AN OPTIMAL CONTROL APPROACH FOR E-RUMOR

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ABSTRACT
Social networks have a significant role in spreading rumors. Such phenomena of e-rumor are big challenges for communities, organizations and states, since the spread of rumors can rapidly jeopardise their public opinion and their economic and financial markets. In these last decades, many mathematical theories have been developed on this topic both in algebraic and numerical terms. In the present work, an optimal control approach is applied on a rumor's dynamical model in order to minimize the spread. At the end of the paper some numerical results are given.

KEYWORDS: e-rumor, optimal control, Cauchy-Lipschitz's theorem, Pontryagin's maximum principle

MSC: 49K15

RESUMEN
Las redes sociales juegan un papel fundamental en la propagación de un rumor. Por su rápida difusión, influye en la imágen pública de instituciones con las consecuentes afectaciones económicas y financieras que esto puede provocar. En los últimos años, se ha estudiado el fenómeno usando herramientas algebraicas y numéricas. Este trabajo se utilizan técnicas d econtorl optimal para el análisis del modelo dinámico correspondiente a la minimización de la difusión del rumor. Se muestran también algunos ejemplos numéricos.

1. INTRODUCTION

With the growing phenomenon of online social networks, a new kind of rumor, e-rumor as one refers to, occurs and progress significantly. Since it can be dangerous for a society, the understood of rumor's diffusion on complex networks is an issue of great importance in social, computer sciences, etc. This problem is a challenge for a long time and becomes more and more important with the development of new technologies. Many network models have been constructed and studied, taking into account the spread's process of rumor. The construction of these models is based on the one of a ring lattice with N nodes, in which each node is connected to its nearest neighbor and takes into account the probability that a node can infect another. Mainly inherited from the susceptible-infected-removed (SIR) model of epidemics, the DK model proposed by Daley and Kendall in [1, 2] subdivides the population of a network into three groups: the ignorants, the spreaders and the stiflers. These last ones are those who know the rumor but choose not to spread it. The previous model has been modified in [6] in order to take into account that the rumor can be propagated

only with a direct contact of someone with a spreader. But these two last models have been proved not to be suitable for rumor spreading in large-scale social networks. Let us note also that these previous studies show that the rumor spreading depends greatly on the average degree of networks.
Then, J. Huang and X. Jin modified a traditional rumor model and tested a random and then a targeted immunization strategy in [3]. Let us underline that, in [3] , the authors considered also a network with the three categories mentioned bellow. In order to introduce their immunization strategy, they divided the stiflers into two subcategories, one corresponding to the individuals who accept the rumor but lose interest to spread it and another one corresponding to the individuals who oppose the rumor and can be regarded as vaccinated nodes. The random immunization strategy has been done to immunize nodes chosen randomly whereas the targeted one concentrated on the choice of nodes with highest degree as immune nodes. Once again, these two types of immunization are effective when the proportion of individuals is not too large.
The idea of the present paper is based on the following. Since the models of e-rumor are inherited from SIR models, and there exists in the literature various contributions using the optimal control theory for epidemiological SIR models, as [4] and [5] for example, it seems natural to try an optimal control approach for e-rumor.
For this, we start with the model proposed by J. Huang and X. Jin in [3] which is a dynamical system, then we modify it to be more realistic, taking into account the dependance in time of the different parameters. The optimal control theory is used in order to minimize in particular the density of spreaders, with a control which is one of the parameters of the model affecting on the propagation of the rumor. Consequently, the first part is dedicated to show the existence of an optimal control for a model of spreading rumor on network introduced in this paper. The characterization of this control will be done in the second part. The third part of the paper will be devoted to numerical experiments, trying to illustrate the behavior of the density of spreaders and ignorants, in a first step, when the number of individuals of the network changes and, in a second step, when one parameter susceptible to be the optimal control varies. Throughout this paper, the notation $\dot{X}(t)$ will denote the derivative of X with respect to the time variable t.

