this batch that are served prior to the test unit, in other words, $D = D_1 + D_2$. Let $W(x)$ and $W_i(x)$ be the d.f of 'D' and 'D_i' respectively and $W^*(\theta)$ and $W_i^*(\theta)$ be the LST of $W(x)$ and $W_i(x)$ respectively for i =1, 2. Now since the $W_i^*(\theta)$ are mutually independent of each other, therefore we may write

$$W^*(\theta) = W_1^*(\theta)W_2^*(\theta) \tag{7.1}$$

Let $B(x)$ be the d.f of the modified service time distribution and $B^*(\theta)$ be its LST. Denote $\beta^*(\theta)$ be the LST of the d.f of the total service time of all customers belonging to the same arrival group. Then

$$\beta^*(\theta) = \sum_{k=1}^{\infty} a_k [B^*(\theta)]^k = X[B^*(\theta)].$$

In our system, since we take our modified service time as service time therefore we may write

$$B^*(\theta) = [(1-p) + pV^*(\theta)]S_1^*(\theta)S_2^*(\theta) .$$

In fact the concept of a modified service time was first introduced by Keilson and Servi (1986) for an GI/G/1 queueing system. Subsequently, in a series of papers Keilson and Servi (1987, 1989), Servi (1986) and Ramaswamy and Servi (1988) utilized the concept of modified service time for M/G/1 queueing system.

Clearly, the first two moments of the modified service time can be computed as follows

$$E(B) = -\left.\frac{dB^*(\theta)}{d\theta}\right]_{\theta=0} = E(S_1) + E(S_2) + pE(V)$$

$$E(B^2) = -\left.\frac{d^2B^*(\theta)}{d\theta^2}\right]_{\theta=0} = E(S_1^2) + E(S_2^2) + 2E(S_1)E(S_2) + p[E(V^2) + 2E(V)\{E(S_1) + E(S_2)\}]$$

Again, it is well known that the LST of $W_1^*(\theta)$ on M/G/1 queue [e.g. see Kleinrock (1975), page – 200] is given by

$$W_1^*(\theta) = \frac{\theta(1-\rho^*)}{\theta - \lambda\{1 - B^*(\theta)\}}; \qquad (7.2)$$

where $\rho^* = \lambda E(B)$ is the utilization factor of the M/G/1 queueing system.

To obtain the delay 'D_1', consider a batch as a single super customer. Then the LST of the waiting time distribution of the first member of the batch in which the test unit arrive can be obtained from the corresponding expression of an M/G/1 queue with $B^*(\theta)$ replace by $\beta^*(\theta)$. That is $\rho^* = \lambda E(X)[E(S_1) + E(S_2) + pE(V)]$ and replacing $B^*(\theta)$ by $\beta^*(\theta)$ in (7.2), we get

$$W_1^*(\theta) = \frac{\theta(1-\rho^*)}{\theta - \lambda[1 - X\{\{(1-p) + pV^*(\theta)\}S_1^*(\theta)S_2^*(\theta)\}]} \qquad (7.3)$$

Now to obtain $W_2^*(\theta)$, let us define the following events

$E_{i,r}$: the event that the test customer in the i-th position under the condition that the group of size is 'r'

H_r : the event that the test customer belongs to a group of size 'r' .

Now, as defined by Burke (1975), we have

$$\text{Prob}(E_{i,r}) = 1/r$$

$$\text{Prob}(H_r) = ra_r / E(X) .$$

Thus we have

$$W_2^*(\theta) = \sum_{r=1}^{\infty}\sum_{k=1}^{r} [B^*(\theta)]^{k-1} \frac{1}{r}\frac{ra_r}{E(X)} = \frac{1 - X[B^*(\theta)]}{E(X)[1 - B^*(\theta)]} \qquad (7.4)$$

Now utilizing (7.3), (7.4) in (7.1), we get *LST* of the waiting time distribution as

$$W_1^*(\theta)=\frac{\theta(1-\rho^*)[1-X\{((1-p)+pV^*(\theta))S_1^*(\theta)S_2^*(\theta)\}]}{E(X)[\theta-\lambda[1-X\{\{(1-p)+pV^*(\theta)\}S_1^*(\theta)S_2^*(\theta)\}]][1-\{(1-p)+pV^*(\theta)\}S_1^*(\theta)S_2^*(\theta)]} \tag{7.5}$$

