

A FRAMEWORK FOR THE DERIVATION AND VERIFICATION OF VARIABLES ACCEPTANCE SAMPLING PLANS

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ABSTRACT

Acceptance sampling is a method for verifying quality or performance requirements using sample data. Variables acceptance sampling is an alternative to attributes acceptance sampling, which in many instances requires significantly smaller samples. In this paper, we consolidate the literature on variables acceptance sampling by providing a unified exposition of the approach used to develop such plans. From within this framework, we review the derivation of plans for exponential, normal, gamma, Weibull, and Poisson random variables. We verified these derivations on a set of test problems.

KEYWORDS: quality engineering, reliability, inspection, requirements verification

MSC: 62P30

RESUMEN

El muestreo de aceptación es un método para verificar la calidad o el requerimiento del comportamiento usando datos muestrales. El muestreo de aceptación por variables es una alternativa para el muestreo de aceptación por atributos, el cual en muchos casos requiere de muestras significativamente pequeñas. En este trabajo, consolidamos la literatura sobre el muestreo de aceptación por variables presentando una exposición unificada del enfoque usado para desarrollar tales planes. Desde el interior del marco de trabajo, revisamos la derivación de planes para variables aleatorias con distribución exponencial, normal, gamma, Weibull, y Poisson. Verificamos estas derivaciones sobre un conjunto de problemas de prueba.

1. INTRODUCTION

One of the oldest problems in quality engineering is to assess the acceptability of items that a customer receives from a producer. *Acceptance sampling* is an alternative to 100% inspection applied when inspection is destructive, or when the time and/or cost of 100% inspection are unwarranted or prohibitive. The customer decides the disposition of an incoming lot based on a standard specifying the maximum proportion of nonconforming items in the sample. The decision can be to accept or reject the entire lot, or to continue sampling.

Acceptance sampling also has been adapted to problems not identified with procurement. Smith *et al.* [1] advocated the application of acceptance sampling in the context of water quality assessment. Bayard *et al.* [2] applied the underlying binomial model to the assessment of future aircraft runway-incursion controls. White, *et al.* [3] showed that attributes acceptance sampling can be applied to any sampling experiment, including those commonly employed to verify design requirements using simulation and Monte Carlo methods.

Acceptance sampling by *attributes* (ASA) sentences a lot based a count of the number of nonconforming items relative to the sample size. The inspection variable is binary (pass/fail) and therefore the count is necessarily binomial. ASA can be used with categorical outputs, or with outputs measured on a continuous or discrete scale, by reference to a required limiting value. Conceptually simple, easily applied, and universally applicable, ASA is the first choice for sampling inspection.

Acceptance by *variables* (ASV) is an alternate approach, which in many instances prescribes significantly smaller samples than ASA. ASV requires that inspection variable is measured on a continuous or discrete scale, that the

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distribution of this variable is known a priori and stable, and that a plan exists for this particular distribution. While far more restrictive in its assumptions, ASV should be considered when the larger samples required by ASA are unavailable.

The objective of this paper is to consolidate the scholarly literature on ASV and provide a readable tutorial on the concept and general approach used to develop ASV plans for alternative parent distributions. While ASV in fact employs a consistent procedure for determining sampling plans, this commonality is obscured by different presentation styles and the sometimes vastly different notation adopted by researchers in the field. Here we offer a unified presentation and a consistent notation. In addition, each of the plans described here was implemented in a calculator and verified on a range of sample problems.

2. PROBABILISTIC REQUIREMENTS AND LIMIT STANDARDS

In uncertain environments, requirements verification seeks to determine whether a measurable quantity is *conforming* or *nonconforming*, i.e., to determine whether or not the parent population from which a sample is drawn achieves a specified level of quality or performance. The requirement can be stated probabilistically as an $(I, \square, \square, \square)$ *limit standard* [3] where:

(1) I is the *performance indicator*. The measured quantity may be inherently categorical or qualitative in nature and performance indicated by occurrence or nonoccurrence of some event. Alternately, the measured quantity may be inherently quantitative and performance indicated by success or failure in achieving a limit or tolerance.