2. EXISTENCE OF AN OPTIMAL CONTROL FOR E-RUMOR

First, let us introduce our rumor model. Denoting by $I(.)$, $S(.)$, $RA(.)$ and $RU(.)$ the density of ignorant nodes, spreader nodes, stifler nodes who accept the rumor and stifler nodes who are under the rumor respectively. The evolution equations of these four quantities could be described with the following dynamical system:

$$\begin{cases} \dot{I}(t) = -(\lambda(t) + \alpha(t) + \beta(t))I(t)S(t), \\ \dot{S}(t) = \lambda(t)I(t)S(t) - \theta(t)S(t)(S(t) + RA(t) + RU(t)), \\ \dot{RA}(t) = \theta(t)S(t)(S(t) + RA(t) + RU(t)) + \alpha(t)I(t)S(t), \\ \dot{RU}(t) = \beta(t)I(t)S(t), \end{cases} \tag{1}$$

with the initial conditions $I(0) = \frac{N-1}{N}$, $S(0) = \frac{1}{N}$ and $RA(0) = RU(0) = 0$, where N is the number of persons of the network. For short, in the rest of the paper, we will denote spreader nodes by spreaders, ignorant nodes by ignorants and stifler nodes by stiflers. The parameters α, β, λ and θ are real continuous functions depending on the time and represent in fact the rules of propagation described as follows.

At the fixed time t, when a spreader meets a ignorant, the last one becomes a spreader with rate $\lambda(t)$, or a stifler who accepts the rumor with rate $\alpha(t)$ or a stifler who opposes the rumor with rate $\beta(t)$. If a spreader meets a spreader or a stifler, the spreader becomes a stifler who accepts the rumor with rate $\theta(t)$. Consequently, for all $t \in I\!R_+$, the coefficients $\alpha(t)$, $\beta(t)$, $\lambda(t)$ and $\theta(t)$ are positive real numbers and $\alpha(t) + \beta(t) + \lambda(t) \leq 1$.

Let us remark that, in our model, the parameters α, β, λ and θ depend on the time t, contrary to the rumor model of [3]. This is natural to take into account this consideration which just traduces the fact that the network (and the parameters) evolves when the time changes.

By noting that, at all time t, $I(t) + S(t) + RA(t) + RU(t) = 1$, we can obtain the fourth equation from the three first ones and see that the derivative of $RA(.)$ depends only on $I(.)$ and $S(.)$. Consequently, we can reduce problem (1) to the following one:

$$\begin{cases} \dot{I}(t) = -(\lambda(t) + \alpha(t) + \beta(t))I(t)S(t), \\ \\ \dot{S}(t) = \lambda(t)I(t)S(t) - \theta(t)S(t)(1 - I(t)). \end{cases} \tag{2}$$

This last system can be rewritten on the form:

$$\dot{X}(t) = F(t, X(t), U(t)), X(0) = X_0, \tag{3}$$

where $X(t) = \begin{pmatrix} I(t) \\ S(t) \end{pmatrix}, X_0 = \begin{pmatrix} \frac{N-1}{N} \\ \frac{1}{N} \end{pmatrix}$,

$$F(t, X(t), U(t)) = \begin{pmatrix} -(\lambda(t) + \alpha(t) + \beta(t))I(t)S(t) \\ (\lambda(t) + \theta(t))\, I(t)S(t) - \theta(t)S(t) \end{pmatrix}$$

and $U = \lambda$ is here the chosen control. This choice of such a control is based on the fact that we want to control the parameter leading to an ignorant to become a spreader, after having met one.

Theorem 2.1. *Let I be an interval of $\mathbb{R}$, V and W open sets of $\mathbb{R}^2$ and $\mathbb{R}$ respectively and $F : I \times V \times W \to V$ the function defined as previously. For all fixed control $U = \lambda$, there exists one and only one maximal solution $([0, t_m(U)], X_U(.))$ of the Cauchy problem (3), with $t_m(U) \in \mathbb{R}_+ \cup \{+\infty\}$.*

Proof. For all $t \in \mathbb{R}_+$, $\lambda(t)$ is a positive real number and $\alpha(t) + \beta(t) + \lambda(t) \leq 1$, which imply that $U(.) = \lambda(.)$ is in $L^\infty(I, \mathbb{R})$. Consequently, for all fixed control U, it is easy to see that the function $(t, X) \mapsto F(t, X, U(t))$ satisfies the assumptions of Cauchy-Lipschitz 's theorem (see [9], p. 205), which completes the proof.

