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A SIMULATION STUDY OF THE LOCAL LINEARIZATION METHOD FOR THE NUMERICAL (STRONG) SOLUTION OF STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY ALPHA-STABLE LÉVY MOTIONS

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ABSTRACT.
A new variant of Local Linearization (LL) method is proposed for the numerical (strong) solution of differential equations driven by (additive) alpha-stable Lévy motions. This is studied through simulations making emphasis in comparison with the Euler method from the viewpoint of numerical stability. In particular, a number of examples of stiff equations are shown in which the Euler method has explosive behavior while the LL method correctly reproduces the dynamics of the exact trajectories.

RESUMEN.
Se propone una nueva variante del método de Linealización Local (LL) para la solución (fuerte) de ecuaciones diferenciales con respecto a procesos de Lévy alfa-estables (aditivo). A través de simulaciones se estudia el método comparación con el método de Euler desde el punto de vista de la estabilidad numérica. En particular, a partir de un número de ecuaciones rígidas ("stiff"), se muestra que el método de Euler tiene un comportamiento explosivo mientras que el método LL reproduce correctamente la dinámica de las trayectorias exactas.

Key words: Stochastic Differential Equations, Local Linearization Method, Stable Distributions, Lévy Process, Numerical Stability.

MSC: 60G52, 60H35.

1 INTRODUCTION

There exist a wide range of methods for computing approximate solutions of stochastic differential equations (SDEs) driven by Brownian motions (see e.g. Kloeden and Platen, 1995). In contrast, up to now the only approach that has been followed for the numerical (strong) solution of stochastic differential equations driven by Lévy processes and semimartingales is the Euler method (see Protter, 1985; Karandikar, 1991; Kohatsu-Higa and Protter, 1991; Kurtz and Protter, P. 1991; Janicki *et al*., 1993; Protter, 1995, Section V.4, and references therein).

In the present paper we carry out a simulation study on a version of the Local Linearization (LL) method for the numerical (strong) solution of stochastic differential equations driven by alpha-stable Lévy motions. In recent years, the LL approach has been developed for a variety of differential equations; e.g., ordinary differential equations (Jiménez *et al.*, 2002), Ito stochastic differential equations (Ozaki, 1992 and Biscay *et al.*, 1996) and random differential equations (Carbonell *et al.*, 2005). The underlying idea of LL method for SDEs driven by Lévy motions is basically that followed for these other types of equations.

The organization of the paper is as follows. In Section 2 the Local Linearization method for SDEs driven by Lévy motions is presented. Algorithmic aspects of the method are discussed in Section 3. Finally, a comparative study between the LL and Euler methods is carried out through simulations in Section 4. Numerical stability of the approximate solutions is emphasized. In particular, two examples of stiff equations are shown in which the Euler method has an explosive behavior for moderate step sizes while the LL method reproduces correctly the underlying dynamics.

2 THE LOCAL LINEARIZATION METHOD

The Local Linearization (LL) method for Ito's SDEs with additive noise is derived from the following steps: (1) the local linearization of the drift coefficient of the SDE over each time interval of the discretization by means of its first order deterministic Taylor expansion or a truncated Ito-Taylor expansion, (2) the analytic computation of the solution of the resulting linear SDE, and (3) the approximation of the Ito's integral involved in the solution obtained in step (2). There are some variations of the LL method that can be seen in Jiménez *et al.*, 1999.

A similar approach will be followed to obtain a version of the LL method for SDEs driven by (additive) Lévy noise. Specifically, we consider the following *d*-dimensional non-autonomous differential equation with additive noise:

$$d\mathbf{X}_t = \mathbf{f}(t, \mathbf{X}_t)dt + \mathbf{G}(t)d\mathbf{L}_t, \qquad \forall t \in [t_0, T] \tag{1}$$

