# DEPENDENCE BETWEEN VOLATILITY PERSISTENCE, KURTOSIS AND DEGREES OF FREEDOM

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**ABSTRACT**
In this paper the dependence between volatility persistence, kurtosis and degrees of freedom from Student's t-distribution will be presented in estimation alternative risk measures on simulated returns.
As the most used measure of market risk is standard deviation of returns, i.e. volatility. However, based on volatility alternative risk measures can be estimated, for example Value-at-Risk (VaR). There are many methodologies for calculating VaR, but for simplicity they can be classified into parametric and nonparametric models. In category of parametric models the GARCH(p,q) model is used for modeling time-varying variance of returns.

**KEY WORDS:** Value-at-Risk, GARCH(p,q), T-Student

**MSC** 91B28

**RESUMEN**
En este trabajo la dependencia de la persistencia de la volatilidad, kurtosis y grados de liberad de una Distribución T-Student será presentada como una alternativa para la estimación de medidas de riesgo en la simulación de los retornos.
La medida más usada de riesgo de mercado es la desviación estándar de los retornos, i.e. volatilidad. Sin embargo, medidas alternativas de la volatilidad pueden ser estimadas, por ejemplo el Valor-al-Riesgo (Value-at-Risk, VaR). Existen muchas metodologías para calcular VaR, pero por simplicidad estas pueden ser clasificadas en modelos paramétricos y no paramétricos . En la categoría de modelos paramétricos el modelo GARCH(p,q) es usado para modelar la varianza de retornos que varían en el tiempo.

## 1. INTRODUCTION

It isn't easy to estimate VaR when stochastic process which generates distribution of returns is not known. Unfortunately the assumption that the returns are independently and identically normally distributed is often unrealistic. Furthermore, empirical research about financial markets reveals following facts about financial time series:

- financial return distributions are leptokurtic, i.e. they have heavy and fat tails,
- equity returns are typically negatively skewed and
- squared return series shows significant autocorrelation, i.e. volatilities tend to cluster

According to first two facts it is important to examine which probability density function capture heavy tails and asymmetry the best. According to the third fact it is important to correctly specify conditional mean and conditional variance equations from GARCH family models. Therefore, high kurtosis exists within financial time series of high frequencies (observed on daily or weekly basis). This confirms the fact that distribution of returns generated by GARCH(p,q) model is always leptokurtic, even when normality assumption is introduced.
It is important to note that kurtosis is both a measure of peakdness and fat tails of the distribution.

Hence, in this paper distributional properties of returns generated using GARCHH(1,1) model with high volatility persistence will be compared to distributional properties of returns generated by the same model but with low volatility persistence. The effect between differences in distributional properties on alternative risk measures will be also examined.

## 2. KURTOSIS OF GARCH(1,1) PROCESS

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If it is assumed GARCH(1,1) process:[3]

$$\begin{aligned} r_t &= \varepsilon_t \\ \varepsilon_t &= u_t \sqrt{\sigma_t^2} \quad ; \; u_t \approx i.i.d.N(0,1), \\ \sigma_t^2 &= \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \end{aligned} \tag{1}$$

the second moment of innovation process $\{\varepsilon_t\}$ equals:

$$E[\varepsilon_t^2] = Var[\varepsilon_t] = \frac{\alpha_0}{1-\alpha_1-\beta_1}, \tag{2}$$

while the fourth moment is given as:

$$E[\varepsilon_t^4] = \frac{3\alpha_0^2(1+\alpha_1+\beta_1)}{(1-\alpha_1-\beta_1)(1-3\alpha_1^2-2\alpha_1\beta_1-\beta_1^2)}. \tag{3}$$

From covariance stationary condition of GARCH(1,1) process, and strictly positively conditional variance:

$$\begin{aligned} 1-\alpha_1-\beta_1 &> 0 \\ \alpha_0 &> 0 \end{aligned}, \tag{4}$$

follows that the second moment of $\{\varepsilon_t\}$ process exist. To assure the existence of the fourth moment, apart from conditions in (4), it is necessary in relation (3) to satisfy this restriction:

$$3\alpha_1^2 + 2\alpha_1\beta_1 + \beta_1^2 < 1. \tag{5}$$

Since kurtosis is defined as:

$$k = \frac{E[\varepsilon_t^2]}{(E[\varepsilon_t^4])^2}, \tag{6}$$

then expression (6) becomes:

$$k = \frac{3(1+\alpha_1+\beta_1)(1-\alpha_1-\beta_1)}{1-3\alpha_1^2-2\alpha_1\beta_1-\beta_1^2}. \tag{7}$$