(2) ρ is the *minimum reliability* for the population. This is the minimum, acceptable, long-run proportion of observations on which the population achieves the desired performance. If p is the *failure probability* for the population, then requirement is $p \leq 1 - \rho$.

(3) β is the *maximum acceptable consumer's risk*. This is the probability of incorrectly accepting a nonconforming population as the result of sampling error.

Formally, we represent a measurable quantity as the random variable X with probability distribution $F(x; \theta)$ and density $f(x; \theta)$, where θ is an unknown distribution parameter. Let $\{X_i; i=1, \dots, n\}$ be a random sample with observed values $\{x_i; i=1, \dots, n\}$. The verification problem is to determine whether or not the population as a whole conforms to a specified limit standard, based on the statistics of the sample observed.

3. SAMPLING PLANS AND OPERATING CHARACTERISTICS

A *sampling plan* is the pair (n, κ) , where n is the minimum sample size, i.e., the minimum number of observations required to verify statistically the requirement imposed by standard. κ is a constant factor which is used to assess whether or not the population is conforming. The interpretation of κ depends on whether the characteristic is continuous or discrete, as discussed in Section 4.

For a given distribution, every sampling plan has a unique *operating characteristic* (OC). The OC is a function which defines the probability of accepting a population P_a for every value of the failure probability p , i.e., the OC is the function $P_a(p)$ where $p \in [0, 1]$. A sampling plan is derived by first defining two *operating points*, $(p_0, 1 - \alpha)$ and (p_1, β) , where $p_0 < p_1$ and α and β are small probabilities. We then require that the OC must (minimally) satisfy the following inequalities

$$P_a(p_0) \geq 1 - \alpha \tag{1}$$

$$P_a(p_1) \leq \beta \tag{2}$$

Alternately, it is sometimes more convenient to express these in terms of the rejection probabilities as the power conditions $P_r(p_0) = 1 - P_a(p_0) \leq \alpha$ and $P_r(p_1) = 1 - P_a(p_1) \geq 1 - \beta$. Note that when $p_1 = 1 - \rho$ and β is the consumer's risk, inequality (2) enforces the limit standard. Under this plan we will accept a population with failure probability p_1 as conforming with small probability β . Inequality (1) captures the competing good—under this plan we will accept a population with probability p_0 as conforming with high probability $1 - \alpha$.

The probability of incorrectly rejecting a conforming system, α , is called the *producer's risk*. When given as a percentage, $100\%(1 - p_0)$ is called the *acceptable quality level* (AQL) and $100\%(1 - p_1)$ is called the *lot tolerance percent defective* (LTPD). Note that the limit standard does not specify the $(p_0, 1 - \alpha)$ operating point. It is common to

develop a range of plans with the required consumer's risk β and differing α . Plans with larger sample sizes n then have smaller corresponding α .

4. HYPOTHESIS TESTING AND ACCEPTANCE LIMITS

The underlying problem can be framed as an hypothesis test for which we intend to enforce both significance and power requirements. The null and alternate hypotheses are

$$H_0: p = p_0 \text{ and } H_1: p = p_1 > p_0$$

Under H_0 fail to reject the population as nonconforming and under H_1 we reject the population as nonconforming. Inequality (1) establishes the significance of test as α and inequality (2) establishes the power of the test as $1-\beta$, where α and β are the producer's and consumer's risk, respectively.

We use the sample data to choose between the null and alternate hypotheses. To accomplish this, we need to determine the critical value of an appropriate test statistic. Denote the test statistic as the *acceptance limit* $A(n, \kappa)$, where the arguments are the parameters of the sampling plan. For a continuous quantity, $\kappa=k$ and the acceptance limit and typically has the form

$$A(n, \pm k) = \hat{\theta}(n) \pm k\hat{\sigma}(n)$$

where the plus is used with an upper bound and the minus with a lower bound. It follows that

$$k = \frac{A(n, \kappa) - \hat{\theta}(n)}{\hat{\sigma}(n)} \quad (3)$$

where $\hat{\theta}$ and $\hat{\sigma}$ are estimators for the unknown parameter and the standard deviation of the population.