where $\mathbf{X}_{t_0} = \mathbf{X}_0 \in \Re^d$, and $\mathbf{L}_t$ a standard *m*-dimensional alpha-stable Lévy motion. That is, $\mathbf{L}_t$ is an adapted stochastic process $\mathbf{L}_t = \{\mathbf{L}_t(\omega) : t \geq 0, \omega \in \Omega\}$, $\mathbf{L}_0 = 0$ a.s., it has independent increments, it is continuous in probability, and for all $t \geq s$ the components of the vector $\mathbf{L}_t - \mathbf{L}_s$ are independent with $S_\alpha\left((t-s)^{1/\varepsilon}, \beta, \mu\right)$ distributions (see Samorodnitsky, 1994 and Protter, 1995 for more details on standard definitions and notations concerning stable distributions and Lévy processes). In this work we set $\beta = \mu = 0$, i.e., $\mathbf{L}_t$ is a symmetric alpha-stable Lévy motion.

It is also assumed that $\mathbf{f}(t, \mathbf{X}_t) : \Re_+ \times \Re^d \to \Re^d$ is a differentiable function and satisfies a Lipschitz-type condition, and $\mathbf{G}(t)$ is a matrix function with values in $\Re^{d \times m}$.

The SDEs (1) can also be written in integral form:

$$\mathbf{X}_t = \mathbf{X}_0 + \int_0^t \mathbf{f}(s, \mathbf{X}_{s-})ds + \int_0^t \mathbf{G}(s)d\mathbf{L}_s. \tag{2}$$

Let $\pi = \{t_i, i = 1, 2, \cdots k_N : 0 = t_0 < t_1 < \cdots < t_{k_N} = T\}$ be a sequence of non-random numbers; i.e., a non-random partition of the interval [0, *T*] or *time discretization*. The LL approximation will be derived on the basis of the first-order Taylor expansion of the drift coefficient $\mathbf{f}(t, \mathbf{X}_t)$ over each time interval $]t_n, t_{n+1}]$ around the point $]t_n, \mathbf{X}_{t\,n}]$. For this, write the equation (2) as

$$\mathbf{X}_t = \mathbf{H}_{\mathbf{n.t}} + \int_{t_n}^t \mathbf{J}_\mathbf{n} \mathbf{X}_{s-} ds, \tag{3}$$

where

$$\mathbf{J_n} = \mathbf{J}(t_n, \mathbf{X}_{t_n}) = \frac{\partial}{\partial x}\mathbf{f}(t_n, \mathbf{X}_{t_n})$$

$$\mathbf{H_{n.t}} = \mathbf{X}_{t_n} + \int_{t_n}^{t} (\mathbf{f}(s, \mathbf{X}_{s-}) - \mathbf{J_n}\mathbf{X}_{s-}) ds + \int_{0}^{t} \mathbf{G}(s) d\mathbf{L}_s,$$

Then, by the variation of constants formula (see, e. g., Theorem 56 in Protter, 1995) the solution of (3) can be expressed as

$$\mathbf{X}_t = e^{(t-t_n)\mathbf{J_n}} \mathbf{X}_{t_n} + \int_{t_n}^{t} e^{(t-s)\mathbf{J_n}} \{\mathbf{f}(s, \mathbf{X}_{s-}) - \mathbf{J}_n \mathbf{X}_{s-}\} ds + \int_{t_n}^{t} e^{(t-s)\mathbf{J_n}} \mathbf{G}(s) d\mathbf{L}_s \tag{4}$$

From this, several variants of the LL method are obtained by means of suitable approximations of the integrals in (4). One of them is the following. Fix $n \le k_N - 1$ and suppose that the approximation **Y** has been defined on $]t_{n-1}, t_n]$ starting at $Y_{t_0} = Y_0 = X_0$. For $s \in]t_n, t_{n+1}]$. Consider the linear approximation

$$\mathbf{f}(s, Y_s) \approx \mathbf{f_n} + \mathbf{J_n}(\mathbf{Y}_s - \mathbf{Y}_{t_n}) + \mathbf{d_n}(s - t_n),$$

where

$$\mathbf{f_n} = \mathbf{f}(t_n, \mathbf{Y}_{t_n}) \qquad \mathbf{d_n} = \frac{\partial}{\partial t}\mathbf{f}(t_n, \mathbf{Y}_{t_n}).$$

