COMPUTING THE STRONG ALLIANCE POLYNOMIAL OF A GRAPH

W. Carballosa[*,*], J. C. Hernández-Gómez[**,†], O. Rosario[***, ‡] and Y. Torres-Nuñez[****,§]

[*]Cátedra CONACYT, Universidad Autónoma de Zacatecas, México.
[**]Facultad de Matemáticas, Universidad Autónoma de Guerrero, México.
[***]Departamento de Matemáticas, Universidad Carlos III de Madrid, Spain.
[****]Humboldt International University, Miami, Fl., USA.

ABSTRACT
We introduce the strong alliance polynomial of a graph. We obtain some properties of the strong alliance polynomial of a graph and its coefficients. In particular, we compute strong alliance polynomial for path, cycle, complete, start, complete bipartite and double star graphs; some of them verify unimodality.

KEYWORDS: Alliances in Graphs; Strong Defensive Alliances; Polynomials of Graph; Alliance Polynomials; Unimodal Polynomial.

MSC: 05C69; 11B83.

RESUMEN
Este trabajo introduce el polinomio de alianzas fuerte de un grafo. Se obtienen propiedades del polinomio de alianza fuerte de un grafo y sus coeficientes. Particularmente, se calculan los polinomios de alianza fuerte para los grafos camino, ciclo, completo, estrella, completo bipartito y doble estrella; varios de los cuales satisfacen unimodalidad.

1. INTRODUCTION.

The study of the mathematical properties of alliances in graphs were started by P. Kristiansen, S. M. Hedetniemi and S. T. Hedetniemi in 2004, see [12]. The alliances in graphs is a topic of recent and increasing interest in graph theory; see, for instance [5, 8, 10, 15, 16, 18, 19, 20, 21].

Some parameters of a graph G allow to define polynomials on the graph G, for instance, the parameters associated to matching sets [7, 9], independent sets [4, 11], domination sets [1, 2], chromatic numbers [14, 22] and many others. In [6], the authors use the exact index of alliance by define the exact alliance polynomial of a graph. In this work we choose the cardinality of the strong alliance by define the strong alliance polynomial of a graph (see Section 2).

A *graph* G is an ordered pair $\big(V(G), E(G)\big)$ consisting of a set $V(G)$ of *vertices* and a set $E(G)$, disjoint from $V(G)$, of *edges*, together with an *incidence function* ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. If e is an edge and u and v are vertices such that $\psi_G(e) = \{u, v\}$, then e is said to *join* u and v, and the vertices u and v are said *adjacent*. If neither a vertex is adjacent himself nor two edges with the same image by ψ_G then G is said *simple*. We refer to [3] for more background and further results on graphs.

Throughout this paper, $G = (V, E)$ denotes a simple graph of order $|V| = n$ and size $|E| = m$. We denote two adjacent vertices u and v by $u \sim v$. For a nonempty set $X \subseteq V$, and a vertex $v \in V$, $N_X(v)$ denotes the set of neighbors that v has in X: $N_X(v) := \{u \in X : u \sim v\}$, and the degree of v in X will be denoted

---

[*]waltercarb@gmail.com.

[†]carloshg@cimat.mx.

[‡]orosario@math.uc3m.es.

[§]yadiratn83@gmail.com.

by $\delta_X(v) = |N_X(v)|$. We denote the degree of a vertex $v_i \in V$ by $\delta(v_i) = \delta_G(v_i)$ (or by δ_i for short), the minimum degree of G by δ and the maximum degree of G by Δ. The subgraph induced by $S \subset V$ will be denoted by $\langle S \rangle$ and the complement of the set $S \subset V$ will be denoted by $\overline{S}$. This notation is often used by papers on graphs, see ,$e.g.$, [5, 20, 21].