For I a *lower specification limit* with value x_{\min} , the desired performance on the i^{th} observation is achieved if and only if $x_i \geq x_{\min}$. For the sample as a whole, we reject the null hypothesis if $A(n, -k) \leq x_{\min}$. That is, the acceptance limit is smaller than the specified limit. For I an *upper specification limit* with value x_{\max} , the desired performance on the i^{th} observation is achieved if and only if $x_i \leq x_{\max}$. For the sample as a whole, we reject the null hypothesis if $A(n, -k) \geq x_{\max}$. That is, the acceptance limit is larger than specified limit. The effect in either case

is to move the critical point of the acceptance limit away the estimate of the unknown quantity in the direction of the specified limit by k standard deviations. Intuitively, this hedge is intended to compensate for error in the estimate for a sample of n observations in a statistically exact way. (Note that in this paper we will not address the third case, where two limits are specified. In this case I is a *tolerance interval*, $x_{\max} \geq x_i \geq x_{\min}$. The derivation of sampling plans for tolerance is modestly more complicated than that described in Section 5. The interested reader is referred to the considerable literature on tolerance intervals.)

For a discrete population characteristic, the acceptance limit has the form

$$A(n, c) = Y(n) - c$$

where Y is some function of X with nonnegative integer values. It follows that

$$c = Y(n) - A(n, c) \quad (4)$$

For a lower specification limit on Y , the *acceptance number*, c , is the minimum acceptable value of $Y(n)$ and the rejection criteria is $A(n, c) < 0$. For an upper specification limit on Y , c is the maximum acceptable value of $Y(n)$ and the rejection criteria is $A(n, c) > 0$.

5. GENERAL PROCEDURE FOR DEVELOPING VARIABLES ACCEPTANCE SAMPLING PLANS

For a continuous distribution with lower specification limit, the procedure comprises four steps:

(1) Determine the value of θ_0 (typically the mean) for the null probability distribution with failure probability $p_0 = \Pr[X \leq x_{\min}] = F(x_{\min}; \theta_0)$; determine the value of θ_1 for the alternate distribution with failure probability $p_1 = \Pr[X \leq x_{\min}] = F(x_{\min}; \theta_1)$. This is accomplished using the inverse distribution. As shown in Figure 1, the requirement that $p_0 < p_1$ implies that $\theta_0 > \theta_1$.

(2) Determine the minimum value of n from the sampling distributions for θ_0 and θ_1 such that both inequalities (1) and (2) are satisfied.

(3) Determine the maximum acceptance limit, A_0 , from the null sampling distribution with $\alpha = \Pr[\hat{\theta} \leq A_0] = F(\hat{\theta}; \theta_0, n)$; determine the minimum acceptance limit, A_1 , from the alternate sampling distribution with $\beta = \Pr[\hat{\theta} \leq A_1] = F(\hat{\theta}; \theta_1, n)$; as illustrated in Figure 2.

(4) Determine the factors k_0 and k_1 corresponding to A_0 and A_1 , respectively, from Equation (3). In general, these factors are not equal. A conservative choice is to use the larger of the two values as the factor k for the sampling plan. Alternately, the k is sometimes taken as the average of these two values.

For I an upper specification limit, the failure probability of the population is $p = \Pr[X > x_{max}] = 1 - F(x; \theta)$. The procedure is identical with this single exception. For a discrete distribution, instead of the factors k_0 and k_1 , at Step 4 the constant c is determined.

