NO-SHOW PARADOXES IN CONDORCET VOTING METHODS: A COMPUTATIONAL EXPERIMENT*

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ABSTRACT
A No-show Paradox can be described in a voting context as the fact that there is a voter who would be better not voting, since she prefers the winner resulting from the election when she abstains to the winner resulting when she votes honestly. Variations of the No-show Paradox, that affects ordinal aggregation methods have been identified and analysed in the context of voting methods. A method affected by one of these types of paradox can be manipulated in the sense that a voter may obtain better results when she does not show her preferences. A stronger version of the paradox, called Strong No-show Paradox, says that there is a voter whose favourite candidate loses the election if she votes honestly, but gets elected if she abstains. All Condorcet and scoring run-off methods are known to be affected by at least one version of the paradox and almost all of Condorcet ones are known to be affected by the strong versions (Pérez, 2001). The practical relevance of these paradoxes in the evaluation of a voting method depends, at least in part, on the probability of occurrence of a situation when the paradox effectively happens. As an extension of our first research step (Pérez **et al**., 2001), we present now the results obtained through a random simulation that explores the ocurrence of some versions of the paradox in some of the best known Condorcet voting methods.

Key words: Abstention, Paradox, Voting, Condorcet correspondences.

MSC: 91B12, 91B14, 68U20.

RESUMEN
En el contexto de los métodos de votación, la Paradoja de la Abstención se produce cuando existe al menos un votante que obtiene un mejor resultado si decide abstenerse y no votar, pues el candidato ganador que resulta en ese caso le es más preferido que aquel que resultaría si hubiese decidido votar honestamente (es decir, expresando sus verdaderas preferencias). En este trabajo se identifican y analizan algunas de las paradojas de la abstención, que afectan a los métodos ordinales de agregación de preferencias, en el contexto de los métodos de votación. Que un método de votación sufra algún tipo de paradoja de la abstención significa que ese método es de algún modo manipulable, pues evidencia la posibilidad de que un votante obtenga un mejor resultado si no vota expresando sus verdaderas preferencias. Una versión fuerte de la paradoja, denominada Paradoja Fuerte de la Abstención, se produce cuando existe un votante cuyo candidato favorito no resulta elegido si vota honestamente, pero si es elegido sí se abstiene. Se ha demostrado que todos los métodos de votación Condorcet y todos los métodos de votación Posicionales con Descarte están afectados por al menos alguna versión de la paradoja de la abstención y que casi todos los métodos Condorcet están sometidos a las versiones fuertes de la paradoja (Pérez, 2001). La importancia práctica de estas paradojas en la evaluación de los métodos de votación puede ser determinada, al menos parcialmente, mediante la probabilidad de ocurrencia de situaciones en las que la paradoja se produce de un modo efectivo. Como ampliación de nuestra primera investigación (Pérez **et al**., 2001), en este trabajo se presentan los resultados obtenidos a través de simulaciones aleatorias, explorando la ocurrencia de distintas versiones de la paradoja para algunos de los métodos de votación Condorcet más conocidos.

1. INTRODUCTION

The No Show Paradoxes, also called the Abstention Paradoxes (going to vote causes a worse result, from the point of view of the voter, than abstaining), are known to be very common in Condorcet voting methods. A mild version of the paradox was shown in Moulin (1988) to affect all of them, while some stronger versions were shown in Pérez (2001) and Jimeno **et al**. (2003) to affect all or almost all of them. However, it is worth to notice that the important family of positional voting methods (which includes the Borda and the Plurality methods) is free from these paradoxes. Only the run-off versions of these methods are known to be affected (see Lepelley and Merlin, 2001, for a probabilistic analysis of the paradox incidence on those methods).

Although we know that, as a consequence of some impossibility results, like those of Arrow and of Gibbard-Satherwhite, all ordinal voting methods are affected by some form of paradox (a failing to satisfy some intuitively reasonable property), it is important to know which methods are affected by every type of paradox and also to know the frequency of situations in which the paradox is present.

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In this paper we explore, through random simulations, the frequency of situations where different versions of the paradox (two weak versions, called weak optimist and weak pessimist, and a strong one) occur for the cases of five known Condorcet voting correspondences (Fishburn-Miller, Black, Copeland, Minmax and Top Cycle) and for some combinations of the parameters m (number of voters) and n (number of candidates).

Section 2 presents the terminology. Section 3 defines some indicators to measure the frequency of occurrence of the paradoxes and specifies the random experiment. Section 4 shows some of the results obtained in the simulation exercise, and presents some final comments and conclusions.

2. TERMINOLOGY AND REVIEW OF RESULTS

2.1. Terminology

Let $X = \{x_1, x_2,...,x_n\}$ be a set of two or more candidates. The preferences of any voter are assumed to be a complete order L over X. In general, we suppose this order to be linear (strict and complete). We say $L = x > y > z > t...$ or $L = x\ y\ z\ t...$ to denote the preference linear order in which x is the most preferred candidate, y is the second one, and so on, and a L b means that a is preferred to b in L.

Given the set of candidates $X = \{x_1, x_2, ..., x_n\}$, and any finite set $V = \{1, 2, ..., m\}$ with one or more voters, we call a **Situation** any pair (X, p), where p is a preference profile over X from V, that is to say, an m-tuple of orders over X, each one meaning the preferences of a voter over X. We call **unanimous** any situation (X, p), where all voters in profile p have exactly the same preferences over candidates in X.

