ON SOME STATISTICAL PROPERTIES OF THE APPORTIONMENT INDEX

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ABSTRACT
The purpose of this paper is to initiate a statistical analysis of the *Apportionment index*, a measure used in the context of forest harvesting to evaluate the fit between demand and supply distribution of logs. There have been some attempts to understand this index, but a serious theoretical foundation is still lacking. We briefly review the available literature and then proceed to investigate the properties of the index from a distributional point of view. This is mainly an exploratory article and we focus only on the cases of two and three log classes, i.e., locations. In the case of two locations we use the beta distribution for the random relative output variables; with three locations the random relative outputs are assumed to follow a singular Dirichlet distribution. Using this formulation it is possible to understand the statistical properties of the apportionment index.

Key words: Mean deviation, beta distribution, Dirichlet distribution, exchangeable distributions, median.

RESUMEN
El propósito de este trabajo es comenzar el análisis estadístico del índice del prorrateo, esta es una medida usada en el contexto de la tala en bosque para evaluar el ajuste entre la demanda y distribución del suministro de leña. Ha habido algunos esfuerzos por entender este índice, pero se necesita una base teórica seria. Discutimos brevemente la literatura existente y procedemos a investigar las propiedades del índice desde un punto de vista distribucional. Éste es fundamentalmente un artículo exploratorio y sólo enfocamos los casos de dos y tres clases de leña, es decir, locaciones. En el caso de dos clases usamos la distribución beta para las variables aleatorias relativas a la salida (output) del rendimiento; en tres locaciones al azar se asume que los rendimientos relativos siguen la distribución de Dirichlet singular. Usando esta formulación es posible entender las propiedades estadísticas del índice del prorrateo.

MSC: 62P12.

1. INTRODUCTION

Modern sawmills attempt to develop their production strategies based on customer demands in terms of the distribution of logs of various diameter - length specifications. The quality of the actual harvesting operation has mainly been measured by calculating the relationship between the demand log distribution and the actual production distribution. A very commonly adopted practice in Scandinavia is to measure the bucking outcome by the so-called *apportionment degree*, or *apportionment index* (AI). This measure was developed by the Swedish mathematician Bergstrand in the mid 1980s, when the first steps were taken in developing automatic bucking systems for forest harvesters. The main idea is to compare the relative proportions of the demand and target distributions (e.g. Bergstrand 1990). While there have been attempts to understand this index (Kivinen **et al**. 2003 and Nummi **et al**. 2004), a serious theoretical foundation is still lacking. Several extensions of this measure are proposed in Kirkkala et al. (2000) and Malinen & Palander (2004).

Our interest here is in the statistical analysis of the apportionment index AI. This kind of analysis may have many potential applications in harvesting. For example, in harvesting planning we may have many possible output distributions (stands) and we should be able to select the optimal one for a given target. Further, we may have many possible targets and we wish to know which is optimal for a given output. A proper understanding of the statistical properties of this measure is thus of great importance.

In Section 2 we initiate a statistical study of the AI based on the joint distribution of the random component outputs in the output matrix. In section 2.1 we take the analysis of the two log classes as a starting-point. in Section 2.2 we extend the analysis to the case of three locations and finally in section 3 some observations are made on the future course of action.

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Before closing the section, we record the general definition of AI. Suppose X refers to the *proportional* output distribution and θ refers to the *proportional* demand distribution. Then the AI is defined as

$$AI = 1 - \frac{1}{2}\sum_i \sum_j | X_{ij} - \theta_{ij} | . \quad (1)$$

After some simple manipulations we can show that AI can also be written as

$$AI = \sum_i \sum_j \min\{X_{ij} . \theta_{ij}\}. \quad (2)$$

since $\sum_i \sum_j X_{ij} = \sum_i \sum_j \theta_{ij} = 1$ and $\min(a, b) = \frac{a+b}{2} - \frac{|a-b|}{2}$ for two real numbers a and b.