After some rearrangement in (7) we can write:

$$k = 3 + \frac{6\alpha_1^2}{1-3\alpha_1^2-2\alpha_1\beta_1-\beta_1^2}. \tag{8}$$

From relation (8) follows that distribution of returns generated from GARCH(1,1) process always results in excess kurtosis, i.e. Fisher's kurtosis ($k > 3$) even normality assumption is introduced, if and only if conditions in (4) are satisfied.[4] These conditions also could be satisfied when parameter $\alpha_1 = 0$. Only in that case

[3] Engle assumes multiplicative structure of innovation process.

[4] Normality assumption is often introduced, because the parameters are estimated using maximum likelihood method (MLE). When normality assumption is not satisfied estimates are called quasi-maximum likelihood estimates, and robust standard errors should be used.

innovations distribution would be normally shaped ( $k = 3$ ). Therefore, the kurtosis is very sensitive on value of parameter $\alpha_1$.

Empirical research also shows that kurtosis increases much intensively with larger parameter $\alpha_1$ in comparison to parameter $\beta_1$.

## 3. DEGREES OF FREEDOM ESTIMATION

Generally, there are three parameters that define a probability density function: (a) location parameter, (b) scale parameter and (c) shape parameter. The most common measure of location parameter is the mean. The scale parameter measure variability of probability density function (*pdf*), and the most commonly used is variance or standard deviation. The shape parameter (kurtosis and/or skewness) determines how the variations are distributed about the location parameter.

If the data are heavy tailed, the VaR calculated using normal assumption differs significantly from Students t-distribution. Therefore, we find that kurtosis and degrees of freedom from Student's distribution are closely related.

Probability density function of noncentral Student t-distribution is given as:

$$f(x) = \frac{\Gamma\left(\frac{df+1}{2}\right)}{\Gamma\left(\frac{df}{2}\right)\sqrt{\pi \cdot \beta \cdot df}}\left(1+\frac{(x-\mu)}{\beta \cdot df}\right)^{-\frac{1+df}{2}}, \tag{9}$$

where $\mu$ is location parameter, $\beta$ scale parameter and $df$ shape parameter, i.e. degrees of freedom, and $\Gamma(\cdot)$ is gamma function. Standard Student's t-distribution assumes that $\mu = 0, \beta = 1$, with integer degrees of freedom. However, degrees of freedom can be estimated as non-integer, relating to kurtosis:

$$k = \frac{6}{df-4} + 3 \qquad \forall df > 4 \,. \tag{10}$$

From relation (10) it's obvious that standard t-distribution has heavier tails than normal distribution when $4 < df \leq 30$. Hence, if empirical distribution is more leptokurtic estimated degrees of freedom would be smaller.

The second and fourth central moment of function (9) are defined as:

$$\begin{aligned} \mu_2 &= E\left[(x-\mu)^2\right] = \frac{\beta \cdot df}{df-2} \\ \mu_4 &= E\left[(x-\mu)^4\right] = \frac{3\beta^2 df^2}{(df-2)(df-4)} \end{aligned}, \tag{11}$$

with Fisher's kurtosis:

$$k^* = \frac{\mu_4}{\mu_2^2} - 3 = \frac{6}{df-4}\,. \tag{12}$$

Therefore, we may apply method of moments and get consistent estimators:

$$
\begin{aligned}
\hat{df} &= 4 + \frac{6}{\hat{k}^*} \\
\hat{\beta} &= \left( \frac{3 + \hat{k}^*}{3 + 2\hat{k}^*} \right) \hat{\sigma}^2
\end{aligned}, \tag{13}
$$

where the sample variance is biased estimator of $\beta$.

To get unbiased estimator of standard deviation we use correction factor:

$$
\sqrt{\frac{3 + \hat{k}^*}{3 + 2\hat{k}^*}}, \tag{14}
$$

which is equivalent to:

$$
\sqrt{\frac{\hat{df} - 2}{\hat{df}}}. \tag{15}
$$

In practice, the kurtosis is often larger than six, leading to estimation of non-integer degrees of freedom between four and five. However, kurtosis will depend on volatility persistence. Volatility persistence is defined as the sum of parameters $\alpha_1 + \beta_1$ in GARCH(1,1) model.