Now differentiating equation (7.5) with respect to 'θ' and taking limit as $\theta\to 0$, we get the mean waiting time for this $M^x/(G_1,G_2)/V_s/1$ *(BS)* queue as

$$E(W)=-\left.\frac{dW_Q^*(\theta)}{d\theta}\right]_{\theta=0}$$

$$=\frac{\lambda E(X)[E(S_1^2)+E(S_2^2)+2E(S_1)E(S_2)+p[E(V^2)+2E(V)\{E(S_1)+E(S_2)\}]]}{2(1-\rho^*)}$$

$$+\frac{\lambda E[X(X-1)][E(S_1)+E(S_2)+E(V)]}{2E(X)(1-\rho^*)}$$

$$=\frac{\lambda E(X)E(B^2)}{2(1-\rho^*)}+\frac{E[X(X-1)]E(B)}{2E(X)(1-\rho^*)} \tag{7.6}$$

Now putting *p* =1 in (7.5), we get

$$W_1^*(\theta)=\frac{\theta(1-\rho^*)[1-X\{V^*(\theta)S_1^*(\theta)S_2^*(\theta)\}]}{E(X)[\theta-\lambda[1-X\{V^*(\theta)S_1^*(\theta)S_2^*(\theta)\}]][1-\{V^*(\theta)S_1^*(\theta)S_2^*(\theta)\}]}$$

$$=\frac{\theta(1-\rho^*)[1-X\{V^*(\theta)B^*(\theta)\}]}{E(X)[\theta-\lambda[1-X\{V^*(\theta)B^*(\theta)\}]][1-\{V^*(\theta)B^*(\theta)\}]} \tag{7.7}$$

where $\rho^*=\lambda E(X)[E(S_1)+E(S_2)+E(V)]\quad(<1)$.

It may be noted here that equation (7.7) is the LST of the waiting time distribution for an $M^X/(G_1,G_2)/1$ type of single vacation queue with limited service i.e. $M^x/(G_1,G_2)/V_s/1$ queue with limited service. In such a model, if there is at least one or a batch of customers in the system at the end of a vacation, the service starts immediately. Otherwise the system becomes idle until a new customer arrives. In this model, the total time required by an unit to complete both the phases of service is then $B=S_1+S_2$ with its LST $B^*(\theta)=S_1^*(\theta)S_2^*(\theta)$.

Further, if we take Prob [X =1] = 1 then E(X) = 1 and consequently (7.7) is simply reduced to

$$W_Q^*(\theta)=\frac{\theta(1-\rho^*)}{\theta-\lambda[1-V^*(\theta)B^*(\theta)]}$$

where $\rho^*=\lambda[E(S_1)+E(S_2)+E(V)]=\lambda[E(B)+E(V)]$.

Note that the LST of the waiting time distribution for an M/G/1 queue with a single vacation and limited service was studied by Takagi (1991) and it verifies equation (6.10(a)) of Takagi (1991) [see page - 230].

8. NUMERICAL EXAMPLE

In order to see the effect of the parameter λ and p on the mean waiting time (M.W.T) in our model, we take the service time distributions are exponential with mean $E(S_i)={}^1\!/\!_{\mu_i}$ and finite second moment $E(S_i^2)={}^2\!/\!_{\mu_i^2}$ for i = 1, 2. The calculation of, $E(X)$, $E(X^2)$, $E(V)$ and $E(V^2)$ can be computed as follows

$$E(X) = \left.\frac{dX(z)}{dz}\right]_{z=1} \quad (8.1)$$

$$E(X) = \left.\frac{dX(z)}{dz}\right]_{z=1} + \left.\frac{d^2X(z)}{dz^2}\right]_{z=1} \quad (8.2)$$

$$E(V) = -\left.\frac{dV^*(\theta)}{d\theta}\right]_{\theta=0} \quad (8.3)$$

$$E(V^2) = (-1)^2 \left.\frac{d^2V^*(\theta)}{d\theta^2}\right]_{\theta=0} \quad (8.4)$$

Now, let us assume that vacation time = 0.2 (constant) then we have $V^*(\theta) = e^{-0.2\theta}$ and therefore from equation (8.3) and (8.4), we get $E(V) = 0.2$ and $E(V^2) = 0.04$. Now, for computing $E(X)$ and $E(X^2)$, let us consider the following two examples .