2. JOINT DISTRIBUTION OF RANDOM OUTPUTS AND STATISTICAL ANALYSIS OF AI

2.1. The case of two locations

Let us assume for simplicity that there are only two locations, labeled “L1" and “L2", and their relative (i.e., proportional) demands are θ and $1 - \theta$, respectively. Let the relative random output generated at location L1 be denoted by X so that in location L2 the output generated is 1 - X. By the definition the AI is given by the formula (1) and in the case of two locations it can be written as

$$AI = 1 - | X - \theta |. \quad (3)$$

Because AI is now a random quantity we may look at the expected value of the AI given by the formula

$$E(AI) = 1 – E [| X - \theta |] . \quad (4)$$

At this stage we note that the relative random output X in L1 is a random variable defined over [0,1] and this explains the random nature of AI in (3). Our purpose is to maximize AI in some sense, since this will suggest maximum apportionment. By reason of the stochastic nature of AI, one possibility would be to attain a heavy right tail distribution for AI so that it will tend to be probabilistically large. In this paper, we use the notion of maximization in the averaged sense, i.e., we aim at maximizing the expectation of AI in (4). This is equivalent to minimizing $E[|X-\theta|]$. Since the mean deviation is least when it is taken about the median, our goal is to recommend a distribution for X for which the median is the *known* target value of θ, say θ_0. Since X is distributed over [0,1], it is natural to express its distribution as a member of the family of beta distributions ($B(x;\alpha,\beta)$) introduced below by the density in (5).

$$f(x; \alpha, \beta) = x^{\alpha-1}(1 - x)^{\beta-1} / B(\alpha, \beta);\ 0 < x < 1, \quad (5)$$

where $B(\alpha, \beta)$ is the beta integral defined for $\alpha > 0$, $\beta > 0$. On this see e.g. Johnson **et al**. (1995), 2nd Edition, pp. 210-211; Kotz and Johnson (1982), Vol.1, pp. 228-229. We may thus seek to use the beta distribution for X with proper choice of the parameters α and β, determined by the condition that θ_0 serves as the median of the X distribution. In effect, we seek a solution for α and β, so as to satisfy

$$0.5 = \int_0^{\theta_0} f(x; \alpha,\beta)\, dx . \quad (6)$$

Since equation (6) does not have an unique solution, it is reasonable to introduce the condition

$$\alpha\beta / [(\alpha + \beta)^2(\alpha + \beta + 1)] = V_0, \quad (7)$$

since Var(X) = Var(1 - X). In the above, V_0 is a pre-specified quantity. From (7), we may readily observe that a solution to α exists provided that

$$V_0 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \leq \frac{1}{4(\alpha+\beta+1)} < \frac{1}{4} .$$

However, equation (6) is not easy to solve analytically, even if it is expressed as a function of one unknown quantity, say α , in view of the above consideration. In Table 1 we display solutions to α and β satisfying (6) for selected values of V_0 and θ_0. Define

$$\Delta(\theta; \alpha, \beta) = E[| X - \theta|], X \sim B(x; \alpha, \beta). \quad (8)$$

Since

$$\Delta(\theta; \alpha, \beta) = \Delta(1 - \theta; \beta, \alpha), \quad (9)$$

in Table 1, we present values θ up to 0.5. Moreover, we also display the efficiency ratio Q $\times$100, where Q = 1/E(AI). The smaller the value of Q the better the degree of apportionment will be attained. In Figure 1, we display Q $\times$ 100 values vs. V_0 for selected values of θ_0.

The purpose of Table 1 is to demonstrate the specification of the parameters of the beta distribution for X when the target (θ) is specified, and we maximize the AI in an averaged sense for a given value of Var(X). Some of the findings are displayed in Figure 1.

Figure 1 shows that for a specified target value θ_0 of θ, Q (reciprocal of averaged AI) increases in V_0. In other words, if we seek to be liberal (by allowing a larger variation in the X distribution by taking a higher value of Var(X)), then we will tend to achieve a poorer apportionment on an average. From the figure we can determine the extent of variation to be allowed in the X distribution to meet any specific value of the averaged AI.

Table 1. Values of α, β and Q $\times$ 100 subject to (6) for a given θ and Var(X).