From this linearization and (4) one obtains the approximation

$$\mathbf{Y}_t = e^{(t-t_n)\mathbf{J_n}} \mathbf{Y}_{t_n} + \int_{t_n}^{t} e^{(t-s)\mathbf{J}_n} \{\mathbf{f}_n - \mathbf{J_n}\mathbf{Y}_{t_n} + \mathbf{d_n}(s - t_n)\} ds + \int_{t_n}^{t} e^{(t-s)\mathbf{J}_n} \mathbf{G}(s) d\mathbf{L}_s$$

In turns, a simple approximation to the second integral in this expression leads to

$$\mathbf{Y}_t = e^{(t-t_n)\mathbf{J_n}} \mathbf{Y}_{t_n} + \int_{t_n}^{t} e^{(t-s)\mathbf{J_n}} \{\mathbf{f_n} - \mathbf{J_n}\mathbf{Y}_{t_n} + \mathbf{d_n}(s - t_n)\} ds + e^{(t-t_n)\mathbf{J_n}} \mathbf{G_n} \{\mathbf{L}_t - \mathbf{L}_{t_n}\},$$

where $\mathbf{G_n} = \mathbf{G}(t_n)$. Finally, the matrix identity

$$e^{(t-t_n)\mathbf{J_n}} \mathbf{Y}_{t_n} - \int_{t_n}^{t} e^{(t-s)\mathbf{J_n}} \mathbf{J_n}\mathbf{Y}_{t_n} ds = \mathbf{Y}_{t_n}$$

allows one to write the LL approximation **Y** in the equivalent form:

$$\mathbf{Y}_t = \mathbf{Y}_{t_n} + \int_{t_n}^{t} e^{(t-s)\mathbf{J}_n} \{\mathbf{f}_n + \mathbf{d}_n(s - t_n)\} ds + e^{(t-t_n)\mathbf{J}_n} \mathbf{G}_n \{\mathbf{L}_t - \mathbf{L}_{t_n}\}. \tag{5}$$

A major advantage of this variant is that it can be easily computed for any alpha stable Lévy process.

3 COMPUTATIONAL ASPECTS

Let $\mathbf{Y}_t$, $t \geq 0$, be the (continuous-time) LL approximation defined by (5). The corresponding LL discretization is defined by evaluating at the discrete times $t = t_n$, $n = 0,1,\cdots,k_N$. In this Section we discussed a specific scheme for implementing the LL discretization.

For this, the LL discretization is decomposed into a recursive part $\mathbf{r_n}$ and a noisy term $\boldsymbol{\xi}_\mathbf{n}$.

$$\mathbf{Y}_{\mathbf{t_{n+1}}} = \mathbf{Y}_{\mathbf{t_n}} + \mathbf{r_n} + \boldsymbol{\xi}_\mathbf{n},$$

where

$$\mathbf{r_n} = \int_{t_n}^{t_{n+1}} e^{(t-s)\mathbf{J_n}} \{\mathbf{f_n} + \mathbf{d_n}(s - t_n)\} ds \tag{6}$$

$$\boldsymbol{\xi}_\mathbf{n} = e^{h\mathbf{J_n}} \mathbf{G_n} \{\mathbf{L}_{t_{n+1}} - \mathbf{L}_{t_n}\}. \tag{7}$$

We will set $h = t_{n+1} - t_n \quad \forall n = 1,2,\cdots,k_N$. Details on the evaluation of these two components are discussed below.

a. Computation of the recursive term

The integral $\mathbf{r_n}$ in (6) can be explicitly computed by means of just a matrix exponential (see Proposition 1 in Jiménez, 2002). Specifically, define

$$\mathbf{C_n} = \begin{pmatrix} \mathbf{J_n} & \mathbf{d_n} & \mathbf{f_n} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \Re^{(d+2)\times(d+2)}$$

Then, (6) can be obtained as a block of the matrix exponential $\exp(h\mathbf{C_n})$ according to the following identity:

$$\begin{pmatrix} \mathbf{F} & \mathbf{b} & \mathbf{r_n} \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = e^{h\mathbf{C_n}},$$

w

here $\mathbf{F} \in \Re^{d\times d}$, $\mathbf{b} \in \Re^d$ and $c \in \Re$ are certain block matrices.