Let us call **Voting Correspondence** (from now VC) any function f which maps any situation (X, p) to a non-empty subset of X, f(X, p). The elements of f(X, p) are the chosen candidates (the winners) over X from the preference profile p. When f(X, p) is required to have only one element for every situation (X, p), f is said to be a **Voting Function**. Since we will consider only anonymous VCs (all voters are equally considered), a preference profile over X from V can also be described by specifying how many of the m voters from V sustain any of the possible preference orders on X.

Given any X, any two disjoint sets of voters $V_1 = \{1, 2, ..., m_1\}$ and $V_2 = \{m_1 + 1, m_1 + 2, ..., m_1 + m_2\}$, and any two preference profiles p_1 and p_2 over X from, respectively, V_1 and V_2, we can merge these two profiles in order to obtain a new profile over X, but originated from $V_1 \cup V_2$ in this case. This new profile will be called $p_1 + p_2$.

Let (X, p) be any situation with n candidates and m voters. Given any two different candidates x, y from X, p(x, y) is computed adding the number of voters in p which strictly prefer x to y and half the number of voters indifferent between x and y. It is obvious that p(x, y) + p(y, x) = m. The square $n \times n$ matrix M_p, whose entries are p(x, y) (and without entries in the main diagonal), will be called the **Comparison Matrix** for (X, p).

For every candidate x, the sum of the off-diagonal row entries in M_p is called the **Borda Score** of x.

A VC f satisfies the **Translation Invariance** property if, for any two situations (X, p) and (X, q), q(x, y) = q(y, x) $\forall x,y \in X$, implies f(X, p) = f(X, p + q). That is to say, adding a group of voters whose preferences, as measured by its comparison matrix, are the same for every candidate, does not change the set of winners.

Candidate x is said to **beat** y, denoted by xW_py, if and only if p(x, y) > p(y, x). If, given a situation (X, p), x beats any other candidate, then x is called the **Condorcet** candidate for this situation. A VC f is called **Condorcet** if, for every situation (X, p), if there is a Condorcet candidate, it will be the only winner. Candidate x is said to **beat indirectly** y, denoted by xWW_py, if and only if there exists a sequence $x_0, x_1,...,x_k$, such that $x_0 = x$, $x_k = y$ and $x_0W_px_1W_p... W_px_k$.

We say x **covers** y if xW_py and (yW_pz implies xW_pz).

Following Fishburn (1977), the correspondence f is said **C1** if, for every situation (X, p), the set f(X, p) depends only on the W_p relation, and the correspondence f is said C2 if it is not a C1-Correspondence and for every situation (X, p), f(X, p) depends only on the Comparison Matrix M_p.

2.2. Some known results

Definition 1:

a) A Voting Function f satisfies the **Participation** property if, for any given pair of situations (X, p) and (X, v), where situation (X, v) is unanimous, f(X, p) = {x} and x is preferred to y in v implies f(X, p+v) ≠ {y}.

b) A VC f satisfies the **VC-Participation** property if for any given pair of situations (X, p) and (X, v), where situation (X, v) is unanimous. If $x \in$ f(X, p) and x is unanimously preferred to y in v, then $y \in$ f(X, p+v) implies $x \in$ f(X, p+v).

In words, when f satisfies Participation, if x is the winner for a situation and a set of identical voters who prefer x to y is added, candidate y will not become the winner. Thus, the new voters could not do better abstaining, because submitting their ballots would never result in the election of a less preferred candidate. Failing to satisfy Participation means that the No Show paradox sets in, because a potential voter could do better abstaining.

On the other hand, VC-Participation is a translation of the Participation property to the Voting Correspondences framework. It says that if candidate x is chosen for a situation and some new voters, with identical preferences such that they strictly prefer x to y, are added, candidate y will not be chosen if she is not accompanied by x. If, on the contrary, y is chosen and x is not, the new voters would have done better abstaining, because submitting their ballots they cause the election of a less preferred candidate. Thus, failing to satisfy this property also means that a No Show paradox sets in.

The incompatibility of the above two properties with the Condorcet property is shown in sections a and b of proposition 1 below, and it was established, respectively, in Moulin (1988) and Pérez (2001).

Proposition 1:

a) No Condorcet Voting Function satisfies the Participation property.

b) No Condorcet VC satisfies the VC-Participation property.

The following property is a weak version of VC-Participation.

Definition 2:

A VC f satisfies the **Positive Involvement** (**PI**) property if and only if, for any given pair of situations (X, p) and (X, v), where situation (X, v) is unanimous, if $x \in$ f(X, p) and x is preferred to any y in v, then $x \in$ f(X, p+v).

In other words, PI requires that, if candidate x is chosen, x will remain being chosen when new identical voters are added for whom x is preferred to any other candidate. The failure by a VC f to satisfy the PI property means that f suffers the **Positive Strong No Show Paradox**.

The following participation-type properties have an intermediate character in relation to the ones already presented. The non-fulfilment of these new properties supposes the appearance of new versions of the paradox.

Definition 3:

a) A VC f satisfies the **Optimistic Involvement** (**OptI**) property if, for any given pair of situations (X, p) and (X, v), where situation (X, v) is unanimous, there is a candidate $y \in$ f(X, p+v) such that no candidate $x \in$ f(X, p) is strictly preferred to y in v.

b) A VC f satisfies the **Pessimistic Involvement** (**PesI**) property if, for any given pair of situations (X, p) and (X, v), where situation (X, v) is unanimous, there is no candidate $y \in$ f(X, p+v) such that every candidate $x \in$ f(X, p) is strictly preferred to y in v.