In our work on defensive alliances we use the terminology of [21]. A nonempty set $S \subseteq V$ is a *defensive k-alliance* in G, with $k \in [-\Delta, \Delta] \cap \mathbb{Z}$, if for every $v \in S$,

$$\delta_S(v) \geq \delta_{\overline{S}}(v) + k. \tag{1}$$

The concept of defensive k-alliance was introduced in [17] as a generalization of defensive alliance defined in [12]. Colloquially speaking, a defensive k-alliance in a graph G is a set S of vertices of G such that every vertex in S has at least k more neighbors in S than it has outside of S. A *defensive alliance* in a graph G is a defensive (-1)-alliance in G, while a *strong defensive alliance* is a defensive 0-alliance. In strong defensive alliance S says that each vertex of S is strongly held from a possible attack by the vertices in $\overline{S}$ to be in numerical superiority over its neighbors in $\overline{S}$. The strong defensive alliances are also known as cohesive set, see e.g., [17]. For graphs having defensive k-alliances, the *defensive k-alliance number* of G, denoted by $\alpha_k(G)$, is defined as the minimum cardinality of a defensive k-alliance in G. In particular, we denote by $\alpha(G)$ the *strong defensive alliance number* (i.e., defensive 0-alliance number) of a graph G.

Throughout this paper we just consider defensive k-alliances with $k = 0$, i.e., strong defensive alliances. Also, we pay special attention to strong defensive alliances with connected induced subgraph. Note that to study $S \subset V$ with $\langle S \rangle$ no connected, can be analyzed separately in each connected components. Throughout this paper we consider strong defensive alliance S with connected induced subgraph.

A finite sequence of real numbers $(a_0, a_1, a_2, ..., a_n)$ is said to be *unimodal* if there is some $k \in \{0, 1, ..., n\}$, called the *mode* of the sequence, such that

$$a_0 \leq ... \leq a_{k-1} \leq a_k \quad \text{and} \quad a_k \geq a_{k+1} \geq ... \geq a_n;$$

the mode is unique if $a_{k-1} < a_k$ and $a_k > a_{k+1}$. A polynomial is called unimodal if the sequence of its coefficients is unimodal.

In the next section, we introduce the strong alliance polynomial and obtain some of its properties. In Section 3, we compute the strong alliance polynomial for some graphs and study its coefficients; in particular, we show that some of them are unimodal.

2. STRONG ALLIANCE POLYNOMIAL

In this section, we state the definition of strong alliance polynomial and some of its properties.

Let G be a graph with order n and strong defensive alliance number $\alpha(G)$. We define the *strong alliance polynomial* of G with variable x as follows:

$$a(G; x) = \sum_{i=\alpha(G)}^{n} a_i(G)x^i, \tag{2}$$

where $a_i(G)$ is the number of strong defensive alliances with cardinality i in G.

Note that for any graph G there is $S \subset V$ where $\langle S \rangle$ is a connected component of G. Thus, S in a strong defensive alliance in G.

Remark 2.1. $a(G; x) \neq 0$ *for every graph G.*

The cycle graph C_4 with 4 vertices, for example, has one strong defensive alliance of cardinality 4, four strong defensive alliances of cardinalities 3 and 2; its strong alliance polynomial is then $a(C_4; x) = x^4 + 4x^3 + 4x^2$.

As another example, it is easy to see that the path graph P_4 with 4 vertices has strong alliance polynomial $a(P_4; x) = x^4 + 2x^3 + 3x^2$.

An *isomorphism of graphs* G_1 and G_2 is a bijection between the vertex sets of G_1 and G_2, $f : V(G_1) \to V(G_2)$ such that any two vertices u and v of G_1 are adjacent in G_1 if and only if (u) and (v) are adjacent in G_2. If an isomorphism exists between G_1 and G_2, then the graphs are called *isomorphic* and we write $G_1 \simeq G_2$, see [3, Section 1.2].

Remark 2.2. *Let G_1 and G_2 be isomorphic graphs. Then $a(G_1; x) = a(G_2; x)$.*

The *disjoint union* of graphs, sometimes referred simply as *graph union* is defined as follows. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with disjoint vertex sets V_1 and V_2 (and hence disjoint edge sets), their union is the graph $G_1 \cup G_2 := (V_1 \cup V_2, E_1 \cup E_2)$. It is a commutative and associative operation, see [3, Section 1.4].

Remark 2.3. *If G is not a connected graph, the problem of determine $a(G; x)$ is reduced to determine the strong alliance polynomials of each connected component. Thus, $a(G; x)$ is the sum of the polynomials of each connected component. In other word, if $G = G_1 \cup G_2 \cup \ldots \cup G_r$, then*

$$a(G; x) = \sum_{i=1}^{r} a(G_i; x).$$

The *n-vertex edgeless graph* or *empty graph* is the complement graph for the complete graph K_n, and therefore it is commonly denoted as E_n for $n \geq 1$. Note that, for $n \geq 1$, we have $a(E_n; x) = nx$.