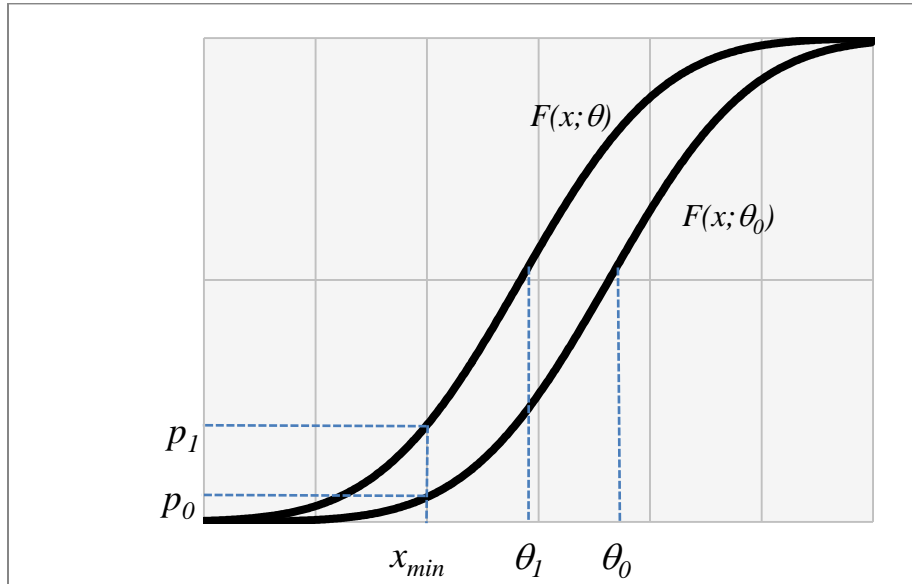


Figure 1. Distribution functions and critical values for X under the null and alternate hypotheses.

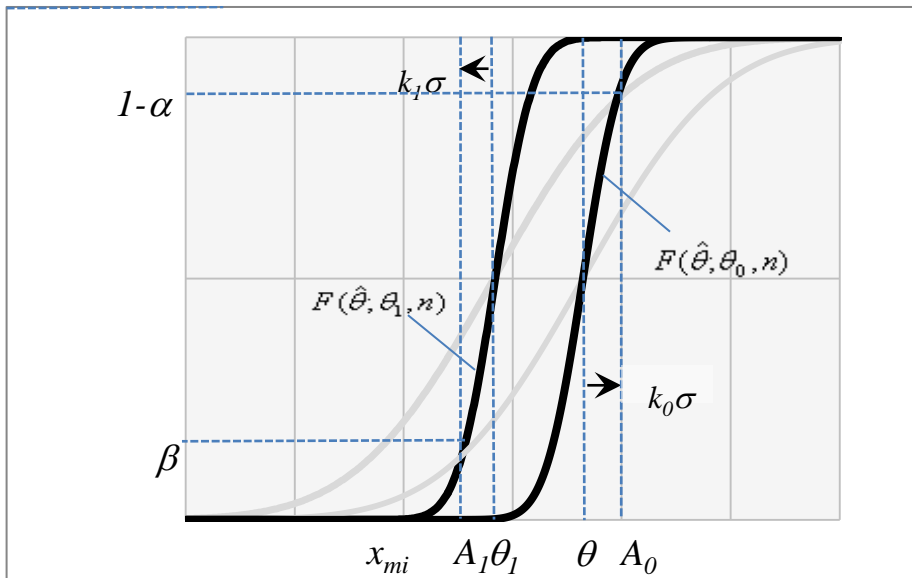


Figure 2. Sampling distributions for the estimator $\hat{\theta}$ under the null and alternate hypotheses.

If the random variable X and the unknown parameter θ can be standardized, it is generally more convenient to use the standardized distribution and standardized sampling distribution to derive sampling plans. This convenience is illustrated in Section 6 for exponential and normal random variables.

6. METHODS IMPLEMENTED

A literature search found derivations of (n, k) sampling plans for exponential, normal, gamma, and Weibull distributions. These derivations follow the general procedure given in Section 5. A derivation also was found for the (n, c) sampling plan for the Poisson distribution. In this section, we outline these derivations with reference to lower specification limits. The corresponding derivations for upper specification limits are easily deduced and are not covered here.

6.1 Exponential

Guenther [4] considers the exponential random variable X with unknown mean μ , support $x \in [0, \infty)$, and distribution function

$$F(x; \mu) = 1 - e^{-x/\mu}$$

Note that exponential and chi-squared distributions are both special cases of the gamma distribution, with $\text{exponential}(\mu) = \text{chi-square}(2) = \text{gamma}(1, \mu)$.