Let $T > 0$, $T \in I$ and let us denote by $\mathcal{U}_T$ the set of all the controls such that the associated trajectory $X_U()$ is well defined, that is $T < t_m(U)$. For all $U \in \mathcal{U}_T$, we define the cost of $U(.)$ by

$$C(U) = \int_0^T \left(aI(t) + bS(t) + \frac{1}{2}cU(t)^2 \right) dt \text{ , with } a, b, c > 0.$$

Note that this choice of cost is motivated by the fact that we want to minimize the density of spreaders, the density of ignorants and the rate λ leading an ignorant to become a spreader, after having met a spreader. We include the density of ignorants in the cost because we consider that an ignorant node is susceptible to become a spreader one.

Theorem 2.2. *Let X_0 be in $\mathbb{R}^2$ such that there is a control $U(.)$ satisfying (3). There exists an optimal control U on $[0, T]$ such that the associated trajectory $X_U(.)$ satisfies (3) and which minimizes the cost $C(.)$.*

Proof. Let us denote by $\mathcal{U}$ the set of all the controls $U : I \to [0, 1]$ satisfying (3). Since $I(t) + S(t) + RA(t) + RU(t) = 1$, we can say that there exists $d > 0$ such that

$$\forall\, U \in \mathcal{U}, \forall t \in [0, T], \quad \|X_U(t)\| \leq d.$$

Moreover, for all $(t, I, S) \in I\!R^3$, the set

$$E(t, I, S) = \left\{ \left(aI + bS + \frac{1}{2}cU^2 + \gamma, -(U + \alpha + \beta)IS, (U + \theta)IS - \theta S \right); U \in [0, 1], \gamma \geq 0 \right\}$$

is convex, due to the convexity of the square function. We omit to put all the t for convenience of writing. The result follows after the existence theorem of optimal trajectory (see [9], p. 94).

3. CHARACTERISATION OF THE OPTIMAL CONTROL FOR E-RUMOR

In this part, we are going to apply Pontryagin's maximum principle (see [7] or [9] p. 103 for more details) in order to characterize the control which minimizes the cost function.

With previous assumptions, if the control $U \in \mathcal{U}$ is optimal on $[0, T]$, that is the associated trajectory $X_U(.)$ is solution of problem (3), with a minimal cost, then there exists an application $P(.) = (P_I(.), P_S(.)) : [0, T] \to I\!R^2$ absolutly continuous called adjoint vector, and a non positive real number P^0, such that $(P(.), P^0)$ is non trivial, and such that, for almost all $t \in [0, T]$,

$$\begin{cases} \dot{I}(t) = \dfrac{\partial H}{\partial P_I}(t, I, S, U, P_I, P_S, P^0), \\ \\ \dot{S}(t) = \dfrac{\partial H}{\partial P_S}(t, I, S, U, P_I, P_S, P^0), \end{cases}$$

and

$$\begin{cases} \dot{P}_I(t) = -\dfrac{\partial H}{\partial I}(t, I, S, U, P_I, P_S, P^0) = P_I(U + \alpha + \beta)S - P_S(U + \theta)S - P^0 a, \\ \\ \dot{P}_S(t) = -\dfrac{\partial H}{\partial S}(t, I, S, U, P_I, P_S, P^0) = P_I(U + \alpha + \beta)I - P_S(U + \theta)I + P_S\theta - P^0 b, \end{cases}$$

where

$$H(t, I, S, U, P_I, P_S, P^0) = -P_I(U + \alpha + \beta)IS + P_S(U + \theta)IS - P_S\theta S + P^0\left(aI + bS + \frac{1}{2}cU^2\right)$$

is the Hamiltonian of the system. We omit again to put all the t for convenience of writing. Moreover, the maximization condition is, almost everywhere on $[0, T]$,

$$H(t, I(t), S(t), U^*(t), P_I(t), P_S(t), P^0) = \max_{v \in [0,1]} H(t, I(t), S(t), v, P_I(t), P_S(t), P^0).$$