A number of algorithms are available to evaluate the matrix exponentials involved in these expressions, e.g. those based on stable Padé approximations with the scaling and squaring method, Schur decomposition, or Krylov subspace methods. The choice of one of them should depend on the size and structure of the Jacobian matrix $\mathbf{J_n}$ (see e.g., Higham, 2004 and references therein).

b. Computation of the noisy term

The evaluation of (7) only requires the simulation of the increments $\mathbf{L}_{t_{n+1}} - \mathbf{L}_{t_n}$. This is specially feasible for any alpha-stable Lévy motion $\mathbf{L}_t$. In this case, said increments are independent random variables with alpha-stable distributions of type $S_\alpha\left((t_{n+1} - t_n)^{1/\alpha},0,0\right)$. Thus, the simulation of the increments reduces to the generation of independent alpha-stable random variables. For this, a number of algorithms are available (see e. g. Janicki, 1994 and Samorodnitsky, 1994, and references therein).

4 SIMULATION STUDY AND SOME EXAMPLES

The implementation of the LL method for a stochastic differential equation driven by alpha stable Lévy motion was implemented in MATLAB following the algorithm described in the previous Section. The diagonal Padé approximation for the exponential matrix is computed by the MATLAB function expm. We generate the required alpha-stable random variables by means of the algorithm described in Janicki *et al*., 1993.

The LL method was introduced above for SDEs driven by additive Lévy motions, i.e., $\mathbf{G}(t)$ in (1) is a function of only the time variable *t*. For equations with additive noise it can be extended the concept of A-stability of integrators borrowed from the numerical theory of ordinary differential equations (see, e.g., Section 8 of Chapter 9 in Kloeden and Platen, 1995). It can be directly demonstrated that, in contrast with the Euler method, the LL method introduced above is A-stable in this sense.

In the following two examples we study in practice through simulations the stable behavior of the LL method for SDEs with alpha-stable additive noise, and compare it with the Euler method. The first example is a non-autonomous stochastic differential equation with additive alpha-stable Lévy noise, which is defined by the following equation:

$$dX_t = -t^2 X_t dt + (3/2)\exp\left\{-\left(t^3 - t_0^3\right)/3\right\}/\left(t+1\right) dL_t$$

over $\left[t_0, T\right] = \left[0,9\right]$ with initial value $X_{t_0} = 1$. This equation has trajectories that (a.s.) tend to zero as the time increases. However, Figure 1 shows that for a moderate step size ($h = 2^{-4}$) the Euler discretization results in an explosive behavior. This is illustrated for two values of alpha: α=1.5 and α=0.7. On the contrary, the LL method provides good approximations with the expected limit behavior. This illustrates that the LL method shows more numerical stability than the Euler method, a fact also confirmed by the other examples below.