Var(X) = 0.01										
θ	.05	.1	.15	.2	.25	.3	.35	.4	.45	.5
α	.60	1.24	2.16	3.34	4.71	6.21	7.76	9.29	10.73	12.0
β	6.31	8.69	10.77	12.39	13.48	14.05	14.13	13.77	13.04	12.0
100Q	107.4	108.2	108.5	108.6	108.7	108.7	108.8	108.8	108.8	108.8
Var(X) = 0.05										
θ	.05	.1	.15	.2	.25	.3	.35	.4	.45	.5
α	.27	.41	.56	.74	.94	1.16	1.38	1.60	1.81	2.0
β	1.42	1.71	1.93	2.11	2.23	2.30	2.30	2.25	2.15	2.0
100Q	117.6	119.7	121.0	121.8	122.3	122.7	122.9	123.0	123.1	123.1
Var(X) = 0.10										
θ	.05	.1	.15	.2	.25	.3	.35	.4	.45	.5
α	.18	.25	.31	.37	.43	.50	.57	.63	.70	.75
β	.60	.69	.75	.79	.82	.84	.84	.82	.79	.75
100Q	129.0	132.0	134.0	135.4	136.5	137.3	137.9	138.3	138.5	138.5
Var(X) = 0.20										
θ	.05	.1	.15	.2	.25	.3	.35	.4	.45	.5
α	.073	.084	.092	.099	.10	.11	.11	.12	.12	.13
β	.11	.12	.13	.13	.13	.13	.13	.13	.13	.13
100Q	162.5	166.3	168.8	170.8	172.3	173.4	174.3	174.9	175.2	175.3

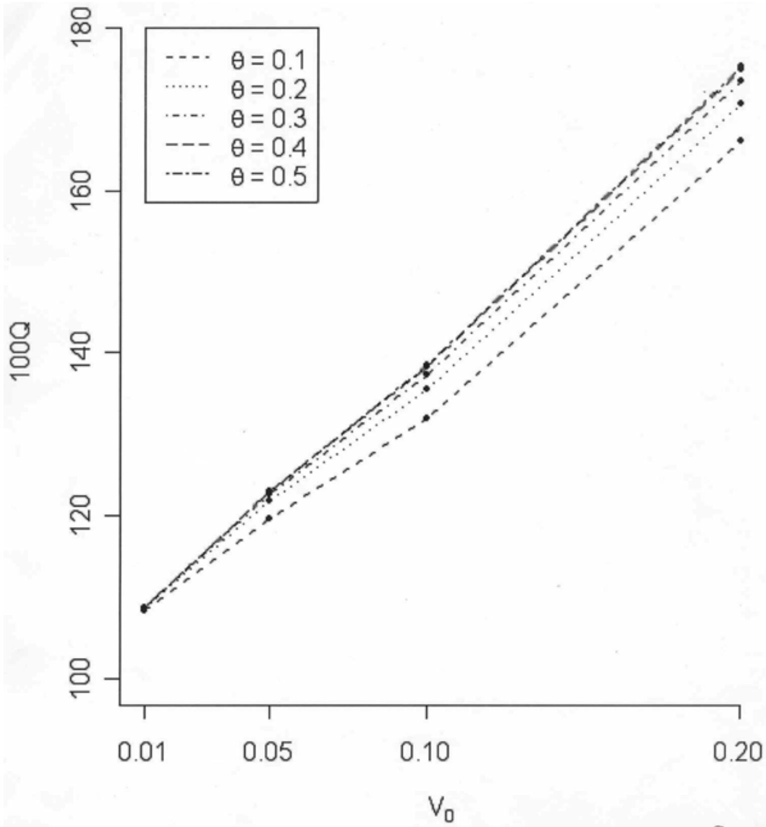

Figure 1. Graph showing $Q \times 100$ vs. V_0 for some selected values of θ in the case of two locations.

2.2. The case of three locations

We now pass on to a discussion for three locations with relative demands to be denoted by θ_1, θ_2 and θ_3 subject to the total being 1. We will also denote the corresponding random relative outputs by the variables X_1, X_2 and X_3, respectively. Note that X_1, X_2 and X_3 are non-negative random variables subject to the normalizing constraint $X_1 + X_2 + X_3 = 1$. Therefore, as a natural generalization of the beta distribution for two locations, we here adopt 3-variate singular Dirichlet distribution to describe the joint distribution of the Xs. We take the parameters of the Dirichlet distribution as α_1, α_2 and α_3 and denote the distribution by Dir[$\alpha_1, \alpha_2, \alpha_3$] and assume ($X_1$,$X_2$) to follow this distribution, unless otherwise stated. For the sake of completeness, we display the density underlying Dir[$\alpha_1, \alpha_2, \alpha_3$] below.

$$f(x_1, x_2; \alpha_1, \alpha_2, \alpha_3) = x_1^{\alpha_1-1} x_2^{\alpha_2-1} (1 - x_1 - x_2)^{\alpha_3-1} / D(\alpha_1, \alpha_2, \alpha_3),$$

$$0 < x_1, x_2 < x_1 + x_2 \leq 1, \tag{10}$$

where $D(\alpha_1, \alpha_2, \alpha_3) = \Gamma(\alpha_1, \alpha_2, \alpha_3) / \Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)$. Here $\Gamma(\alpha)$ refers to the standard Gamma integral defined for $\alpha > 0$ as $\int_0^\infty e^{-x} x^{\alpha-1} dx$. See Johnson & Kotz (1972), pp 231-235; Kotz & Johnson (1982), Vol.2, pp. 386-387.