If $P^0 \neq 0$, by noting that $\dfrac{\partial H}{\partial U} = 0$ if and only if $U = \dfrac{(P_I - P_S)IS}{cP^0}$ and that the control $U = \lambda$ has to be positive and satisfy $\alpha(t) + \beta(t) + U(t) \leq 1$, for all t, we obtain the following optimal control:

$$U^* = \max\left(\min\left(\frac{(P_I - P_S)I^*S^*}{cP^0}, 1 - \alpha - \beta\right), 0\right),$$

with

$$\begin{pmatrix} \dot{I}^*(t) \\ \dot{S}^*(t) \end{pmatrix} = \begin{pmatrix} -(U^*(t) + \alpha(t) + \beta(t))I^*(t)S^*(t) \\ (U^*((t) + \theta(t))\, I^*(t)S^*(t) - \theta(t)S^*(t) \end{pmatrix},$$

the initial condition $X^*(0) = X_0$ and the transversality condition $P_I(T) = P_S(T) = 0$, since we did not fix a final state.

Conversely, if $P^0 = 0$, then

$$H(t, I, S, U, P_I, P_S, 0) = -P_I(U + \alpha + \beta)IS + P_S(U + \theta)IS - P_S\theta S,$$

and the optimal control U^* is defined from

$$H(t, I, S, U^*, P_I, P_S, 0) = -P_I(\alpha + \beta)IS + P_S\theta S(I - 1) + \max_{v\in[0,1]}(P_S - P_I)vIS.$$

More precisely, the optimal control depends on the sign of $(P_S - P_I)$ and we have

$$U^*(t) = \begin{cases} 0 \text{ if } P_S < P_I, \\ ? \text{ if } P_S = P_I, \\ 1 \text{ if } P_S > P_I. \end{cases}$$

Let us remark that if $P_S = P_I$ at all time $t \in [0, T]$, the transversality condition implies that $P_I(T) = P_S(T) = 0 = P^0$, which is impossible and thus U^* could only be equal to 0 or 1, that is the optimal control problem is purely bang-bang.

4. NUMERICAL RESULTS

There are two types of numerical methods in optimal control, the direct methods and the indirect ones. The direct ones consist to discretize the state and the control, by changing the problem to one of non linear optimization. The indirect methods are based on the maximum principle and consist to solve a limit values problem by a shooting method. The limit values of the adjoint vector are defined in this case from the transversality conditions. But these last ones depend on the initial conditions of the state variables, and are not always complete.

In our case, we have an initial condition on the state variables and a final condition on the adjoint vector. In order to solve numerically our problem, we adapt an algorithm introduced in [4] and which is based on a Gauss-Seidel type method. This simple and practical algorithm consists to discretize the state variables by increasing the indices, since we have an initial condition and then discretize the adjoint variables by decreasing the indices since we have a final condition.

For these numerical experiments we fix $\alpha = \beta = \theta = 0.25$ and choose $P^0 = -1$, $a = b = c = 1$. Let us remark that other choices of parameters would be possible and do not really change the conclusion. Our aim is to illustrate the behavior of the density of spreaders S and the density of ignorants I, solutions of our problem, for different sizes of network and different values of λ, compared with the optimal control.

In a first time, we illustrate the behavior of the number of ignorants I and the number of spreaders S for different values of N (the number of persons of the network), when λ is the optimal control founded previously. The values of N are chosen arbitrarily but others values do not change the appearance of the phenomenon.

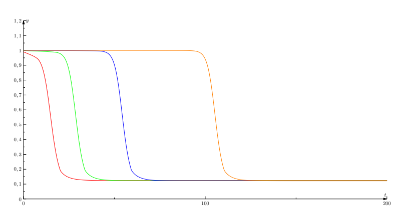

Figure 1: I for $N = 10^2$ (red), $N = 10^3$ (green), $N = 10^4$ (blue), $N = 10^5$ (orange).