Denote a random variable distributed chi-squared with v degrees of freedom as Y . X can be standardized as

$$Y_2 = \frac{2}{\mu} X$$

so that $p = \Pr(X \leq x_p) = \Pr(Y_v \leq y_{2;p})$ and

$$y_{2;p} = \frac{2}{\mu} x_{\min} \tag{5}$$

where p is either p_0 or p_1 . Similarly, the estimator for the mean μ is sample mean $\bar{X}(n)$, which can be standardized as

$$Y_{2n} = \frac{2n}{\mu} \bar{X}(n) \tag{6}$$

Substituting Equation (5) into Equation (6) yields

$$y_{2n;p} = \frac{2n}{\mu} \bar{x}(n) = \frac{ny_{n;p}}{x_{\min}} \bar{x}(n)$$

where p is either p_0 or p_1 . Inequalities (1) and (2) imply

$$y_{2n;p_0} = \frac{2n}{\mu} \bar{x}(n) = \frac{y_{2;p_0}}{x_{\min}} \bar{x}(n) \leq y_{2n;\alpha}$$

$$y_{2n;p_1} = \frac{2n}{\mu} \bar{x}(n) = \frac{y_{2;p_1}}{x_{\min}} \bar{x}(n) \geq y_{2n;(1-\beta)}$$

Together these in turn imply that

$$\frac{y_{2n;\alpha}}{y_{2;p_0}} \geq \frac{\bar{x}(n)}{x_{\min}} \geq \frac{y_{2n;(1-\beta)}}{y_{2;p_1}}$$

or

$$\frac{y_{2;p_1}}{y_{2;p_0}} \geq \frac{y_{2n;(1-\beta)}}{y_{2n;\alpha}} \tag{7}$$

The minimum required sample size is the smallest value of the integer n satisfying inequality (7), which can be determined by table lookup.

With n known, the k -factor can be computed. The estimator for both the mean and standard deviation is the sample mean, $\bar{X}(n)$, therefore the acceptance limit is

$$A(n, -k) = \hat{\mu}(n) - k' \hat{\sigma}(n) = (1 - k) \bar{X}(n) = (1 - k') \frac{\mu}{2n} Y_{2n}$$

It is customary to let $k=1-k'$ and use the k -factors and

$$k_0 = \frac{ny_{2;p_0}}{y_{2n;\alpha}}$$

$$k_1 = \frac{ny_{2;p_1}}{y_{2n;(1-\beta)}}$$

6.2 Normal (σ -known)

The derivation of sampling plans for measurable quantities distributed $N(\mu, \sigma^2)$ is widely published ([4], [5], [6], [7] and [8] among others) and the basis for standards MIL-STD-414 and ANSI/ASQC Z1.9, and ISO 2859. Consider the normal random variable X with unknown mean μ , known standard deviation σ , support $x \in (-\infty, \infty)$, and density function

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2 / (2\sigma^2)}$$

Denote a random variable distributed a standard normal $N(0, 1)$ as Z . X can be standardized as

$$Z = \frac{X - \mu}{\sigma}$$

such that $p = \Pr(X \leq x_p) = \Pr(Z \leq z_p)$ where

$$z_p = \frac{A - \bar{x}(n)}{\sigma} \tag{8}$$

The central limit theorem tells us that the mean estimator $\bar{X}(n)$ is distributed $N(\mu, \sigma^2/n)$, which can be standardized as

$$\bar{Z}(n) = \frac{\bar{X}(n)}{\sigma/\sqrt{n}} \tag{9}$$

where $\bar{Z}(n)$ also is a standard normal $N(0, 1)$ deviate.