In words, OptI means that, when a voter (or several identical voters) is added to the initial situation, we can be sure that at least one of the chosen candidates from the new situation is not worse, for that voter, than the best of the elected ones before. If this property is satisfied, we can ensure that there is no optimistic voter will expect a preferable result by abstaining.

In the same way, PesI means that, when a voter (or several identical voters) is added to the initial situation, we can be sure that at least one of the chosen candidates from the new situation is not worse, for that voter, than the worst of the elected ones before.

We will call **Optimistic and Pessimistic No Show Paradoxes** to the failure in the fulfilment of OptI and PesI properties, respectively.

The incompatibility of the above two properties with the Condorcet property, shown in sections a and b of proposition 2 below, and the implied relations between VC-Participation, Optimistic Involvement (OptI) and Positive Involvement (PI), shown in proposition 3 below, were established in Jimeno **et al**. (2003).

Proposition 2:

a) No Condorcet VC satisfies the OptI property.

b) No Condorcet VC satisfies simultaneously the PesI and Translation Invariance properties.

Proposition 3:

VC-Participation implies OptI, and OptI implies PI.

We end this section with the lemma 1 below which is useful to identify some important cases in which PI fails (it was proved in Moulin, 1988).

Lemma 1:

Given any Condorcet VC f, any situation (X, p) and any two candidates x and z, if f satisfies PI, then $p(x, z) < \text{Min}_{y\in X}p(z, y)$ implies $x \notin f(X, p)$.

2.3. Incidence of the Paradoxes in Known Condorcet Voting Correspondences

As shown by propositions 1 and 2, all Condorcet VCs suffer from the No Show Paradox (because they fail to satisfy VC-Participation) and from the Optimistic and Pessimistic No Show Paradoxes (because they fail to satisfy OptI and PesI properties).

Nevertheless, there exist some reasonable Condorcet VCs immune to the Positive Strong No Show Paradox: in particular, the Simpson-Cramer **Minmax** correspondence is free from this paradox, because it satisfies PI. However, as it was shown in Pérez (2001), almost all known Condorcet VCs suffer from this paradox, including the **Fishburn-Miller, Copeland, Black** and **Top Cycle** correspondences.

Let us briefly review their definitions:

Minmax:

$$f(X, p) \equiv \{x \in X: \text{Min}_{z\in X-\{x\}}\{p(x, z)\} \geq \text{Min}_{z\in X-\{y\}}\{p(y,z)\}\ \forall y\in X\}$$

Fishburn-Miller:

$$f(X,p) \equiv \{x\in X: \text{ there is no } y\in X \text{ such that } y \text{ covers } x\}$$

Copeland:

$$f(X,p) \equiv \{x \in X: \text{ the number of candidates } y, \text{ for which } xW_p y, \text{ is maximal}\}$$

Black:

$f(X, p) \equiv \{c\}$, if a Condorcet Candidate c exists,

$\{x \in X$: x has a maximal Borda Score$\}$, if there is no Condorcet Candidate.

Top Cycle:

$$f(X,p) \equiv \{x \in X: \text{ there is no } y\in X \text{ such that } y \text{ beats indirectly } x \text{ and } x \text{ does not beat indirectly } y\}$$

In the following sections we analyse the frequency of occurrence of the above mentioned paradoxes for the last four voting correspondences, and, for the MinMax correspondence, we analyse the frequency of occurrence of any of the above paradoxes (except the strong one).

3. INDICATORS FOR THE FREQUENCY OF OCCURRENCE OF THE PARADOX

3.1. Identification of the occurrence of a paradox

The identification of the occurrence of any of the above defined No Show paradoxes (that is to say the identification of the failure in satisfying the corresponding participation property) is accomplished, for a given (X, p) situation and for an f voting correspondence, by means of comparing the result f(X, p) with f(X, p'), being (X, p') one of the following variants to (X, p):

- (X, p') = (X, p + v), where v represents a profile of added identical voters

- (X, p') = (X, p - u), where u represents a profile of eliminated identical voters

With these variations to a given (X, p) situation, we are modeling the participation and abstention choices of the voters. In this sense, we distinguish two different types of indicators of paradox occurrence: the potential and the effective ones. While the potential indicator requires the addition of voters (nonexistent voters that might exist are involved), the effective indicator requires the elimination of voters (existent or effective voters are not taken into account).

Strong Effective No Show Paradox (SE)

Occurs when there are one or several effective voters with the same preferences such that their favourite candidate is not a winner but would become a winner had these voters abstained. This means an effective breaking of the Positive Involvement property, in such a way that the Positive Strong No Show Paradox is effectively happening.

Definition 3:

Given a situation (X, p) and a voting correspondence f, the **occurrence of the Strong Effective No Show Paradox (SE)** means that there is a set u of identical voters in (X, p) with a favourite candidate x such that

$$x \in f(X, p - u) - f(X, p).$$

Optimistic Effective No Show Paradox (SE)

Occurs when there are one or several effective voters with the same preferences such that if they had abstained there would be a candidate winner strictly better for all of them to any of the actual winning candidates. This means an effective breaking of the Optimistic Involvement property, in such a way that the Optimistic No Show Paradox is effectively happening.

Definition 4:

Given a situation (X, p) and a voting correspondence f, the **occurrence of the Optimistic Effective No Show Paradox (SE)** means that there is a set u of identical voters in (X, p) such that $\exists\, x \in f(X, p - u)$, (x is strictly preferred to y in u, $\forall y \in f(X, p)$).