Corollary 2.1. *Let n be a natural number with $n \geq 1$. If $a(G; x) = nx$, then $G \simeq E_n$.*

Proof. Assume first that G is not isomorphic graph to E_n. Since term nx appear in $a(G; x)$, the graph G has n isolate vertices. So, $a(G; x) = a(E_n; x) + a(G^*; x)$ where G^* is a graph such that $G = E_n \cup G^*$. Thus, $a(G; x) \neq a(E_n; x)$ since Remark 2.1 gives $a(G^*; x) \neq 0$. $\square$

The following proposition shows general properties which satisfy the strong alliance polynomials.

Proposition 2.1. *Let G be a graph. Then, $a(G; x)$ satisfies the following properties:*

i) All real zeros of $a(G; x)$ are non-positive numbers.

ii) The value 0 is a zero of $a(G; x)$ with multiplicity $\alpha(G) > 0$.

iii) $a(G; 1) < 2^n$, and it is the number of strong defensive alliance in G.

Proof. Since the coefficients of $a(G; x)$ are non-negatives, we have (1). We have a common factor $x^{\alpha(G)}$ in $a(G; x)$ and $\alpha(G) > 0$. By (2), we have $a(G; 1) = \sum_{i=\alpha(G)}^{n} a_i(G)$. Thus, $a(G; 1)$ has as upper bound the number of connected induced subgraph in G; this amount is less that 2^n, since we have $2^n - 1$ nonempty subsets of V. $\square$

The following theorem gives some properties on coefficients of the strong alliance polynomial of a graph.

Theorem 2.1. *Let G be a graph. Then, the following properties are satisfied*

i) $0 \leq a_2(G) \leq m$; and $a_2(G) = m$ if and only if G is a cycle graph, a path graph, or disjoint union of some of them.

ii) $a_k(G) = 0$ if and only if there is $v \in S$ such that $\delta_S(v) \leq \left\lfloor \frac{\delta(v)-1}{2} \right\rfloor$ in every $S \subset V(G)$ with $|S| = k$.

iii) $a_n(G) = 1$ if and only if G is connected.

iv) $a_2(G) = 1$ if and only if there exists an unique edge $uv \in E(G)$ with $\delta(u), \delta(v) \leq 2$.

Proof. We prove separately each item.

i) On the one hand, it is clear that $a_k(G) \geq 0$, in particular $a_2(G) \geq 0$. Besides, since $a_2(G)$ is the number of subsets of two vertices which are strong defensive alliance in G, then the amount of these subsets has as upper bound the size of G, and so, $a_2(G) \leq m$.

On the other hand, notice that if G is no connected then the proof can be reduced to analyze each connected component. Without loss of generality we can assume that G is connected. First, we claim that $a_2(G) = m$ if and only if $\delta(v_i) \leq 2$ for every $v_i \in V$. Let $uv \in E$ with $S := \{u, v\}$ a strong defensive alliance in G; hence, we have $\delta_S(u) = 1 = \delta_S(v)$, $1 = \delta_S(u) \geq \delta_{\overline{S}}(u)$ and $1 = \delta_S(v) \geq \delta_{\overline{S}}(v)$, thus, we have $\delta(u) \leq 2$ and $\delta(v) \leq 2$. If $\delta(v_i) \leq 2$ for every $v_i \in V$, then taking $S := \{u, v\} \subset V$ with $u \sim v$ we have that $\delta_S(u) + \delta_{\overline{S}}(u) = \delta(u) \leq 2$, and so, $\delta_{\overline{S}}(u) \leq 1 = \delta_S(u)$; analogously we obtain $\delta_{\overline{S}}(v) \leq \delta_S(v)$; in fact, S is a strong defensive alliance in G. Thus, we have proved the claim. Note that it suffices prove that G is an isomorphic graph to a path or cycle graph if and only if $\delta(v_i) \leq 2$ for every $v_i \in V$. It is clear that, if $G \simeq P_n$ or $G \simeq C_n$ for any $n \in \mathbb{N}$ then $a_2(G) = m$. Assume now that $\delta(v_i) \leq 2$ for every $v_i \in V$. If $\delta(v_1) = 1$, then there is $v_2 \in V$ with $v_1 \sim v_2$. Hence, if $\delta(v_2) = 1$ then $G \simeq P_2$, else $\delta(v_2) = 2$ and there is v_3 with $v_2 \sim v_3$. Now, if $\delta(v_3) = 1$ then $G \simeq P_3$, else $\delta(v_3) = 2$ and iterating this process we obtain a vertex v_n with $\delta(v_n) = 1$, and so, $G \simeq P_n$. Similarly, we may obtain the result when there no is $v \in V$ with $\delta(v) = 1$, i.e., $G \simeq C_n$.