The acceptance criterion is $A(n, -k) = \bar{x}(n) - k\sigma \geq x_{\min}$, so that we will reject the population if

$$\frac{\bar{x}(n) - x_{\min}}{\sigma} < k$$

Inequalities (1) and (2) imply

$$\begin{aligned} \Pr\left[\frac{\bar{X}(n) - x_{\min}}{\sigma} < k\right] &= \Pr\left[\frac{\bar{X}(n) - x_{\min}}{\sigma/\sqrt{n}} < k\sqrt{n}\right] \\ &= \Pr\left[\frac{\bar{X}(n) - \mu}{\sigma/\sqrt{n}} + \frac{\mu - x_{\min}}{\sigma/\sqrt{n}} < k\sqrt{n}\right] \\ &= \Pr\left[\bar{Z} - z_p \sqrt{n} < k\sqrt{n}\right] \\ &= \Pr\left[\bar{Z} < \sqrt{n}(k + z_p)\right] \end{aligned}$$

The acceptance-limit criteria (expressed as power requirements) from inequalities (1) and (2) then imply

$$\sqrt{n}(k + z_{p_0}) \geq z_{\alpha}$$

$$\sqrt{n}(k + z_{p_1}) \geq z_{(1-\beta)} = z_{1-\beta} = -z_{\beta}$$

Together these imply that

$$k_0 = \frac{z_{\alpha}}{\sqrt{n}} k - z_{p_0}$$

$$k_1 = \frac{-z_{\beta}}{\sqrt{n}} - z_{p_1} \quad (10)$$

and therefore

$$n \geq \left[\frac{z_{\alpha} + z_{\beta}}{z_{p_1} - z_{p_0}} \right]^2 \quad (11)$$

The required sample size is the smallest integer value of n satisfying inequality (11). With n known, the k -factors can be computed as from equations (10).

6.3 Normal (σ -unknown)

The more usual case is where σ unknown and must be estimated from the sample data. While an exact approach is available using a non-central t -distribution (see [4] among others), for modest sample sizes excellent approximations for n and k are achieved applying a result by Cramér [9]. Let $\bar{X}(n)$ and $S(n)$ be sample mean and standard deviation, respectively. For n sufficiently large, the distribution of the acceptance limit $A(n, k) = \bar{X}(n) - kS(n)$ is asymptotically normally distributed $N(\mu_A, \sigma_A^2)$, with mean

$$\mu_A = \mu \pm k\sigma$$

and variance

$$\sigma_A^2 = \frac{\sigma^2}{n} e(k, \xi_3, \xi_4)$$

The limiting variance σ^2/n is weighted by the *expansion factor* $e(k, \xi_3, \xi_4)$ [10]. For a *non-Normal* random variable X with mean μ , standard deviation σ , skew ξ_3 , and kurtosis ξ_4 ,

$$e(k, \xi_3, \xi_4) = \left[1 + \frac{k^2}{4} (\xi_4 - 1) \pm k\xi_3 \right] \quad (12)$$

where the plus is used with an upper bound and the minus with a lower bound. For a *Normal* random variable X , the skew is $\xi_3=0$ and the kurtosis is $\xi_4=3$. The expansion factor therefore reduces to [11]

$$e(k, 0, 3) = \left[1 + \frac{k^2}{2} \right]$$

The derivation of n and k then proceeds as in the σ -known case, resulting in

$$n \geq e(k, 0, 3) \left[\frac{z_{\alpha} + z_{\beta}}{z_{p_1} - z_{p_0}} \right]^2$$

and

$$k = - \left[\frac{z_{\alpha} z_{p_1} + z_{\beta} z_{p_0}}{z_{\alpha} + z_{\beta}} \right]^2$$

6.4 Gamma

Takagi [11] considers the gamma random variable X with unknown location $\delta \leq x < \infty$ and estimated shape $\nu > 0$ and scale $\lambda > 0$ (scale) parameters, support $x \in [\delta, \infty)$, and density function

$$f(x; \nu, \lambda, \delta) = \frac{\{(x - \delta) / \lambda\}^{\nu-1}}{\lambda \Gamma(\nu)} e^{-(x-\delta)/\lambda}$$