Pessimistic Effective No Show Paradox (PE)

Occurs when there are one or several effective voters with the same preferences such that they strictly prefer all winning candidates if they had abstained than at least one of the actual winning candidates. This means an effective breaking of the Pessimistic Involvement property, in such a way that the Pessimistic No Show Paradox is effectively happening.

Definition 5:

Given a situation (X, p) and a voting correspondence f, the occurrence of the Pessimistic Effective No Show Paradox (PE) means that there is a set u of identical voters in (X, p) such that $\exists$ x $\in$ f(X, p), (y is strictly preferred to x in u, $\forall$y $\in$ f(X, p - u)).

Strong Potential No Show Paradox (SP)

Occurs when it is possible to consider one or several potential new voters with the same preferences such that their favourite candidate is a winner now but would become a loser had these voters joined the election. This means a breaking of the Positive Involvement property, in such a way that the Positive Strong No Show Paradox is not effectively happening, but would happen if these potential voters had gone voting.

Definition 6:

Given a situation (X, p) and a voting correspondence f, the **occurrence of the Strong Potential No Show Paradox (SP)** means that there is a set v of identical new voters with a favourite candidate x such that x $\in$ f(X, p) - f(X, p + v).

We could define in a similar way the occurrence of Potential Optimistic/Pessimistic No Show Paradoxes.

There are two basic differences between the effective and potential occurrences of a No Show Paradox, both related to the asymmetric nature of adding and deleting voters.

The first difference concerns to the feasibility and probability of occurrence. Since any effective occurrence requires the effective presence of certain voters (whose elimination would produce counterintuitive results), while the corresponding potential occurrence only requires the theoretical possibility of adding certain voters (whose addition would produce counterintuitive results), it is in general less likely to find effective occurrences of a paradox than the corresponding potential ones.

The second difference concerns to the computation difficulty in the task of identifying any occurrence. Since the identification of an effective occurrence requires analysing only the voters present in the situation, while the identification of a potential occurrence requires the exploration of all theoretically possible new voters, the computational complexity in the last case is notably higher. In fact, since every permutation of candidates is a possible representation of the preferences of a new voter, we conjecture that for some voting correspondences such identification problem is NP-hard.

For the case of Condorcet voting correspondences, the lemma 1 of section 2.2 allows identifying many of the potential occurrences (in our opinion, most of them) in a computationally simple way, thus leading to the definition of the following type of occurrence of the Strong No Show paradox, denoted C-SP to remind us that it is valid only for Condorcet VCs.

C-Strong Potential No Show Paradox (C-SP)

Definition 7:

Given a situation (X, p) and a Condorcet voting correspondence f, the **occurrence of the C-Strong Potential No Show Paradox (C-SP)** means that $\exists$ x $\in$ f(X, p) and $\exists$ z $\in$ X such that p(x, z) < Min y $\in$ X p(z, y).

Observe that, because of lemma 1, if there is an occurrence of C-SP, f fails to satisfy PI. The specific reason is that introducing m - 2p(x, z) identical voters with preferences x > z >..., candidate z becomes the Condorcet candidate.

Therefore, the just defined C-SP occurrence is a sufficient condition, for Condorcet voting correspondences, of the occurrence of a SP occurrence. But, even in Condorcet voting correspondences, this way of precipitating the paradox is not the only way, thus the number of C-SP occurrences is just a lower bound of the number of SP occurrences.

The positive side is that the identification of a C-SP occurrence only requires to find a special candidate, if she exists, in the comparison matrix, a computationally easy task.

3.2. Computational Experiment

To explore the frequency of situations where the previous defined paradoxes occur, we have developed an experiment consisting in measuring the occurrences of each of them as a percentage of the total number of generated situations. The sample over which we have worked was randomly generated in de following way:

1. The number n of candidates takes the values 3 to 12.

2. The number m of voters takes the values 5 to 45, with 10 as step.

3. Given n and m, we randomly generate 1000 situations in the following manner: an order of preferences for each voter is randomly obtained, in such a way that to each voter corresponds, equally likely, any of the n! possible orders of candidates in X, independently of the other voters.

The main hypothesis of the random experiment is the Hypothesis of Impartial Culture, which means that the preferences of the voters are independent and uniformly distributed among all possible orders of preferences.

For every of these 1000 random situations, we have computed the chosen candidate set for the voting correspondences: Fishburn-Miller, Copeland, Black, MinMax and Top Cycle, and we analyse if it is one of the above paradox type situation. The percentages of cases of paradox situations are the indicators of frequency. We have computed this frequency for each type of the above defined paradoxes (SE, OE, PE, SP and C-SP).

4. RESULTS OF THE RANDOM SIMULATION

The following tables show a representative sample of results, corresponding to an odd number of voters, odd and even number of candidates, and with the percentages rounded to the first decimal digit when not exact:

4.1. $SE_{n,m}$ indicator (estimator of the frequency of occurrence of Type SE situations):

Table 1. $SE_{n,m}$ indicator for the Fishburn-Miller correspondence.

	5 voters	15 voters	25 voters	35 voters	45 voters
3 candidates	4.6	14	17	20.6	21.6
4 candidates	8.1	18.8	21.4	24.8	23.5
5 candidates	9.4	26.3	23.6	27.6	23.8
6 candidates	10.5	30.1	31.7	30.6	29.1
7 candidates	11.2	32.2	36	36	32.4
8 candidates	14.2	34.4	37.3	38.2	36.9
9 candidates	13.5	36.5	39.2	41.4	38.2
10 candidates	12.2	40	41.6	38.9	43.4
11 candidates	12.1	40.9	41.9	46.7	45.7
12 candidates	13.8	39.1	46.2	46	45.1

Table 2. $SE_{n,m}$ indicator for the Copeland correspondence.