ii) On one hand, if $a_k(G) = 0$ for any $k \in \mathbb{N}$, then there is no $S \subset V$ with $|S| = k$ which is strong defensive alliance in G. Hence, for every $S \subset V$ with $|S| = k$ there is $v \in S$ such that $\delta_S(v) < \delta_{\overline{S}}(v)$, so, $2\delta_S(v) < \delta(v)$ and $2\delta_S(v) \leq \delta(v) - 1$. Thus, since $\delta_S(v)$ in an integer number we obtain $\delta_S(v) \leq \left\lfloor \frac{\delta(v)-1}{2} \right\rfloor$. On the other hand, if there is $v \in S$ with $\delta_S(v) \leq \left\lfloor \frac{\delta(v)-1}{2} \right\rfloor$ in every $S \subset V$ with $|S| = k$, then $\delta_S(v) \leq \frac{\delta(v)-1}{2}$ and since $\delta_S(u) + \delta_{\overline{S}}(u) = \delta(u)$ we conclude that $\delta_S(v) < \delta_{\overline{S}}(v)$, therefore, there is no $S \subset V$ with $|S| = k$ which is strong defensive alliance. Thus, we have $a_k(G) = 0$.

iii) Note that V is a strong defensive alliance if and only if G is connected, thus, $a_n(G) = 1$ if and only if G is connected.

iv) If $a_2(G) = 1$ then there exists an unique edge uv such that $S = \{v, u\}$ is a strong defensive alliance in G, besides, using the same argument in item i) we obtain that $\delta(u) \leq 2$ and $\delta(v) \leq 2$. If $uv \in E$ is the unique edge such that $\delta(u), \delta(v) \leq 2$, then; $S = \{u, v\}$ is a strong defensive alliance, but, any $S_1 = \{u_1, v_1\} \subset V$ with $S_1 \neq S$ is no strong defensive alliance since we have either $u_1 \nsim v_1$, $\delta(u_1) > 2$ or $\delta(v_1) > 2$. So, we have $a_2(G) = 1$.

$\square$

3.   STRONG ALLIANCE POLYNOMIALS OF SOME CLASS OF GRAPHS.

In this Section, we obtain the explicit formula for strong alliance polynomials of some classical class of graphs using combinatorial arguments. Besides, we verify unimodality of strong alliance polynomials of path, cycle, complete and complete bipartite graphs.

Proposition 3.2.

1. $a(P_n; x) = \sum_{i=2}^{n}(n + 1 - i)x^i$, *for* $n \geq 2$.

2. $a(C_n; x) = n\sum_{i=2}^{n-1} x^i + x^n$, *for* $n \geq 3$.

3. $a(K_n; x) = \sum_{k=\lceil \frac{n+1}{2} \rceil}^{n} \binom{n}{k} x^k$, *for* $n \geq 1$.

Proof.

1. We compute the coefficients of $a(P_n; x)$ starting by its strong defensive alliance number $\alpha(P_n)$. Clearly, we have $\alpha(P_n) = 2$. By Theorem 2.1 i) we have $a_2(G) = n - 1$. Assume now that $n \geq 3$ and let k be a natural number with $2 \leq k \leq n$. Since every $S \subset V(P_n)$ with $|S| = k$ which is strong defensive alliance verify that $\langle S \rangle \simeq P_k$, we have that $a_k(P_n)$ is the number of different P_k in P_n. Therefore, we obtain $a_k(P_n) = n - k + 1$.

2. Similarly to the previous item, we have $\alpha(C_n) = 2$. By Theorem 2.1 iii) we have $a_n(C_n) = 1$. Let k be a natural number with $2 \leq k \leq n - 1$. Since every $S \subset V(C_n)$ with $|S| = k$ which is strong defensive alliance verify that $\langle S \rangle \simeq P_k$, we have that $a_k(C_n)$ is the number of different P_k in C_n. Therefore, we obtain $a_k(C_n) = n$.