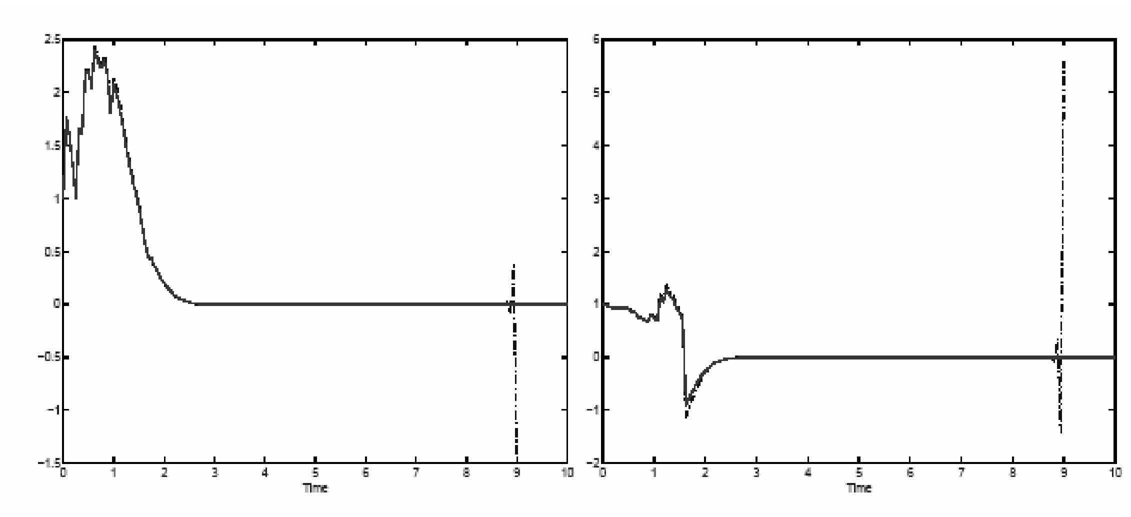

Figure 1. Approximate solutions obtained by the LL (solid line) and Euler (dash-dot line) methods, with α=1.5 (left panel) and α=0.7 (right panel). In both cases, the step size is $h = 2^{-4}$.

Next, we present two classical examples of stiff equations: Van der Pol and Brusselator. The first one is the system of autonomous stochastic differential equations with additive noise:

$$dX_1 = X_2 dt$$
$$dX_2 = C\left[\left(1 - X_1^2\right)X_2 - X_1\right]dt + \sigma\, dL_t$$

over $[t_0, T] = [0,10]$ with initial value $X_{t_0} = [-2;0]$. It is well-known that the ordinary differential equation (ODE) defined by its drift part becomes more difficult to integrate numerically as the parameter *C* increases. Typically, such an ODE is used as a test equation with *C* around the values 10, 100 and 1000. In this work we take smaller values of *C* because of the addition of an alpha-stable Lévy motion makes more difficult the numerical integration. The real constant σ controls the amount of the noise.

The behavior of Euler and LL methods it shown in Figure 2 for *C*=10, and several step sizes $h = 2^{-5}, 2^{-7}$ and 2^{-12}. The results for the smallest one, $h = 2^{-12}$, can be regarded as the exact trajectory for visualization purposes. For the greatest step size $h = 2^{-5}$, the Euler method has an explosive behavior. In contrast, the LL discretization reproduces correctly the dynamics of the system for all the step sizes regarded.

In order to also study the behavior of both discretizations when the parameter *C* changes we compute the discretizations for several values *C*=300, 100 and 10, fixing α=1.75 and the noise level σ =5. It is observed in Figure 3 that for a very small step size ($h = 2^{-12}$) neither the LL nor Euler methods has an explosive behavior, no matter the value of C. However, Figure 4 shows that for a moderate step size $h = 2^{-7}$ it appears an explosive behavior in the Euler discretization for large values of C, namely, C=300 and C=100, while the LL approximation retains its good performance in the whole range of C for