	5 voters	15 voters	25 voters	35 voters	45 voters
3 candidates	4.6	14	17	20.6	21.6
4 candidates	5.8	13.3	15.2	16.8	16.5
5 candidates	4.6	12.9	13.6	18.2	13.8
6 candidates	3.8	13.5	17.2	17.2	15.6
7 candidates	4.2	12.9	15.8	18	15.8
8 candidates	4	11.2	16.3	15.3	16.2

9 candidates	2.8	12.1	15	18.8	17.4
10 candidates	2.5	10.8	12.4	15.4	17.7
11 candidates	2.3	9.8	12.8	15.3	17.8
12 candidates	2.5	8.6	15.3	14.6	17.3

Tabla 3. $SE_{n,m}$ indicator for the Black correspondence.

	5 voters	15 voters	25 voters	35 voters	45 voters
3 candidates	0	2.9	3.9	3.4	3.2
4 candidates	0.4	3.9	5.1	4.9	4.7
5 candidates	1.1	5.1	5.5	6	6.4
6 candidates	1	4.5	5.9	5.6	4.8
7 candidates	1.3	4.6	5.4	6.5	4.3
8 candidates	1.4	4.1	5.7	4.5	5.3
9 candidates	1.8	3.9	4.4	5.9	4.8
10 candidates	1.1	3.6	4.1	4.9	5.2
11 candidates	0.7	4.5	3.8	4.4	4.5
12 candidates	0.9	3	4.9	4.8	4.1

Table 4. $SE_{n,m}$ indicator for the Top Cycle correspondence.

	5 voters	15 voters	25 voters	35 voters	45 voters
3 candidates	0	7.7	14	18.5	20.5
4 candidates	0	6.7	12.2	18.8	21.1
5 candidates	0	2.2	6.3	12.5	13.7
6 candidates	0	0.7	1.7	3.6	5.4
7 candidates	0	0	0.4	0.5	1
8 candidates	0	0	0	0.2	0.1
9 candidates	0	0	0	0	0
10 candidates	0	0	0	0	0
11 candidates	0	0	0	0	0
12 candidates	0	0	0	0	0

As can be seen in tables 1 to 4, the global frequency of occurrences depends on the voting method. The highest frequency corresponds to Fishburn-Miller, then follows Copeland, and the lowest frequency corresponds to Black and (for six or more candidates) to Top Cycle. Minmax is free from this strong paradox.

The frequency pattern, as a function of the number of candidates and voters, also depends on the voting method. For the Fishburn-Miller case the frequency tends to increase with the numbers of candidates and voters, reaching the maximal frequency (46.7%) for n = 11 and m = 35 and the minimal frequency (4.6%) for n = 3 and m = 5. For Copeland the main factor seems to be the number of voters, increasing the frequency with this number. With m = 5 the frequency is lower than 5% while with m = 45 the frequency is higher than 13%.

On the other hand, for the Black case the frequency is low (never higher than 6%) and approximately uniform. Lastly, for the Top Cycle case the frequency tends to increase strongly (for any given number of candidates) with the number of voters, but decreases strongly with the number of candidates. The maximal frequency (21%) is reached for n = 5 and m = 45, but when n>6 the frequency is always lower than 1%.

4.2. $OE_{n,m}$ indicator (estimator of the frequency of occurrence of Type OE situations):

Table 1. $OE_{n,m}$ indicator for the Fishburn-Miller correspondence.

	5 voters	15 voters	25 voters	35 voters	45 voters
3 candidates	4.6	14	17	20.6	21.6
4 candidates	9.8	19.9	22.2	26.1	25.1
5 candidates	12.5	29.2	24.7	28.4	24.4
6 candidates	15.5	35	33.8	32	29.5
7 candidates	18.9	39.6	39.6	37.2	33.8
8 candidates	21.6	41.1	41.3	42	38.4
9 candidates	25.2	43.8	45.7	44.5	39.4
10 candidates	21.2	48.9	48.7	42	44.9
11 candidates	22.6	52	49.9	50.3	49
12 candidates	25.9	49.2	53.4	50.8	49.5

Table 2. $OE_{n,m}$ indicator for the Copeland correspondence.

	5 voters	15 voters	25 voters	35 voters	45 voters
3 candidates	4.6	14	17	20.6	21.6
4 candidates	6.7	15.1	17	21.2	21.7
5 candidates	7	17.4	17.5	22.3	18.9
6 candidates	5.9	19.3	23.8	21	20.6
7 candidates	7.2	20.7	23.4	25.4	22.2
8 candidates	6.5	19.6	25.7	25.7	24.3
9 candidates	6.5	20.1	24	28.8	24.2
10 candidates	5.3	22.9	23.4	25.6	28.3
11 candidates	5.1	21.5	26	28.1	29.1
12 candidates	6	21.5	29.4	28.8	28.3

Table 3. $OE_{n,m}$ indicator for the Black correspondence.