3. Let $S \subset V(K_n)$. So, we have $\delta_S(v_i) = |S| - 1$ and $\delta_S(v_i) + \delta_{\overline{S}}(v_i) = n - 1$ for every $v_i \in S$. Hence, in order to obtain a strong defensive alliance in K_n it suffices to $\delta_S(v_i) \geq \left\lceil \frac{n-1}{2} \right\rceil$ for every $v_i \in S$, and so, we obtain $\alpha(K_n) = \left\lceil \frac{n+1}{2} \right\rceil$. Furthermore, if $|S| \geq \left\lceil \frac{n+1}{2} \right\rceil$, then S is a strong defensive alliance in K_n; thus, we have $a_k(K_n) = \binom{n}{k}$ for every $\left\lceil \frac{n+1}{2} \right\rceil \leq k \leq n$.

$\square$

The following results show that many strong alliance polynomials are unimodal. Note that all strong alliance polynomials included in Proposition 3.2 satisfice that its coefficients are decreasing.

Corollary 3.2. *The strong alliance polynomials of path, cycle and complete graphs are unimodal.*

Now, we recall that $\binom{z}{y}$ is equal zero if y is no integer.

Theorem 3.2. *For $n, m \geq 1$, we have*

$$a(K_{n,m}; x) = \left(a(K_n; x) + \binom{n}{\frac{n}{2}} x^{\frac{n}{2}} \right) \left(a(K_m; x) + \binom{m}{\frac{m}{2}} x^{\frac{m}{2}} \right).$$

Proof. Denote by N and M the parts of vertices set of the complete bipartite graph $K_{n,m}$, i.e., $N \cup M = V(K_{n,m})$, $|N| = n$ and $|M| = m$. Hence, we have $\delta(v) = m$ for every $v \in N$ and $\delta(w) = n$ for every $w \in M$. Clearly, $S \subset V(K_{n,m})$ is a strong defensive alliance in $K_{n,m}$ if and only if there are $S_N \subset N$ and $S_M \subset M$ with $S = S_N \cup S_M$, $|S_N| \geq \left\lceil \frac{n}{2} \right\rceil$ and $|S_M| \geq \left\lceil \frac{m}{2} \right\rceil$. Previous statement is straightforward since $\delta(v) = |S_M|$ for every $v \in S_N$ and $\delta(w) = |S_N|$ for every $w \in S_M$. Therefore, $\alpha(K_{n,m}) = \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{m}{2} \right\rceil$ and for $\left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{m}{2} \right\rceil \leq k \leq n + m$, we have

$$a_k(K_{m,n}) = \sum_{i=\lceil \frac{m}{2} \rceil}^{k - \lceil \frac{n}{2} \rceil} \binom{m}{i} \binom{n}{k-i}.$$

Then, we obtain

$$a(K_{m,n}; x) = \sum_{k=\lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil}^{n+m} \left(\sum_{i=\lceil \frac{m}{2} \rceil}^{k - \lceil \frac{n}{2} \rceil} \binom{m}{i} \binom{n}{k-i} \right) x^k = \left(\sum_{i=\lceil \frac{n}{2} \rceil}^{n} \binom{n}{i} x^i \right) \left(\sum_{j=\lceil \frac{m}{2} \rceil}^{m} \binom{m}{j} x^j \right).$$

$\square$

The complete bipartite graph $K_{n-1,1}$ is called an n *star graph* S_n, for every $n \geq 2$. We have the following consequence of Theorem 3.2.