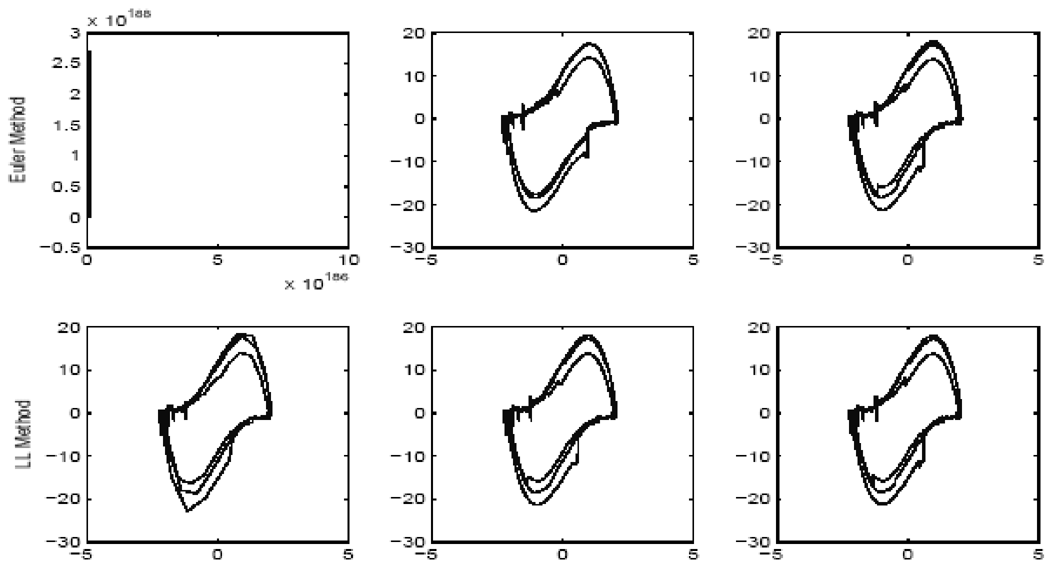

Figure 2. Approximate solutions by LL and Euler methods for the Van der Pol's equation with α=1, σ =1, *C*=10. Results for step sizes $h = 2^{-5}, 2^{-7}$ and 2^{-12} are shown by columns from left to right.

the same step size.

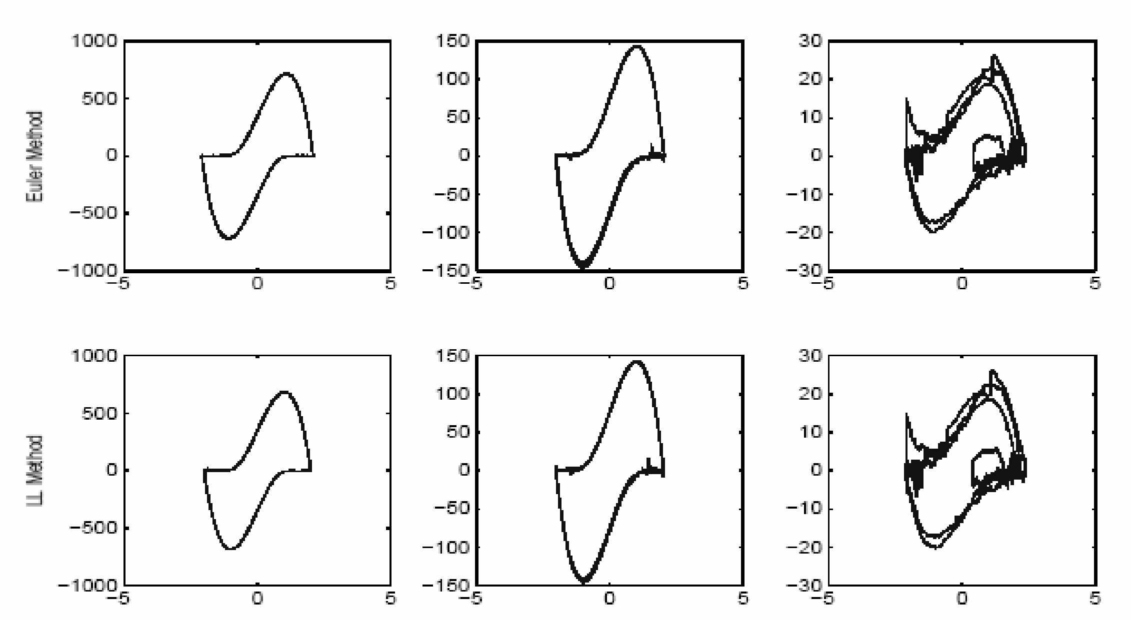

Figure 3. Approximate solutions obtained by LL and Euler methods for the Van der Pol's equation with α=1.75, σ =5, $h = 2^{-12}$. Columns from left to right correspond to *C*=300, 100 and 10.