	5 voters	15 voters	25 voters	35 voters	45 voters
3 candidates	0	2.9	4.2	4	3.8
4 candidates	0.6	4.6	6	6.8	7
5 candidates	1.6	6	6.6	7.9	8.7
6 candidates	1.7	5.4	6.4	6.2	5.8
7 candidates	2.3	6.5	6.7	7.4	5.2
8 candidates	2.5	5.9	6.9	5.1	6.5
9 candidates	3.3	6.2	6.1	7.1	5.9
10 candidates	2.5	5.8	6	6.3	6.1
11 candidates	1.3	7.4	6.2	5.3	5.3
12 candidates	2.4	5.5	7.1	6.1	5.7

Table 4. $OE_{n,m}$ indicator for the MinMax correspondence.

	5 voters	15 voters	25 voters	35 voters	45 voters
3 candidates	0	0	0	0	0
4 candidates	0	0.3	0.6	1.4	1.1
5 candidates	0	0.9	2.3	2.2	2.1
6 candidates	0	2.5	3.6	3.5	3.6
7 candidates	0	2.8	4.1	4.5	5.6
8 candidates	0	4.2	6.7	7.5	6.3
9 candidates	0.1	5.7	6.1	10.1	8.4
10 candidates	0	6.8	8.1	9.4	9.7
11 candidates	0.1	7.2	9.6	10.8	11.7
12 candidates	0	9.3	11.7	11.8	13.5

Table 5. $OE_{n,m}$ indicator for the Top Cycle correspondence.

	5 voters	15 voters	25 voters	35 voters	45 voters
3 candidates	0	7.7	14	18.5	20.5
4 candidates	0	7.3	13.3	20.3	21.8
5 candidates	0	2.9	6.8	13.7	15.5
6 candidates	0	0.7	1.8	3.9	6.9
7 candidates	0	0	0.7	0.5	1.3
8 candidates	0	0	0	0.3	0.1
9 candidates	0	0	0	0	0
10 candidates	0	0	0	0	0
11 candidates	0	0	0	0	0
12 candidates	0	0	0	0	0

We can observe that, analogously to the case of the SE indicator (which computed the frequency of occurrences of a stronger, and consequently rarer, paradox), in the case of the OE indicator, the highest frequency corresponds to Fishburn-Miller, then follows Copeland, and lastly follows Black and (for seven or more candidates) Top Cycle.

The frequency pattern for the OE indicator, as a function of the number of candidates and voters, also depends on the voting method. For the Fishburn-Miller case the frequency increases with n and m, reaching its maximum (51%) for n = 12 and m = 35 and its minimum (5%) for n = 3 and m = 5. For the Copeland case the frequency increases mainly with m, remaining lower than 8% for m=5 and higher than 18% for m = 45.

For the Black case the frequency is also low (never higher than 9%) and approximately uniform. For Top Cycle the frequency increases with m (reaching a maximum of 22% for n = 4 and m = 45) and decreases strongly with n (for n > 9 the frequency almost vanishes). Lastly, with respect to Minmax, the only method free from the strong paradox (null frequencies for the SE indicator), the frequency is in general low (with a maximum of 13% for n = 12 and m = 45), increasing slowly with the numbers of candidates and voters.

It is worth noting that, being the Optimistic paradox (whose degree of incidence is measured by OE) weaker than the Positive Strong paradox (incidence measured by SE), the frequencies obtained by the OE indicator must be higher than those obtained by the SE indicator, but they happen to be only slightly higher.

4.3. $PE_{n,m}$ indicator (estimator of the frequency of occurrence of Type PE situations):

Table 1. $PE_{n,m}$ indicator for the Fishburn-Miller correspondence.

	5 voters	15 voters	25 voters	35 voters	45 voters
3 candidates	0	0	0	0	0
4 candidates	0.1	1.5	3.5	6	8.6
5 candidates	0	0.8	3.1	3.4	5.8
6 candidates	0	0.4	0.3	1.4	1.7
7 candidates	0	0	0.4	0.4	0.3
8 candidates	0	0.1	0.5	0.2	0.5
9 candidates	0	0.1	0.5	0.6	1
10 candidates	0	0.2	0.1	0.6	0.9
11 candidates	0	0.2	0.3	0.9	0.8
12 candidates	0	0.2	0.4	1.1	1.3

Table 2. $PE_{n,m}$ indicator for the Copeland correspondence.

	5 voters	15 voters	25 voters	35 voters	45 voters
3 candidates	0	0	0	0	0
4 candidates	0.9	3.9	6.1	8.2	10.8
5 candidates	1.1	8.8	10.8	12.2	14.9
6 candidates	1.9	12.6	15.2	14.6	16.6
7 candidates	3.5	13.8	19	19.6	19
8 candidates	3.4	16.7	20.8	21.5	21.8
9 candidates	5.4	19.1	23.4	23.5	23.1
10 candidates	4.2	18.4	21.7	25.7	26.1
11 candidates	4.2	18.7	21.7	25.4	25.9
12 candidates	6.1	21.1	26.6	26.9	28.1

Table 3. $PE_{n,m}$ indicator for the Black correspondence.

	5 voters	15 voters	25 voters	35 voters	45 voters
3 candidates	0	1.1	2.4	2.9	2.8
4 candidates	0	2.8	4.8	5.7	6.2
5 candidates	0.3	3.3	5	6.3	7.4
6 candidates	0.7	4.4	5.4	5.2	5.4
7 candidates	0.8	4.5	5	6.3	3.9
8 candidates	0.7	4.3	5.6	4.4	5.1
9 candidates	1.5	3.7	4.9	6	5.2
10 candidates	1.1	4.5	4.6	5.8	5.4
11 candidates	0.9	5.3	5	4.5	4.7
12 candidates	1.4	4.2	5.9	5.3	5.3

Table 4. $PE_{n,m}$ indicator for the MinMax correspondence.