The moments for this distribution are

$$\mu = \delta + \nu\lambda$$

$$\sigma^2 = \nu\lambda^2$$

$$\xi_3 = 2/\sqrt{\nu} \tag{13}$$

$$\xi_4 = 3 + 6/\nu$$

With the same normal approximation $N(\mu_A, \sigma_A^2)$ for $A(k, n)$, the expansion factor given in Equation (12) is computed using the gamma moments in Equation (13). The derivation is essentially the same as that for the normal distribution, resulting in

$$n \geq e(k, \xi_3, \xi_4) \left[\frac{z_\alpha + z_\beta}{t_{p_1} - t_{p_0}} \right]^2 \tag{14}$$

and

$$k = - \left[\frac{z_\alpha t_{p_1} + z_\beta t_{p_0}}{z_\alpha + z_\beta} \right]^2 \tag{15}$$

Here

$$t = \frac{X - \delta}{\nu}$$

is a standard gamma deviate with density

$$f(t; \nu) = \frac{t^{\nu-1}}{\Gamma(\nu)} e^{-t}$$

for which $\mu_T=0$ and $\delta_T=1$. The expansion factor in equation (12) is computed using the Weibull moments in Equations (10). The expressions for n and k are again given by Equations (14) and (15), where it is understood that t is the Weibull deviate.

6.5 Weibull

Takagi [11] also considers the Weibull random variable X with unknown location $\delta \leq x < \infty$ and estimated shape $\nu > 0$ and scale $\lambda > 0$ (scale) parameters, support $x \in [\delta, \infty)$, and density function

$$f(x; \nu, \lambda, \delta) = \frac{\nu}{\lambda} \left(\frac{x - \delta}{\lambda} \right)^{\nu-1} e^{[-(x-\delta)/\lambda]^\nu}$$

The moments for this distribution are

$$\mu = \delta + \lambda\mu_T$$

$$\sigma^2 = \lambda^2 \sigma_T^2$$

$$\xi_3 = \left[\Gamma(3b+1) - 3\Gamma(2b+1) + 2\Gamma^3(b+1) \right] / \sigma_T^3 \tag{16}$$

$$\xi_4 = \left[\Gamma(4b+1) - 4\Gamma(3b+1)\Gamma(2b+1) + 6\Gamma(2b+1)\Gamma^2(b+1) \right] / \sigma_T^4$$

Here

$$T = \frac{(x - \delta)}{\lambda}$$

is a standard Weibull random variable with density

$$f(t; \nu) = \nu t^{\nu-1} e^{-t^\nu}$$

and mean and variance

$$\begin{aligned} \mu_T &= \Gamma(b + 1) \\ \sigma_T^2 &= \Gamma(2b + 1) - \Gamma^2(b + 1) \end{aligned}$$

where $b=1/\nu$. The expansion factor in Equation (12) is computed using the Weibull moments in equations (16). The expressions for n and k are again given by Equations (14) and (15), where it is understood that t is the Weibull deviate.

6.6 Poisson

Guenther [4], [12] considers the Poisson random variable X with unknown mean μ , support $x \in \{0, 1, 2, \dots\}$, and distribution function

$$F(x; \mu) = e^{-\mu} \sum_{i=0}^x \frac{\mu^i}{i!},$$

The means μ_0 and μ_1 can be determined from the inverse distribution. Alternately, Guenther exploits the relationship between Poisson and chi-squared distributions and calculates the means using the expression

$$\mu = \frac{\chi_{2x_{min}; 1-p}^2}{2}$$

We can think of X as a discrete random variable representing the number of usable items in a lot and specification limit x_{min} as the specified minimum number of usable items in each lot. If we obtain a sample of n lots, the total number of usable items obtained is the sum

$$Y = \sum_{j=0}^n X_j$$

which is distributed Poisson with a mean of $n\mu$, i.e., $F(y; n\mu)$. Because we are working with discrete random variables (as is the case in attributes acceptance sampling), the rejection criterion is based on the minimum total number of usable items in all of the lots, $y \geq c+1=d$, instead of the distance k . Inequalities (1) and (2) are

$$\begin{aligned} P_a(p \geq p_0) &= 1 - F(y_d; n\mu_0) \geq 1 - \alpha \\ P_a(p \geq p_1) &= 1 - F(d; n\mu_1) \leq \beta \end{aligned}$$

or more simply the power requirements $F(d; n\mu_1) \geq 1 - \beta$ and $F(d; n\mu_0) \leq \alpha$.