Corollary 3.3. *For $n \geq 2$, we have*

$$a(S_n; x) = \begin{cases} x \, a(K_{n-1}; x), & \text{if } n \text{ is even,} \\ x \, a(K_{n-1}; x) + \binom{n-1}{\frac{n-1}{2}} x^{\frac{n+1}{2}}, & \text{if } n \text{ is odd.} \end{cases}$$

Remark 3.4. *The strong alliance polynomials of star graphs are unimodals. Note that its coefficients are a decreasing subsequence of binomial numbers.*

Let us consider two star graphs S_r, S_t with r, t vertices. Denote by $v_r \in V(S_r)$ and $w_t \in V(S_t)$ its central vertices. We define the double star graph $S_{r,t}$ as the graph obtained by disjoint union of these star graphs with addition one edge joining v_r and w_t, see Figure 1.
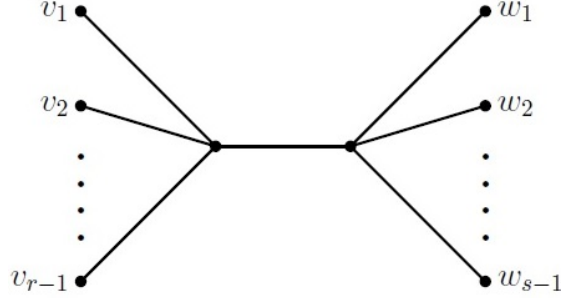

Figure 1: Double star graph $S_{r,t}$ with $r + t$ vertices.

Theorem 3.3. *For $r, t \geq 3$, we have*

$$
a(S_{r,t}; x) = a(S_r; x) + a(S_t; x) - \binom{r-1}{(r-1)/2} x^{(r-1)/2} - \binom{t-1}{(t-1)/2} x^{(t-1)/2}
$$
$$
+ \left(\binom{r-1}{\frac{r}{2}-1} x^{\frac{r}{2}} + a(S_r; x) \right) \left(\binom{t-1}{\frac{t}{2}-1} x^{\frac{t}{2}} + a(S_t; x) \right).
$$

Proof. Colloquially speaking, we say that $S_{r,t}$ is formed by two stars S_r and S_t (without to include the edge joining its central vertices). Denote by v_r and w_t the centers of $S_{r,t}$ with $\delta(v_r) = r$ and $\delta(w_t) = t$ (the center of S_r and S_t, respectively). Let $S \subset V(S_{r,t})$ be a strong defensive alliance in $S_{r,t}$. Then $\{v_r, w_t\} \cap S \neq \emptyset$, thus, we have either (1) $v_r \in S$ and $w_t \notin S$, (2) $v_r \notin S$ and $w_t \in S$ or (3) $v_r \in S$ and $w_t \in S$. Clearly, this cases are disjoints and cases (1) and (2) are symmetric cases.

In order to analyze the firsts cases we can assume that $v_r \in S$ and $w_t \notin S$ (case (1)). Since $\delta(v_r) = r$, we have $\delta_S(v_r) \geq \lceil \frac{r}{2} \rceil$, and so, $|S| \geq \lceil \frac{r+2}{2} \rceil$. Besides, it is a simple matter to every $S^* \subset V(S_{r,t})$ with $v_r \in S$, $w_t \notin S$ and $|S^*| \geq \lceil \frac{r+2}{2} \rceil$ is a strong defensive alliance in $S_{r,t}$. In fact, we obtain in $a(S_{r,t}; x)$, from case (1), as addend

$$
\sum_{i=\lceil \frac{r}{2} \rceil}^{r-1} \binom{r-1}{i} x^{i+1} = a(S_r; x) - \binom{r-1}{(r-1)/2} x^{(r-1)/2}.
$$

So, by symmetric we have the analogous result by case (2), *i.e.*, these strong defensive alliances provide in $a(S_{r,t}; x)$ as addend

$$
a(S_t; x) - \binom{t-1}{(t-1)/2} x^{(t-1)/2}.
$$

In order to finish the proof, we now consider case (3), *i.e.*, $v_r \in S$ and $w_t \in S$. Obviously, $S \subset V(S_{r,t})$ is a strong defensive alliance in $S_{r,t}$ if and only if there are $S_1 \subset V(S_r)$ and $S_2 \subset V(S_t)$ with $S = S_1 \cup S_2$, $|S_1| \geq \lceil \frac{r}{2} \rceil$ and $|S_2| \geq \lceil \frac{t}{2} \rceil$. Thus, we obtain in $a(S_{r,t}; x)$, from case (3), as addend

$$
\left(\sum_{i=\lceil \frac{r}{2} \rceil -1}^{r-1} \binom{r-1}{i} x^{i+1} \right) \left(\sum_{j=\lceil \frac{t}{2} \rceil -1}^{t-1} \binom{t-1}{j} x^{j+1} \right),
$$

or what is the same $\left(\binom{r-1}{\frac{r}{2}-1}x^{\frac{r}{2}} + a(S_r;x)\right)\left(\binom{t-1}{\frac{t}{2}-1}x^{\frac{t}{2}} + a(S_t;x)\right)$. $\qquad\square$