	5 voters	15 voters	25 voters	35 voters	45 voters
3 candidates	0	0	0	0	0

4 candidates	0	0.5	0.6	0.8	1.2
5 candidates	0	2	1.8	1.5	1.9
6 candidates	0	3.3	3.9	3.4	3.6
7 candidates	0	4.7	2.9	4.6	5.2
8 candidates	0	5.7	6.4	6.6	6.5
9 candidates	0.1	8	9.3	9	8.3
10 candidates	0	8	9.7	8.5	8
11 candidates	0.1	10.9	10.7	9.8	9.3
12 candidates	0	12.1	12.1	11.5	11.4

Table 5. $PE_{n,m}$ indicator for the Top Cycle correspondence.

	5 voters	15 voters	25 voters	35 voters	45 voters
3 candidates	0	0	0	0	0
4 candidates	0.6	3.8	5.6	5.6	4.9
5 candidates	0	6.8	9.3	8.5	10.7
6 candidates	1.5	8.9	10.9	12.3	14.5
7 candidates	2.5	12.3	15	18.2	16.3
8 candidates	4.3	14.6	15.7	18	17.8
9 candidates	3.9	13.7	17.4	16.7	17.8
10 candidates	4.4	14.7	15.4	20.3	19.2
11 candidates	3.4	14	15.3	18.7	17
12 candidates	3.8	14	13.4	16.5	17.3

The behaviour shown by the five methods with respect to the Pessimistic paradox (measured by the PE indicator) greatly differs from that observed with respect to the Optimistic and the Strong paradoxes (measured by the OE and SE indicators). Now we find that the higher frequency of occurrences corresponds to Copeland, followed by Top Cycle, MinMax, Black and, lastly, Fishburn-Miller.

The frequency pattern for the PE indicator, as a function of the number of candidates and voters, depends on the voting method in the following way. For the Fishburn-Miller case the frequency increases only m, reaching its maximum (9%) for m = 35 and its minimum (0.1%) for m = 5. For the Copeland case, and leaving aside the case of 3 candidates, for which the frequency is 0%, the frequency increases with n and m, reaching its maximum (28%) for n = 12 and m = 45 and its minimum (1%) for n = 4 and m = 5.

For the Black case the frequency is low (never higher than 8%) and increasing slowly with n and m. For Minmax, the frequency is in general low (slightly higher than for Black) generally increasing with the number of candidates. And for Top Cycle the frequency increases with m and n.

It is worth noting the very different behaviour of Top Cycle with respect to the other methods. A possible conjecture (to be analysed in a deeper way) to explain this fact is the very low resoluteness exhibited by this methods in some situations, in which all or almost all candidates are selected as winners.

4.4. $C\text{-}SP_{n,m}$ indicator (estimator of the frequency of occurrence of Type C-SP situations):

Table 1. $C\text{-}SP_{n,m}$ indicator for the Fishburn-Miller correspondence.

	5 voters	15 voters	25 voters	35 voters	45 voters

3 candidates	2	5.3	9.5	7.5	7.7
4 candidates	5.8	12.6	14.8	16	16.2
5 candidates	6.9	19.2	21.3	21.3	21.6
6 candidates	12.2	26.3	26.8	25.5	28.6
7 candidates	12.8	30.8	32.4	32.6	32.9
8 candidates	17.7	34.1	38.6	37.6	38.1
9 candidates	18.6	38.8	41.6	41.8	41.5
10 candidates	21.4	39.1	41.2	44.9	44
11 candidates	24.3	43	46	47.2	48.9
12 candidates	26.4	46.7	47.4	48.9	53

Table 2. C-$SP_{n,m}$ indicator for the Copeland correspondence.

	5 voters	15 voters	25 voters	35 voters	45 voters
3 candidates	2	5.3	9.5	7.5	7.7
4 candidates	4.2	9.7	11.2	12	13
5 candidates	3	11.5	12.8	11.6	17.3
6 candidates	4	12.7	13.9	14.1	16.9
7 candidates	4.4	13.5	15.7	16	16.3
8 candidates	5.4	14.7	16	17.3	19.1
9 candidates	5.7	13.9	16.9	17.8	18.5
10 candidates	6.4	13.5	15.6	20.5	18.6
11 candidates	5.4	15.9	17.4	16.9	20.4
12 candidates	4.9	15.5	17.9	18.8	20.6

Table 3. C-$SP_{n,m}$ indicator for the Black correspondence.

	5 voters	15 voters	25 voters	35 voters	45 voters
3 candidates	0	0.2	1.3	0.9	1.1
4 candidates	0	1.6	2.1	2.9	3.3
5 candidates	0.3	3.1	5	3.6	5.5
6 candidates	0.1	4	4.8	4.8	5.9
7 candidates	0.3	3.9	6.2	5.8	6.9
8 candidates	1	5.6	6.6	7.8	9.8
9 candidates	0.5	5	7.3	8.3	8.6
10 candidates	1.8	6.1	8.6	10.1	9.2
11 candidates	1.5	7.5	10.1	8.4	11.4
12 candidates	1.6	8.2	8.4	10.6	11.8

Table 4. C-$SP_{n,m}$ indicator for the Top Cycle correspondence.