This time our requirements are fairly stringent so far as attainment of the absolute maximum of the expected AI is concerned. In the case of three locations the AI is given by the formula

$$AI = 1 - \frac{1}{2} \left[| X_1 - \theta_1 | + | X_2 - \theta_2 | + | X_3 - \theta_3 | \right] \tag{11}$$

and the expected AI is

$$E(AI) = 1 - \frac{1}{2} [E[|X_1 - \theta_1|] + E[|X_2 - \theta_2|] + E[|X_3 - \theta_3|]]. \tag{12}$$

Note that now $X_3 = 1 - X_1 - X_2$ and $\theta_3 = 1 - \theta_1 - \theta_2$ but for simplicity we remain in the notations X_3 and θ_3. To maximize (12) we need to minimize

$$\psi(\alpha) = E[\,|(X_1 - \theta_1)|\,] + E[\,|(X_2 - \theta_2)|\,] + E[\,|(X_3 - \theta_3)|\,] \tag{13}$$

We know that for any individual term above, minimization is achieved by taking the demand parameter θ_i as the median of the corresponding output distribution of X_i. However, it is *not* possible to attain this feature for all three terms simultaneously. To see this, we refer to Statement II in the Appendix.

Note that in a Dirichlet distribution, each marginal distribution is beta. Hence in order for all the terms to be simultaneously minimized to attain the least possible value corresponding to the median in each case, we must have

(i) $0.5 = \int_0^{\theta_1} f(x_1;\alpha_1,\alpha_2+\alpha_3)\,dx_1 = \Delta(\theta_1;\alpha_1,t_1\alpha_1)$, where $t_1 = \dfrac{\alpha_2+\alpha_3}{\alpha_1}$;

(ii) $0.5 = \int_0^{\theta_2} f(x_2;\alpha_2,\alpha_1+\alpha_3)\,dx_2 = \Delta(\theta_2;\alpha_2,t_2\alpha_2)$, where $t_2 = \dfrac{\alpha_1+\alpha_3}{\alpha_2}$;

(iii) $0.5 = \int_0^{\theta_3} f(x_3;\alpha_3,\alpha_1+\alpha_2)\,dx_3 = \Delta(\theta_3;\alpha_3,t_3\alpha_3)$, where $t_3 = \dfrac{\alpha_1+\alpha_2}{\alpha_3}$.

Now, appealing to Statement II in the Appendix, we must have

$$\theta_1(\alpha_2+\alpha_3) < (1-\theta_1)\alpha_1;\ \theta_2(\alpha_1+\alpha_3) < (1-\theta_2)\alpha_2;\ \theta_3(\alpha_1+\alpha_2) < (1-\theta_3)\alpha_3. \tag{14}$$

This pre-supposes that each θ_i is less than 0.5, which will be assumed throughout.

Summing over all the conditions and re-writing the inequality, we obtain

$$(\theta_1+\theta_2+\theta_3)(\alpha_1+\alpha_2+\alpha_3) < (\alpha_1+\alpha_2+\alpha_3),$$

i.e. $(\theta_1+\theta_2+\theta_3) < 1$ which is a contradiction. Therefore, we must have equality in each of the requirements above. This means that

$$0.5 = \int_0^{\theta} f(x;\alpha,t\alpha)\,dx \tag{15}$$

is to be satisfied for a finite α while $t = (1-\theta)/\theta$. This is again a contradiction, as indicated in the Appendix.