A sequence $a_0, a_1, \ldots, a_n, \ldots$ of nonnegative real numbers is called a *logarithmically concave* sequence, or a *log-concave* sequence for short, if $a_i^2 > a_{i-1}a_{i+1}$ holds for $i > 1$. Furthermore, it follows easily that every log-concave sequence is unimodal. Menon in 1969 proves that the binomial convolution of two log-concave sequences is a log-concave sequence (see [13]). Hence, we have the following consequence of Theorems 3.2 and 3.3.

Corollary 3.4. *The strong alliance polynomials of complete bipartite and double star graphs are unimodal (log-concave).*

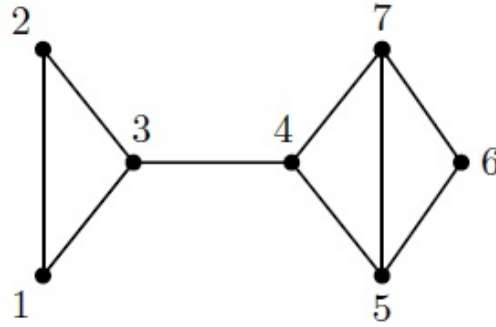The Example 3.1 shows that no all strong alliance polynomial are unimodal.

Figure 2: Graph which its alliance polynomial is not unimodal.

Example 3.1. *Let us consider G the graph in Figure 2. It is easy to compute the coefficients of its strong alliance polynomial.*

$a_2(G) = 1$ *By Theorem 2.1 (iv).*

$a_3(G) = 3$ *Since the strong defensive alliances with cardinality 3 are $\{1,2,3\}$, $\{4,5,7\}$ and $\{5,6,7\}$.*

$a_4(G) = 1$ *Since the strong defensive alliance with cardinality 4 is $\{4,5,6,7\}$.*

$a_5(G) = 4$ *Since the strong defensive alliances with cardinality 5 are $\{1,3,4,5,6\}$, $\{1,3,4,5,7\}$, $\{2,3,4,5,6\}$ and $\{2,3,4,5,7\}$.*

$a_6(G) = 5$ *Since the strong defensive alliances with cardinality 6 are $\{2,3,4,5,6,7\}$, $\{1,3,4,5,6,7\}$, $\{1,2,3,4,6,7\}$, $\{1,2,3,4,5,7\}$ and $\{1,2,3,4,5,6\}$.*

$a_7(G) = 1$ *By Theorem 2.1 (iii).*

Thus, we obtain $a(G;x) = x^2 + 3x^3 + x^4 + 4x^5 + 5x^6 + x^7$. Note that, its coefficients verify $a_2(G) < a_3(G)$, $a_3(G) > a_4(G)$ and $a_4(G) < a_5(G)$.

As usual, we define the graph $G \setminus e$ as the graph with $V(G \setminus e) = V(G)$ and $E(G \setminus e) = E(G) \setminus \{e\}$, where $e \in E(G)$. Analogously, for $r \geq 2$ we define $G \setminus \{e_1, \ldots, e_r\}$ as the graph with $V(G \setminus \{e_1, \ldots, e_r\}) = V(G)$ and $E(G \setminus \{e_1, \ldots, e_r\}) = E(G) \setminus \{e_1, \ldots, e_r\}$, where $\{e_1, \ldots, e_r\} \subset E(G)$.

Theorem 3.4. *Let $n \geq 3$. Then, we have the following statements*

121

1. *If n is odd, then $a(K_n \setminus e_1; x) \neq a(K_n; x)$.*

2. *If n is even, then $a(K_n; x) = a(K_n \setminus e_1; x) = \ldots = a(G \setminus \{e_1, \ldots, e_{\frac{n}{2}-1}\}; x)$, where edges e_i are not common endpoints.*

Note that Theorem 3.4 (2) provides different graphs which have same strong alliance polynomials.

Proof.