	5 voters	15 voters	25 voters	35 voters	45 voters
3 candidates	2	5.3	9.5	7.5	7.7
4 candidates	7.3	13.8	15.3	16.4	16.9

5 candidates	11.5	21.8	23.2	22.7	23
6 candidates	19.8	30.4	29.7	27.7	29.7
7 candidates	21.6	34.5	34.9	35	36.2
8 candidates	30.4	39.8	42.1	41.1	40.1
9 candidates	31.1	44.1	45	44.1	44
10 candidates	37.1	44.7	44.4	47.4	46.3
11 candidates	41.1	47.4	50.1	49.9	51.4
12 candidates	43.8	51.8	50.8	51	54.8

MinMax is free from these potential occurrences. As we see in tables 1 to 4 for the other methods, the frequency of those occurrences, as measured by the C-$SP_{n,m}$ indicator, is for Black consistently lower than for Copeland, and it is for Copeland consistently lower than for Fishburn-Miller and for Top Cycle.

This frequency trend is analogous (except for the case of the Top Cycle) to those observed for the SE and OE indicators. This fact can be, at least partly, explained by means of two reasons. The first reason is that the discrimination ability or resoluteness is higher for Black, medium for Copeland, and lower for Fishburn-Miller (and we conjecture that, in general, the discrimination ability is inversely correlated with the frequency of occurrence of no-show paradoxes in Condorcet correspondences). The second reason is that the Black correspondence is, by definition, the most similar one to the Borda correspondence, which is completely free from any of these paradoxes.

Although Top Cycle has a similar behaviour than Fishburn-Miller in this case, the Top Cycle correspondence seems to have a different nature, a different logic in the process of selecting the winners, because its definition is inconsistent with the covering and domination relations that are on the basis of (or at least are consistent with) the definitions of the other methods. Let us also note that Copeland and Fishburn-Miller are refinements of Top Cycle (every winner in Copeland or Fishburn-Miller is a winner in Top Cycle).

Because of computational difficulties already alluded, we have not yet completed the results corresponding to the SP indicator, corresponding to occurrences of Strong Potential No Show Paradox. But some partial results obtained for 3 to 6 candidates show results for SP very similar to those of C-SP (the biggest difference in the percentages is lower than 0.5 % and there seems to be no identifiable pattern in these differences, perhaps caused mainly by the random nature of the results). Therefore, the C-SP indicator seems to be a very convenient proxy for the SP indicator.

Finally, it is worth noting that there are some asymmetries between the potential and effective indicators of the strong paradox, with respect to the different methods. In Fishburn-Miller and Black methods, the frequency for C-SP is lower than that for SE for the cases with a few number of candidates, and higher for the cases with many candidates. For the Copeland method, the C-SP frequency is higher for the many voter cases. And for the Top Cycle method, the C-SP frequency behaves in a radically different way than the SE one, increasing strongly with the number of candidates, and in a lesser extent with the number of voters (reaching a percentage of 55% when $n = 12$ and $m = 45$).

SOME CONCLUDING COMMENTS

The found frequencies of occurrence of these paradoxes are not globally negligible. They depend strongly on the paradox versions, on the methods and on the numbers of candidates and voters, but in many typical situations (about 6 candidates and about 25-45 voters) the strongest version of the paradox occurs in about 2-6 % of the situations for two methods and in about 15-30 % of the situations for the other two methods. On the other hand, the frequencies for the other versions are in general slightly higher.

Although it is necessary a deeper and more detailed analysis, it seems to be in general, for Condorcet voting correspondences, a negative correlation between the discrimination ability of these correspondences and the frequency of occurrence of no-show paradoxes. The five methods approximately ordered by decreasing discrimination ability are: Top Cycle and Fishburn-Miller, Copeland, Minmax and Black. Therefore, for one of the least resolute methods, Fishburn-Miller, the frequency is consistently higher than for Copeland, and for Copeland, the frequency is consistently higher than for Black, the most resolute. The cases of Minmax (free from some of the paradoxes) and Top Cycle (whose logic in selecting a winner is different from that of the other methods), are somewhat special.

The research can be extended in two directions:

- Other methods. It may be interesting to compare these frequencies with the corresponding to some Non-Condorcet methods, like Plurality Runoff and others.

- Other versions of the paradox. It may also be interesting to compute the frequencies corresponding to some other known versions of the paradox, like those in which ties are allowed in the preferences of voters, studied in Pérez (2001) and Jimeno **et al**. (2003).

REFERENCES

FISHBURN, P.C. (1977): "Condorcet Social Choice Functions". **SIAM Journal on Applied Mathematics**, 33: 469-489.

JIMENO, J.L.; J. PÉREZ and E. GARCÍA (2003): "Some results concerning No Show Paradoxes". communication to the **XXVIII Simposio de Análisis Económico**, Sevilla (Spain).

LEPELLEY, D. and V. MERLIN (2001): "Scoring run-off paradoxes for variable electorates". **Economic Theory** 17, 53-80.

MOULIN, H. (1988): "Condorcet's Principle Implies the No Show Paradox". **Journal of Economic Theory** 45: 53-64.

PÉREZ, J. (2001): "The strong No Show Paradoxes are a common flaw in Condorcet voting correspondences", **Social Choice and Welfare** 18: 601-616.

PÉREZ, J.; E. MOKOTOFF and J.L. JIMENO (2001): "The Frequency of Occurrence of No Show Paradoxes in Some Ordinal Aggregation Methods", **Sociedad Argentina de Información e Investigación Operativa**, Buenos Aires (Argentina), September, 3, 107-116.