Simultaneous minimization of all three terms in (13) to respective absolute minimum must therefore be ruled out. From now onwards, we assume $\theta_1 \le \theta_2 \le \theta_3$ without any loss of generity. However, a unique choice of α_is may be made by selecting them in the ratio of the θ_is and by equating the *highest marginal variance* of the X_is to a given quantity V_0. In other words, we may start with $[\alpha, a\alpha, b\alpha]$, where $a = \theta_2/\theta_1$, $b = \theta_3/\theta_1$ $(1 \le a \le b)$, and seek to choose α using the variance requirement V_0 on the largest of $V(X_1)$, $V(X_2)$ and $V(X_3)$, i.e., on $V(X_3)$. We can compute the value of AI and hence that of Q and examine its behaviour for variations in α for given (a, b), i.e. for given θ_1, θ_2 and θ_3. In Tables 2, 3 and 4 we show the values of Q x 100 and $\psi(\alpha)$ in (13) for different values of θ_1, θ_2, θ_3 as a function of V_0 (or α).

Tables 2, 3 and 4 and also Figure 2 show that for higher values of V_0, equal-demand distribution seems to yield less satisfactory results. We should therefore look at this case more carefully. In Table 5 we compare the relative ratios of Q_i and Q_j by using the measure $RR(i, j) = \dfrac{|Q_i - Q_j|}{\frac{1}{2}(Q_i + Q_j)} \times 100\%$, where indices i and j correspond to the settings in Tables 2, 3 and 4, respectively. Table 5 and also Figure 2 reveal that there is little difference among the realized Q values for different demand distributions whenever V_0 is appreciably small. The comparison also shows that settings 1 and 2 likewise differ little, but a slight difference is obtained for large values of V_0. For settings 1 and 3, and also for settings 2 and 3, the difference is fairly large except for notably small values of V_o.

The values of $\Psi(\alpha)$ and Q $\times$ 100 for different values of θ_1, θ_2, θ_3 as a function of V_0 (or α).

Table 2. $\theta_1 = 0.17$, $\theta_2 = 0.38$, $\theta_3 = 0.45$

$$a = \frac{\theta_2}{\theta_1} = 2.23529,\ b = \frac{\theta_3}{\theta_1} = 2.64706$$

V_0	α	$\psi(\alpha)$	**100Q**
.01	4.04	0.219	112.3
.02	1.93	0.312	118.5
.03	1.23	0.385	123.8
.04	0.88	0.447	128.8
.05	0.67	0.504	133.6
.075	0.39	0.625	145.5
.10	0.25	0.733	157.8
.125	0.17	0.831	171.1
.15	0.11	0.923	185.7
.175	0.070	1.011	202.1
.20	0.040	1.095	221.0

Table 3. $\theta_1 = 0.1$, $\theta_2 = 0.45$, $\theta_3 = 0.45$

$$a = \frac{\theta_2}{\theta_1} = 4.5,\ b = \frac{\theta_3}{\theta_1} = 4.5$$

V_0	α	$\psi(\alpha)$	**100Q**
.01	2.38	0.209	111.6
.02	1.14	0.296	117.4
.03	0.73	0.364	122.3
.04	0.52	0.422	126.8
.05	0.47	0.474	131.1
.075	0.23	0.588	141.6
.10	0.15	0.687	152.4
.125	0.098	0.778	163.7
.15	0.065	0.864	176.1
.175	0.041	0.946	189.8
.20	0.024	1.025	205.2

Table 4. $\theta_1 = \frac{1}{3},\ \theta_2 = \frac{1}{3},\ \theta_3 = \frac{1}{3}$

$a = \frac{\theta_2}{\theta_1} = 1.0,\ b = \frac{\theta_3}{\theta_1} = 1.0.$

V_0	α	$\psi(\alpha)$	**100Q**
.01	7.07	0.215	113.7
.02	3.37	0.345	120.8
.03	2.14	0.426	127.1
.04	1.52	0.496	133.0
.05	1.15	0.560	138.8
.075	0.65	0.699	153.9
.10	0.41	0.823	170.0
.125	0.26	0.938	188.3
.15	0.16	1.045	209.5
.175	0.090	1.148	234.8
.20	0.037	1.247	265.8