In Figure 1, we notice that when N grows, the number of ignorants I, which is a decreasing function as expected, takes more time before decreasing to its smallest constant value. This is natural because this

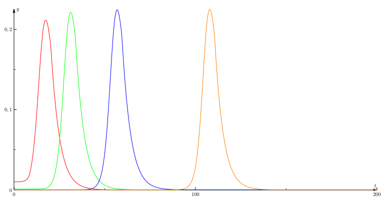

Figure 2: S for $N = 10^2$ (red), $N = 10^3$ (green), $N = 10^4$ (blue), $N = 10^5$ (orange).

means that if the intensity of contacts between ignorants and spreaders is small, it takes longer to get people informed. In Figure 2, we see that the number of spreaders S reaches a maximum before decreasing. and when N grows, S takes more time before reaching its peak which increases with N. We can remark also that for smaller values of N, the maximum of S is smaller, which seems again natural because there are less possible contacts between ignorants and spreaders. By comparing Figures 1 and 2, we can notice that the value of t such that $S(t)$ begins to decrease to zero corresponds to the moment where the number of ignorants remains constant, that is the moment where the rumor begins to disappear.
In a second time, we illustrate for different values of λ, the behavior of the number of ignorants I and the number of spreaders S, on a network of $N = 10^3$ persons. Constant intensities functions $\lambda(.)$ will be compared with the computed optimal control $\lambda = U^*$.

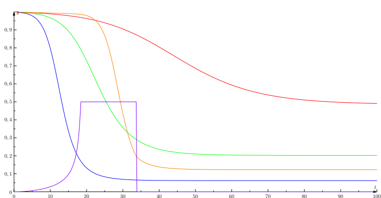

Figure 3: I for $\lambda = 0.1$ (red), $\lambda = 0.25$ (green), $\lambda = 0.49$ (blue), $\lambda = U^*$ (orange); U^* (purple).

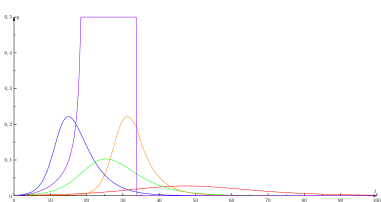

Figure 4: S for $\lambda = 0.1$ (red), $\lambda = 0.25$ (green), $\lambda = 0.49$ (blue), $\lambda = U^*$ (orange); U^* (purple).

In Figure 3, we remark that the function I takes more time to reach its smallest value when λ is small. The controlled I and the situation for $\lambda = 0.49$, giving better results, seem to be more interesting. We remark in Figure 4 that, for little values of λ (less than 0.25 for example), the maximum for S seems to be less that the maximum for bigger values of λ. For these little values, the function S takes more time to reach the null value. For the particular value $\lambda = 0.49$, the maximum and the decreasing of S to 0 are reached more quickly than for the controlled S. However the controlled case is really the best situation according to the chosen objective function we have to minimize which includes a combination of the number of ignorants and the one of spreaders.

5. CONCLUSION

In this paper, the authors introduced a rumor's model which seems to be more complete than the others existing in the literature, in the sense that the different parameters are not fixed and depend on the time. This consideration is natural because the network is not static and is susceptible to know some modifications along the time. Taking into account this consideration, we can mention the recent contribution of E. Stattner, M. Collard and N. Vidot in [8]. For our model, we showed existence and gave the characterization of the optimal parameter λ, that is the rate for which an ignorant becomes a spreader, after having met one. Numerical results are given in order to illustrate the behavior of both spreaders and ignorants in different situations. The numerical results are not strange and seem to be very conform to the reality. An important challenge for our society would be to find some realistic actions in order to reduce or to stop rumors. In a feature work, we plan to study, via an optimal control approach, the efficiency of some external actions on spreaders and on ignorants in order to reach the previous objective.

Acknowledgment. The authors would like to thank Mr Raphael Pasquier, of the C3I center of the University of "Antilles et de la Guyane", who helped us to perform the different figures. We also thank the anonymous referee for his valuable remarks and comments which improved the presentation of this manuscript.

RECEIVED: JULY, 2014.
REVISED: JANUARY, 2015.

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