Again exploiting relationship between Poisson and chi-squared distributions, the power requirements imply

$$\frac{\chi_{2d; \alpha}^2}{2\mu_0} \geq n \geq \frac{\chi_{2d; 1-\beta}^2}{2\mu_1}$$

This expression can be solved for a (d, n) sampling plan by enumeration. The value of integer values of d is increased until an integer value of n is found which satisfies inequalities 1 and 2.

7. CONCLUSIONS

The work reported in this paper is the first phase of a project intended to make ASV a practical alternative to ASA when appropriate variables plans are available. We initially reviewed the literature, discovering published plans for exponential, normal, gamma, and Weibull random variables. With the exception of normal plans, our search for off-the-shelf ASV plan calculators was fruitless.

To perfect our own understanding and to facilitate the future development of variable plans for a wider selection of distributions, we introduced a consistent notation and interpreted the procedure for developing plans within the common framework of hypothesis testing ([4], [12]). The result presented here is a consolidation of the existing literature. In reviewing the methods presented, we also verified the mathematical derivations.

The completion of this foundational research allowed us to implement plan calculators and test the accuracy these plans empirically. Testing is necessary since, as we have shown here, the majority of variables plans are based on approximating the standard deviation by the employing an expansion factor to the sample standard deviation. In the

final phase of this project, we will address the practical application of ASV when distributional forms are themselves uncertain.

RECEIVED OCTOBER 2012

REVISED FEBRUARY, 2013

Acknowledgements: This work was supported under contract by the NASA Engineering Safety Center.

REFERENCES

- [1] SMITH, E. P., ZAHKAN, A., MAHMOUD, M., AND YE, K. (2003): Evaluation of water quality using acceptance sampling by variables. **Environmetrics**, 14, 373–386.
- [2] BAYARD, R., CVIJETIC, M., GRAHAM, J., MARTA MORALES, M., AND NAFF, B. (2007): **A Systems Approach to Runway Incursion Prevention**, unpublished report, Department of Systems and Information Engineering, University of Virginia, Charlottesville, VA. Third-place prize in the 2007 FAA Airport Design Competition for Universities.
- [3] WHITE, K.P., JR., JOHNSON, K.L., AND CREASEY, R.R., JR. (2009): Attribute acceptance sampling as a tool for verifying requirements using Monte Carlo simulation. **Quality Engineering**, 21, 203-214.
- [4] GUENTHER W. C. (1977): **Sampling Inspection in Statistical Quality Control**. Macmillan, New York.
- [5] BOWKER A. H. AND GOODE, H.P. (1952): **Sampling Inspection by Variables**. McGraw-Hill: New York.
- [6] KAO, J. H. K. (1971): MIL-STD-414: Sampling procedures and tables for inspection by variables for percent defective. **Journal of Quality Technology**, 3, 28–37.
- [7] LIEBERMAN G. J., AND RESNIKOFF, G.J. (1955): Sampling plans for inspection by variables. **Journal of the American Statistical Association**, 50, 457–516.
- [8] MONTGOMERY, D. C. (2005): **Introduction to Statistical Quality Control**. 5th ed., Wiley, New York.
- [9] CRAMÉR H. (1945): **Mathematical Models of Statistics**. Princeton University Press, Princeton, NJ.
- [10] TAKAGI, K. (1972): On designing unknown-sigma sampling plans on a wide class of non-normal distributions, **Technometrics**. 14, 669-678.
- [11] WALLIS, W.A. (1947): Use of variables in acceptance inspection by percent defective, in **Selected Techniques of Statistical Analysis**. McGraw-Hill, New York.
- [12] GUENTHER W. C. (1972): Variables sampling plans for the Poisson and binomial. **Statistica Neerlandica**, 26, 113-120.