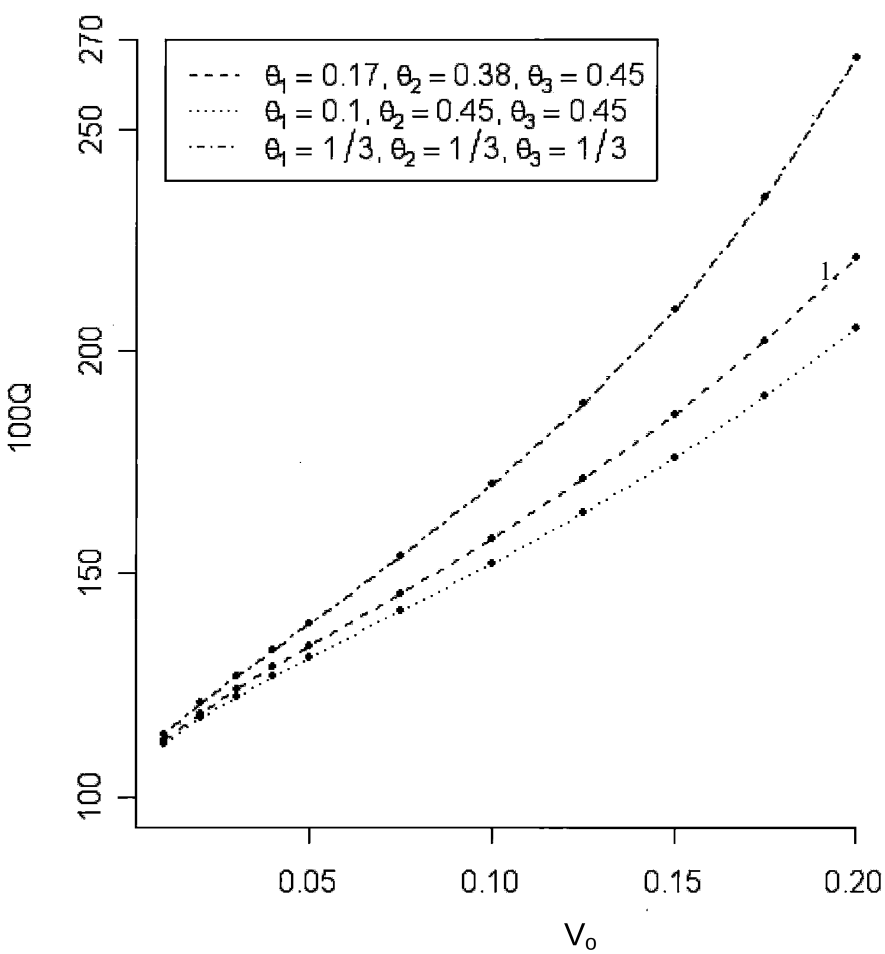

Figure 2. Graph displaying $Q \times 100$ vs. V_0 for different settings in Example 2.1.

Table 5. Relative ratio of Q values defined as $RR(i,j) = \frac{|Q_i - Q_j|}{\frac{1}{2}(Q_i - Q_j)} \times 100\%$, where i and j refer to two different demand distributions.

V_0	RR(1,1)	RR(1,3))	RR(2,3)
.01	0.60	1.26	1.86
.02	0.95	1.95	2.89
.03	1.27	2.58	3.85
.04	1.58	3.21	4.79
.05	1.89	3.85	5.74
.075	2.69	5.60	8.29
.10	3.51	7.43	10.94
.125	4.38	9.58	13.94
.15	5.30	12.06	17.33
.175	6.30	14.96	21.21
.20	7.42	18.41	25.74

The computations in this section were carried out using Mathematica (Wolfram, 1999) and R software (Venables & Ripley 2002) environments. The software code is available from the authors on request.

3. DISCUSSION

In this paper we have initiated a statistical study of the Apportionment Index (AI) by considering it as a random variable and by seeking maximization of its average value. We have developed the necessary theory and computational aspects for this problem in the case of two and three locations. Our results rely on a specification of the distribution of the underlying random variable(s) which is taken to be beta (Dirichlet). The general case is yet to be taken up. There is also scope for a Bayesian analysis of this problem by considering the target parameters (θ) to be random and taking appropriate priors for them.

4. ACKNOWLEDGMENTS

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APPENDIX

STATEMENT I

Define

$$\Delta(r;\alpha,t\alpha)=\frac{\int_0^r x^{\alpha-1}(1-x)^{t\alpha-1}dx}{\int_0^1 x^{\alpha-1}(1-x)^{t\alpha-1}dx}\ . \tag{16}$$

Then $\Delta(r;\alpha,t\alpha)\uparrow$ in t, $\forall\alpha>0$, $\forall r\in(0,1)$.

Proof.