Example 1: Here we consider that the batch size distribution follows a discrete Uniform distribution with probability function $a_k = 1/N$ for k=1,2,…,N and $a_k = 0$ for $k \geq (N+1)$, then X(z) will be $X(z) = \frac{z(1-z^N)}{N(1-z)}$ and therefore from (8.1) and (8.2) $E(X) = \frac{(N+1)}{2}$ and $E(X^2) = E(X)\frac{(2N+1)}{3}$.

Example 2: Here we consider that the batch size distribution is a displaced Geometric distribution with probability function $a_k = (1-\theta)\theta^{k-1}$ for $k \geq 1$, then X(z) will be $X(z) = \frac{(1-\theta)z}{(1-\theta z)}$ and therefore from (8.1) and (8.2) we have $E(X) = \frac{1}{(1-\theta)}$ and $E(X^2) = E(X)\frac{(1+\theta)}{(1-\theta)}$.

Now for some specific values of *N*, we can calculate E(X) and E(X^2) in Example 1. To make a comparative study of the uniform batch size and the Geometric batch size for M.W.T. of our model, we can calculate the value of θ as $\theta = 1 - [1/E(X)]$. Further for computational convenience we arbitrarily choose $\mu_1 = 10$, $\mu_2 = 20$, $\lambda = 0.6$ and E(X) = 1, 1.5, 2.0, 2.5 and 3.0 [i.e. *N* = 1, 2, 3, 4 and 5 for Example 1] for both Uniform and Geometric distribution cases but we vary p from 0 to 1.0 such that the utilization factor $\rho^* < 1$ always satisfied. Now based on our result found in (7.6), we make the following tables for the computed values of M.W.T.

Table 1. Expected mean for the uniform distributed waiting time.

p	E(X) = 1	E(X) = 1.5	E(X) = 2	E(X) = 2.5	E(X) = 3.0
0	0. 03963	0. 20377	0. 41406	0. 69318	1. 08153
0. 1	0. 05012	0. 23513	0. 47781	0. 81009	1. 29286
0. 2	0. 06093	0. 26814	0. 54673	0. 94133	1. 54339
0. 3	0. 07207	0. 30289	0. 62147	1. 08968	1. 84521
0. 4	0. 08355	0. 33954	0. 70282	1. 25872	2. 21572
0. 5	0. 09540	0. 37825	0. 79167	1. 45313	2. 68155
0. 6	0. 10762	0. 41921	0. 88911	1. 67906	3. 28483
0. 7	0. 12024	0. 46261	0. 99647	1. 94486	4. 09696
0. 8	0. 13329	0. 50867	1. 11532	2. 26210	5. 24904
0. 9	0. 14677	0. 55766	1. 24764	2. 64733	7. 01104
1. 0	0. 16071	0. 60985	1. 39583	3. 12500	10. 04168

Table 2. Expected mean for the displaced Geometric distributed waiting time.

p	E(X) = 1	E(X) = 1.5	E(X) = 2	E(X) = 2.5	E(X) = 3.0
0	0. 03963	0. 27226	0. 57031	0. 96591	1. 51631
0. 1	0. 05012	0. 31006	0. 65097	1. 11779	1. 79599
0. 2	0. 06093	0. 34979	0. 73817	1. 28827	2. 12758
0. 3	0. 07207	0. 39164	0. 83274	1. 48098	2. 52698
0. 4	0. 08355	0. 43579	0. 93566	1. 70058	3. 01740
0. 5	0. 09540	0. 48242	1. 04808	1. 95313	3. 63393
0. 6	0. 10762	0. 53175	1. 17138	2. 24662	4. 43237
0. 7	0. 12024	0. 58402	1. 30720	2. 59192	5. 50721
0. 8	0. 13329	0. 63950	1. 45759	3. 00404	7. 03198
0. 9	0. 14677	0. 69851	1. 62500	3. 50447	9. 36397
1. 0	0. 16071	0. 76137	1. 81250	4. 12500	13. 37500

The above tables clearly show that as p increases the mean waiting time increases for both Uniform and Displaced Geometric distributions. But the rate of increase of M.W.T for Displaced Geometric distribution is slightly greater than the Uniform distribution because the coefficient of variation of the Geometric distribution is less than the Uniform distribution. Further, it is observed that for a single unit arrival case i.e. $E(X)=1$ the numerical values of M.W.T for both the cases are exactly same and hence the original model reduces to $M/(G_1,G_2)/V_S/1(BS)$ queue.

ACKNOWLEDGMENT

The authors are thankful to the referees and Editor for their comments to revise the paper in the present form.

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