More generally, for SDEs with multiplicative noise (i.e., when $\mathbf{G}$ is a function $\mathbf{G}(t, X_t)$ of both *t* and X_t), a version of the LL method can be also defined simply by replacing $\mathbf{G}_n$ in (5) and (7) by $\mathbf{G}_n = \mathbf{G}(t_n, X_{t_n})$. Stability of integrators of SDEs in case of multiplicative noise is still a topic of current research. Several concepts of stability have been proposed for this case such as mean-squared stability, etc. (see, e.g., Shurz, 2002). The LL method introduced in the present paper does not satisfy the conditions of such concepts. It theoretically guarantees numerical stability only for equations with additive noise.

However, it is worth of noting that the LL method also shows a very stable performance in some examples of SDEs with multiplicative noise. Likely, this occurs when the linear drift part of the equation has a dominant role in the dynamics of the system. We will illustrate it through the following well known outstanding example: the Brusselator's equation with multiplicative noise (Arnold *et al*., 1999)

$$
\begin{aligned}
dX_1 &= \left\{(c_1 - 1)X_1 + c_1 X_1^2 + X_2(1 + X_1)^2\right\}dt + c_2 X_1(1 + X_1)dL_t \\
dX_2 &= (X_1 + 1)\left\{-c_1 X_1 - X_2(1 + X_1)\right\}dt - c_2 X_1(1 + X_1)dL_t .
\end{aligned}
$$

We consider this equation over $[t_0, T] = [0,20]$ with initial value $X_{t_0} = [-0.5;0]$. Here, $\mathbf{c} = [c_1; c_2]$ is a vector of parameters. The system has a dynamics determined by c_1: its random attractor is a stationary point for $c_1 < 2$, and a cycle for $c_1 > 2$. The parameter c_2 controls the amount of noise. In the simulations shown in Figure 5 we set $c_1 = 2.5$ and $c_2 = 0.6$. Likewise in the previous examples, for a range of moderate to large step sizes it is observed an explosive behavior of Euler method while the LL approximation reproduces correctly the path of the SDE.