Taking the first derivative and requiring it to be positive, we end up, after simplification, with the inequality

$$\left(\int_0^r x^{\alpha-1}(1-x)^{t\alpha-1}\alpha\left(-\log(1-x)\right)dx\right)\left(\int_0^1 x^{\alpha-1}(1-x)^{t\alpha-1}dx\right)<$$

$$<\left(\int_0^r x^{\alpha-1}(1-x)^{t\alpha-1}dx\right)\left(\int_0^1 x^{\alpha-1}(1-x)^{t\alpha-1}\alpha\left(-\log(1-x)\right)dx\right), \tag{17}$$

which is equivalent to

$$\sum_{j=1}^{\infty}\frac{1}{j}\left(\int_0^r x^{\alpha+j-1}(1-x)^{t\alpha-1}dx\right)\left(\int_0^1 x^{\alpha-1}(1-x)^{t\alpha-1}dx\right)<\sum_{j=0}^{\infty}\frac{1}{j}\left(\int_0^r x^{\alpha-1}(1-x)^{t\alpha-1}dx\right)\left(\int_0^1 x^{\alpha+j-1}(1-x)^{t\alpha-1}dx\right)\ . \tag{18}$$

Then, comparing term by term, we require

$$\left(\int_0^r x^{\alpha+j-1}(1-x)^{t\alpha-1}dx\right)\left(\int_0^1 x^{\alpha-1}(1-x)^{t\alpha-1}dx\right)<\left(\int_0^r x^{\alpha-1}(1-x)^{t\alpha-1}dx\right)\left(\int_0^1 x^{\alpha+j-1}(1-x)^{t\alpha-1}dx\right), \tag{19}$$

which can be expressed as

$$\left(\int_0^r x^{\alpha+j-1}(1-x)^{t\alpha-1}dx\right)\left(\int_0^r x^{\alpha-1}(1-x)^{t\alpha-1}dx+\int_r^1 x^{\alpha-1}(1-x)^{t\alpha-1}dx\right)<$$

$$<\left(\int_0^r x^{\alpha-1}(1-x)^{t\alpha-1}dx\right)\left(\int_0^r x^{\alpha+j-1}(1-x)^{t\alpha-1}dx+\int_r^1 x^{\alpha+j-1}(1-x)^{t\alpha-1}dx\right)\ . \tag{20}$$

We can now simplify the above and demand:

$$\left(\int_0^r x^{\alpha+j-1}(1-x)^{t\alpha-1}dx\right)\left(\int_r^1 x^{\alpha-1}(1-x)^{t\alpha-1}dx\right) < \left(\int_0^r x^{\alpha-1}(1-x)^{t\alpha-1}dx\right)\left(\int_r^1 x^{\alpha+j-1}(1-x)^{t\alpha-1}dx\right). \qquad (21)$$

Note next that on the left-hand side, $x^j \le r^j$, while on the right, $x^j \le r^j$, and this is true for all j = 0, 1, 2,… Therefore

$$\text{left-side integral} < r^j\left(\int_0^r x^{\alpha-1}(1-x)^{t\alpha-1}dx\right)\left(\int_0^1 x^{\alpha-1}(1-x)^{t\alpha-1}dx\right),$$

while

$$\text{right-side integral} > r^j\left(\int_0^r x^{\alpha-1}(1-x)^{t\alpha-1}dx\right)\left(\int_r^1 x^{\alpha-1}(1-x)^{t\alpha-1}dx\right).$$

Hence the claim is settled.

It thus follows that $\Delta(r; \alpha, t_1\alpha) \le \Delta(r; \alpha, \alpha) \le \Delta(r; \alpha, t_2\alpha)$, $\forall t_1 < 1 < t_2$, $\forall\alpha > 0$, $\forall r \in (0,1)$.

STATEMENT II

With the Δ function defined as in Statement I, $\Delta(r; \alpha, t\alpha) = 0.5$ is possible only when $t \lessgtr (1 - r)/r$, according as $r \lessgtr 0.5$ whatever be the value of α.

A satisfactory analytical proof of Statement II has so far eluded us. However, we have carried out extensive numerical computations and our results support the claim. In Table 6 we display some of the computations (see also Figure 3).