If we rearrange condition variance equation of GARCH(1,1) model as follows:

$$
\sigma_t^2 = \alpha_0 + \alpha_1 \left( \varepsilon_{t-1}^2 - \sigma_{t-1}^2 \right) + \left( \alpha_1 + \beta_1 \right) \sigma_{t-1}^2, \tag{16}
$$

then the sum of parameters $\alpha_1 + \beta_1$ shows the time which is needed for shocks in volatility to die out. If this sum is close to 1 long time is needed for shocks to die out. However, if the sum is equal to unity the covariance stationary condition is not satisfied and GARCH(1,1) model follows integrated GARCH process of order one, i.e. IGARCH(1,1).

If we substitute $\sigma_t^2 = \varepsilon_t^2 - v_t$ than stationary condition occurs from ARMA(1,1) representation of GARCH(1,1) model:

$$
\varepsilon_t^2 = \alpha_0 + \left( \alpha_1 + \beta_1 \right) \varepsilon_{t-1}^2 + v_t - \beta_1 v_{t-1}. \tag{17}
$$

## 4. EFFECT OF VOLATILITY PERSISTENCE ON RISK MEASURES ESTIMATION

In this paper two GARCH(1,1) process are simulated. One with volatility persistence of 90% ($\alpha_0 = 0.0001$, $\alpha_1 = 0.1$, $\beta_1 = 0.8$), and the other with volatility persistence of 70% ($\alpha_0 = 0.0001$, $\alpha_1 = 0.2$, $\beta_1 = 0.5$). Based on generated returns from GARCH(1,1) models, excess kurtosis, degrees of freedom of Student's t-distribution and standard deviation correction factors are presented in table 1.

From table 1. it's obvious when volatility persistence is high (long time is needed for shocks in volatility to disappear) the excess kurtosis is also high, indicating that distribution of returns is heavy tailed. Therefore, the assumption of Student's t-distribution with 9.5 degrees of freedom would be much appropriate in comparison to normal distribution assumption.

When volatility persistence is low (70%) the Student's t-distribution can be approximated with normal distribution as degrees of freedom becomes larger, and no correction of standard deviation is needed (correction factor converges to unity).

**Table 1: Volatility persistence, excess kurtosis, degrees of freedom with correction factors based on simulated returns of generated GARCH(1,1) models**

| volatility persistence | excess kurtosis | degrees of freedom | correction factor of standard deviation |
|---|---|---|---|
| 70% | 0.141 | 46.5 | 0.978 |
| 90% | 1.082 | 9.5 | 0.889 |

Simulation results of generated GARCH(1,1) processes are presented on figure 1.