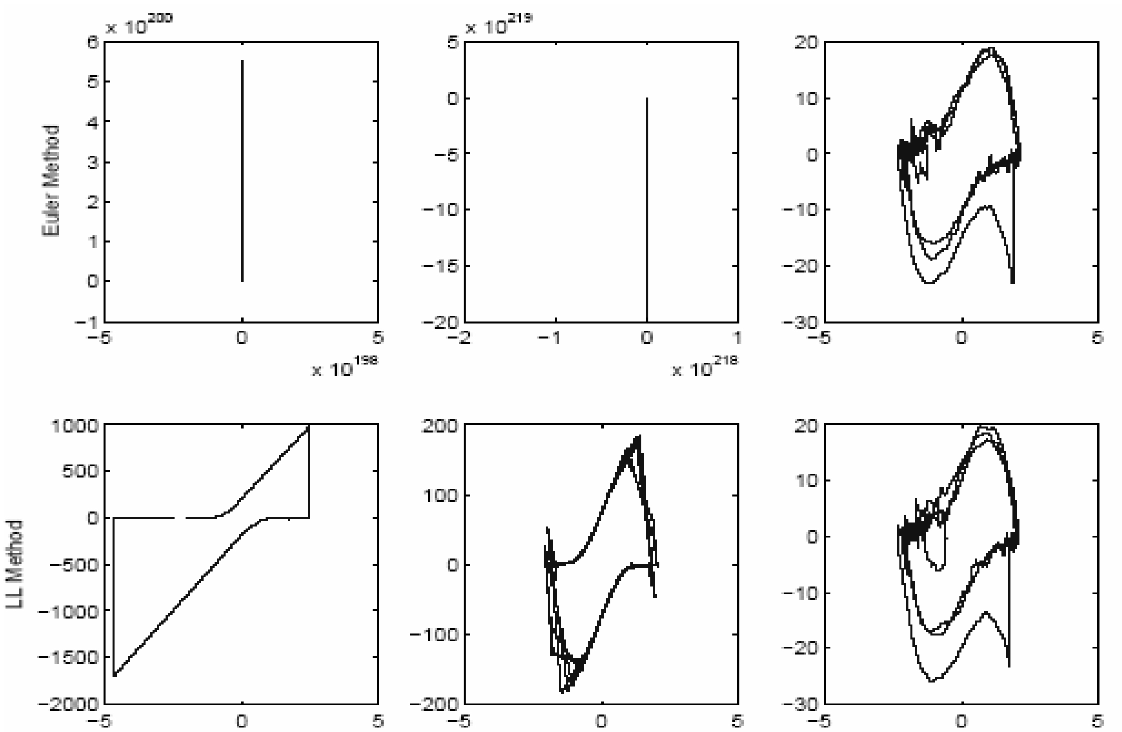

Figure 4. Approximate solutions by means of the LL and Euler methods for the Van der Pol's equation with α=1.75, σ =5, $h = 2^{-7}$. Columns from left to right correspond to *C*=300, 100 and 10.

5 CONCLUSIONS

We extended the LL method to cover SDEs driven by alpha-stables Lévy motion and proposed a numerical scheme for its implementation.

The simulation study demonstrates that the LL method has considerably more numerical stability than the Euler method. In particular, for not small step sizes the latter has an explosive behaviour while the former reproduces correctly the dynamics of the exact trajectories.

In spite of the fact that the LL scheme proposed is only for SDEs with additive Lévy noise, a simple variant of the method to cover multiplicative noise can also provides numerically stable results in some situations with multiplicative noise. This is illustrated through the Brusselator example.

The theoretical development of other variants of the LL method, as well of its main theoretical properties, is the subject of ongoing work by the authors. The practical result discussed in the present paper greatly encourages this research.

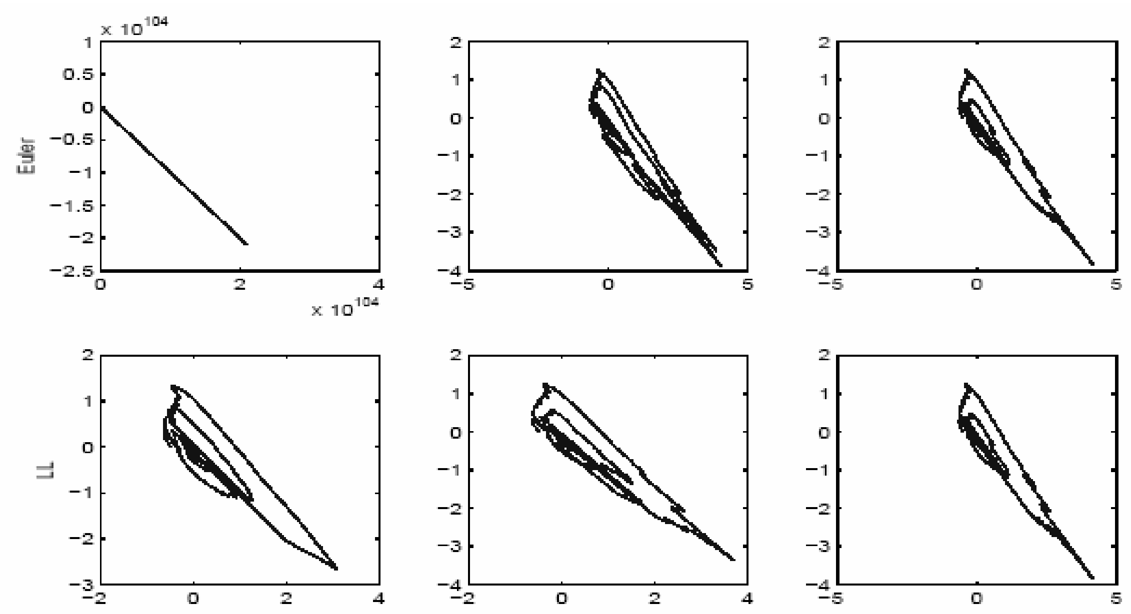

Figure 5. Approximate trajectories computed by means of the LL and Euler methods for the Brusselator's equation with α=1, $c_1 = 2.5$ and $c_2 = 0.6$. Results for step sizes $h = 2^{-4}, 2^{-7} \text{ and } 2^{-12}$ are presented in the columns from left to right.

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