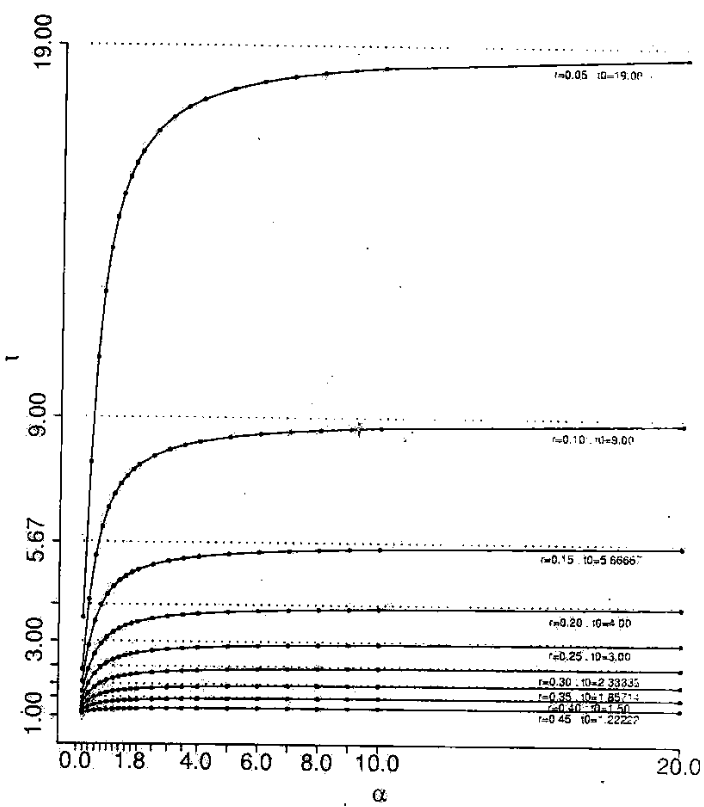

Figure 3.
Graph showing $t(\alpha)$ as a function of α satisfying the equation

$$\int_0^r \frac{1}{B(\alpha,t\alpha)} x^{\alpha-1}(1-x)^{t\alpha-1}dx = 0.5$$

for some selected values of r

(r = 0.05, 0.1,…,0.45).
The values of $t_0(r) = \frac{1}{r} - 1$
are also indicated along the t-axis.

Table 6. Table showing selected values of r and t for different values of α. The values of $t_0(r) = \frac{1}{r} - 1$ are also shown in the table.

$$\int_0^r \frac{1}{B(\alpha, t\alpha)} x^{\alpha-1}(1-x)^{t\alpha-1}dx = 0.5$$

α		r								
		.05	.10	.15	.20	.25	.30	.35	.40	.45
0.2	t	3.642	2.240	1.919	1.647	1.465	1.331	1.226	1.139	1.065
0.4		7.787	4.126	2.883	2.247	1.855	1.585	1.386	1.231	1.105
0.6		10.583	5.310	3.544	2.655	2.117	1.754	1.491	1.290	1.131
0.8		12.340	6.068	3.973	2.923	2.290	1.866	1.560	1.329	1.147
1.0		13.513	6.579	4.265	3.106	2.409	1.943	1.609	1.357	1.159
1.2		14.344	6.942	4.474	3.238	2.496	2.000	1.645	1.377	1.168
1.4		14.960	7.213	4.629	3.337	2.560	2.042	1.671	1.392	1.175
1.6		15.433	7.421	4.750	3.413	2.611	2.075	1.692	1.405	1.180
1.8		15.809	7.587	4.845	3.474	2.651	2.102	1.709	1.414	1.184
2.0		16.113	7.721	4.923	3.524	2.684	2.123	1.723	1.422	1.188
2.5		16.670	7.967	5.066	3.615	2.744	2.163	1.748	1.437	1.194
3.0		17.048	8.134	5.163	3.677	2.785	2.190	1.766	1.447	1.199
3.5		17.320	8.255	5.233	3.722	2.815	2.210	1.778	1.454	1.202
4.0		17.526	8.346	5.286	3.755	2.837	2.225	1.788	1.460	1.204
5.0		17.817	8.475	5.361	3.803	2.869	2.246	1.801	1.465	1.208
6.0		18.011	8.561	5.411	3.836	2.891	2.260	1.810	1.473	1.210
7.0		18.151	8.623	5.447	3.859	2.906	2.271	1.817	1.477	1.212
8.0		18.256	8.670	5.474	3.876	2.918	2.278	1.822	1.479	1.213
9.0		18.338	8.706	5.495	3.890	2.927	2.284	1.826	1.482	1.214
10.0		18.404	8.735	5.512	3.901	2.934	2.289	1.829	1.483	1.215
20.0		18.701	8.867	5.589	3.950	2.967	2.311	1.843	1.492	1.219
	t_0	19.00	9.00	5.66667	4.00	3.00	2.33333	1.85714	1.50	1.22222