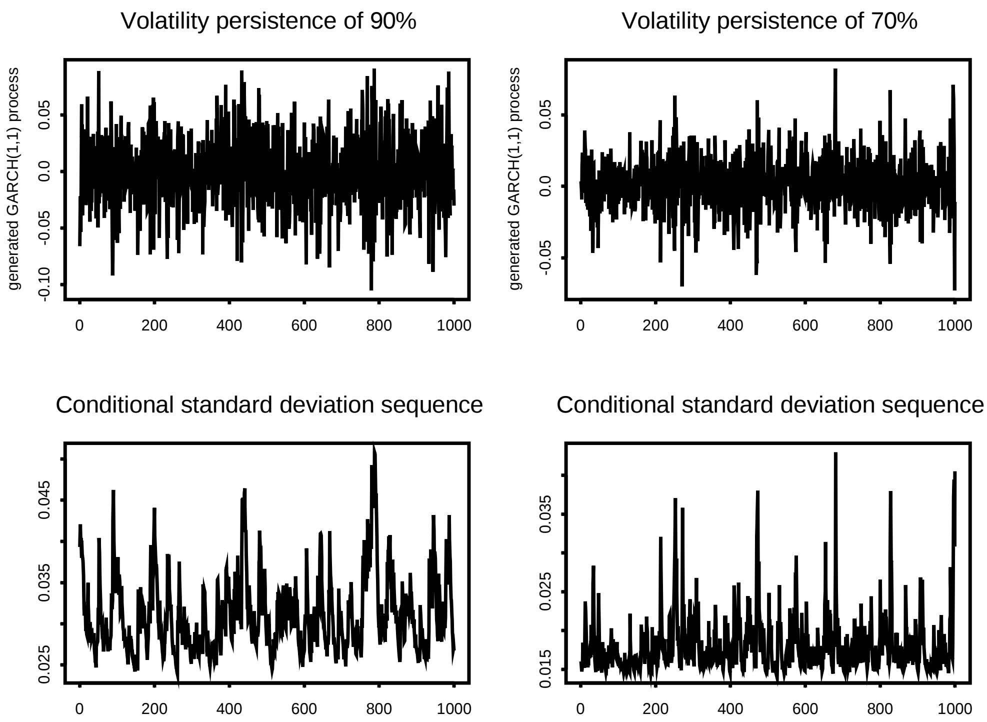



**Figure 1: Generated GARCH(1,1) processes with conditional standard deviations**

From figure 1. it can be shown the difference of generated GARCH(1,1) processes. Therefore, when volatility persistence is 90% reaction of volatility on past market movements are low, and shocks in volatility disappears slowly. When volatility persistence is 70% reaction of volatility on past market movements are much intensively, and shocks in volatility disappears quickly.

In table 2. sample quantiles, sample moments and "Jarque-Bera" normality tests are presented for two generated GARCH(1,1) processes.

Testing results in table 2. shows that normally distributed assumption can not be accepted at empirical p-value less than 1% within volatility persistence of 90%. Hence, we conclude that in case of high volatility persistence Student's t- distribution would be more adequate with non-integer degrees of freedom estimated in table 1.

Correctly specified conditional distribution is very important, not only in estimation parameters in GARCH(1,1) model, but also in risk management. Based on returns distribution assumption different risk measures can be defined.
Standard deviation of returns, i.e. volatility, is the most used risk measure based on alternative risk measure can be calculated, such as VaR, CVaR. Even so, VaR is proposed, by Basel Committee on Banking Supervision in 1996, as the basis for calculation of capital requirements, within establishing banks internal risk models.
VaR is defined as the maximum potential loss of financial instrument with a given probability (usually 1% or 5%) over a certain time period.

**Table 2: Sample quantiles, sample moments and JB-normality test of two generated GARCH(1,1) processes**

| Volatility persistence of 90% | Volatility perzistence of 70% |
|---|---|
| Sample Quantiles:<br>min 1Q median 3Q max<br>-0.0729 -0.0100 0.00147 0.0131 0.08234<br><br>Sample Moments:<br>mean std skewness kurtosis<br>0.001443 0.01871 -0.02773 4.082 | Sample Quantiles:<br>min 1Q median 3Q max<br>-0.105 -0.0195 -0.000696 0.0204 0.09091<br><br>Sample Moments:<br>mean std skewness kurtosis<br>-0.0000607 0.03121 -0.08823 3.141 |
| Null Hypothesis: data is normally distributed<br><br>Test Stat 48.906<br>p.value 0.000<br><br>Dist. under Null: chi-square with 2 df.<br>Total Observ.: 1000 | Null Hypothesis: data is normally distributed<br><br>Test Stat 2.1238<br>p.value 0.3458<br><br>Dist. under Null: chi-square with 2 df.<br>Total Observ.: 1000 |

When normality assumption is introduced VaR can be estimated as:

$$VaR_t(\alpha) = \mu_t + z\sigma_t . \tag{18}$$

In relation (18) $\alpha$ is given probability, $z$ is standardized value, $\mu_t$ is conditional mean and $\sigma_t$ is conditional standard deviation. As the conditional mean and conditional standard deviation are time varying, they can be described using GARCH(1,1) model. In this paper it is assumed that conditional mean equals zero.

When assumption of Student's t-distribution is introduced VaR can be calculated as:

$$VaR_t(\alpha) = \mu_t + t_\alpha^{df} \cdot \sigma_t \sqrt{\frac{df-2}{df}} . \tag{19}$$

In relation (19) $t_\alpha^{df}$ is critical value of t-distribution depending on given probability and estimated degrees of freedom, while $\sqrt{(df-2)/df}$ is correction factor for unbiased standard deviation estimation from sample.
Based on Value-at-Risk another risk measure can be defined - Conditional Value-at-Risk (CVaR). CVaR is expected loss (negative return) under Value-at-Risk region:

$$CVaR_t(\alpha) = E\left[r_t \middle| r_t \le VaR_t(\alpha)\right]. \tag{20}$$

According to definition (20) it is evident this relation:

$$CVaR_t(\alpha) \le VaR_t(\alpha) . \tag{21}$$

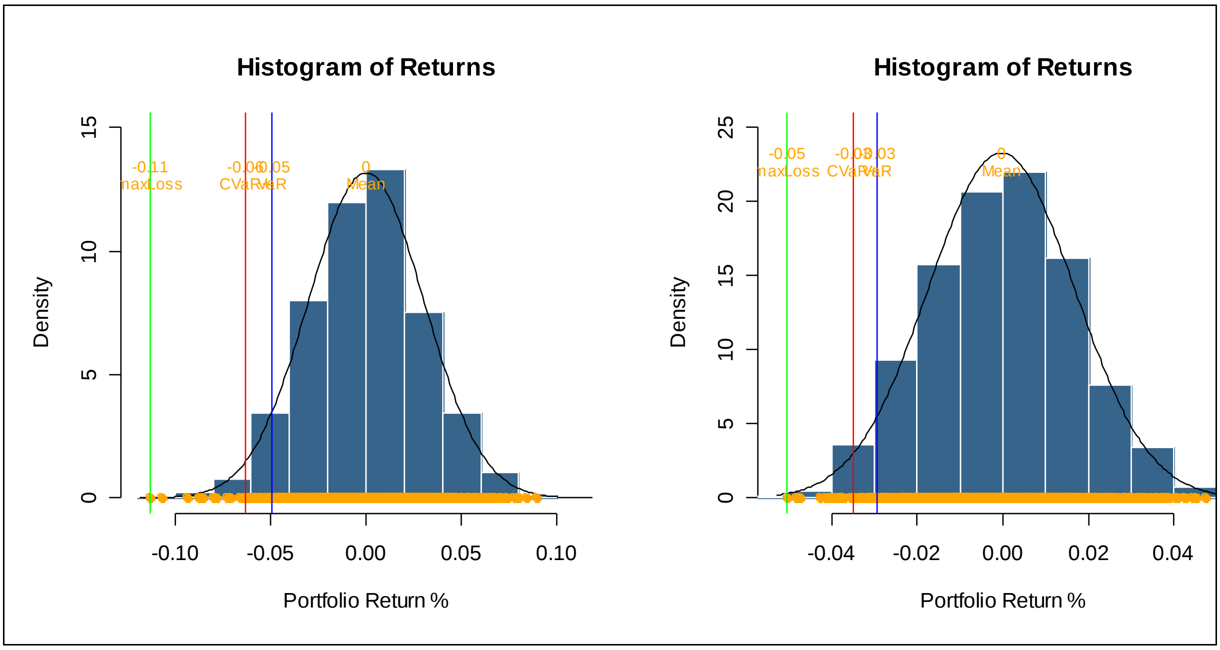
Therefore, CVaR is conditional expectation under interval $\left\langle -\infty, VaR_t(\alpha)\right]$:

$$\int_{-\infty}^{z} xf(x)dx , \tag{22}$$

where $z$ is left percentile of distribution, when normality assumption is introduced.

Conditional distributions of returns generated from GARCH(1,1) processes, with corresponding risk measures are presented on figure 2.

**Figure 2 Conditional distributions of returns with corresponding risk measures at probability level of 5%**




Risk measures of two conditional distributions of returns at 5% probability level are also given in table 3. From table 3. it can be concluded that risk measures under distribution with heavier tails (distribution generated using GARCH(1,1) process with volatility persistence of 90%) are much higher in comparison to distribution which is generated using GARCH model with volatility persistence of 70%. This confirms that Student's t-distribution is more adequate in risk estimation when fat tails are present. These risk measures can reach more extremely values at lower probability level, i.e. 1%.

**Table 3: Corresponding risk measures of two generated conditional distributions of returns**

| Risk measures | Distribution of returns generated using GARCH(1,1) model with 90% volatility persistence | Distribution of returns generated using GARCH(1,1) model with 70% volatility persistence |
|---|---|---|
| VaR | -0.0490300 | -0.02948700 |
| CVaR | -0.0633653 | -0.03480367 |
| max Loss | -0.1129725 | -0.05038104 |

## 5. CONCLUSION REMARKS

If the data (returns) are heavy tailed, the VaR calculated using Normal assumption differs significantly from Student's t-distribution. The fact that kurtosis and degrees of freedom from Student's distribution are closely related is used in estimation procedure of GARCH(1,1) model. The comparison procedure of Value at Risk estimation is established with assumption that returns follows extreme value distribution, precisely Student's t-distribution with non-integer degrees of freedom.

By forecasting Value at Risk investor can protect himself "a priori" from estimated market risk, using financial derivatives, i.e. options, forwards, futures and other instruments. In that sense we find financial econometrics as the most useful tool for modeling conditional mean and conditional variance of nonstationary financial time series, i.e. time series with high frequencies.

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