1. Let p be the natural number such that $n = 2p + 1$. By Proposition 3.2 (3), we have $\alpha(K_n) = p + 1$ and every $S \subset V(K_n)$ with $|S| \geq p + 1$ is strong defensive alliance in K_n; in particular, $a_{p+1}(K_n) = \binom{n}{p+1}$. Denote $e_1 := [u, v]$. However, we choice $S \subset V(K \setminus e_1)$ with $|S| = p + 1$ and $\{u, v\} \subset S$, but S is no strong defensive alliance since $\delta_S(v) = p - 1 < p = \delta_{\overline{S}}(v)$. Thus, $a_{p+1}(K_n \setminus e_1) < \binom{n}{p+1}$ and so, $a(K_n \setminus e_1; x) \neq a(K_n; x)$.

2. Let p be the natural number with $n = 2p$. Consider G_r a graph obtained from K_n deleting $1 \leq r \leq p$ edges without common vertex. Our next claim is that any subset of $V(G_r)$ with cardinality greater than $p + 1$ is a strong defensive alliance in G_r. Let us consider $S \subset V(G_r)$ with $|S| \geq p + 1$. Then, we have $\delta_S(v) \geq |S| - 2 \geq p - 1$ and $\delta_{\overline{S}}(v) \leq 2p - |S| \leq p - 1$ for every $v \in S$; in fact, S is a strong defensive alliance in G_r. So, we have proved the claim and obtain $a(K_n; x)$ as addend in $a(G_r; x)$.
   We next claim is that if $r < p$ then $\alpha(G_r) = p + 1$. Seeking for a contradiction, assume that there is a strong defensive alliance $S \subset V(G_r)$ with $|S| \leq p$. Note that if $|S| < p$, then $\delta_S(v) \leq |S| - 1 < p - 1$ and $\delta_{\overline{S}}(v) = \delta(v) - \delta_S(v) \geq 2p - 2 - |S| + 1 > p - 1$ for every $v \in S$; in fact, S is no a strong defensive alliance in G_r. Therefore, it suffices suppose that $|S| = p$. Hence, if $|S| = p$, then $\delta_S(v) \leq p - 1$ and $\delta_{\overline{S}}(v) = n - 2 - \delta_S(v) \geq p - 1$ for every $v \in S$. Thus, $\delta_S(v) = p - 1 = \delta_{\overline{S}}(v)$ and $\delta(v) = n - 2$ for every $v \in S$. So, $r = p$. But, this is the contradiction we were looking for. Then, we obtain

$$a(K_n; x) = a(K_n \setminus \{e_1\}; x) = \ldots = a(K_n \setminus \{e_1, \ldots, e_{p-1}\}; x).$$

$\square$

Note that if $\{e_1, \ldots, e_{\frac{n}{2}}\}$ is a subset of $E(K_n)$ without common endpoints, then $\alpha(K_n \setminus \{e_1, \ldots, e_{\frac{n}{2}}\}) = n/2$ since we may choose a strong defensive alliance S in $K_n \setminus \{e_1, \ldots, e_{\frac{n}{2}}\}$ with $|S| = p$ when $\{e_1, \ldots, e_{\frac{n}{2}}\}$ is a set of edges which have one endpoint in S and the other endpoint in $\overline{S}$. Besides, any $S \subset V(K_n \setminus \{e_1, \ldots, e_{\frac{n}{2}}\})$ with cardinality at most $\frac{n}{2} - 1$ is no strong defensive alliance.

Remark 3.5. *If n is even, then $a(K_n; x) \neq a(K_n \setminus \{e_1, \ldots, e_{\frac{n}{2}}\}; x)$, where $\{e_1, \ldots, e_{\frac{n}{2}}\} \subset E(K_n)$ without common endpoints. For instance, $a(K_4; x) \neq a(C_4; x)$.*

ACKNOWLEDGEMENTS

REFERENCES

[1] AKBARI, S., ALIKHANI, S. and PENG, Y-H. (2010): Characterization of graphs using domination polynomials. **European J. Comb.**, 31, 7, 1714-1724.

[2] AROCHA, J.L. and LLANO, B. (2000): Mean value for the matching and dominating polynomial. **Discuss. Math. Graph Theory**, 20, 1, 57-69.

[3] BONDY, J.A. and MURTY, U.S.R. (2008): **Graph Theory**, Graduate Texts in Mathematics 244, Springer.

[4] BROWN, J.I., DILCHER, K. and NOWAKOWSKI, R.J., (2000): Roots of Independence Polynomials of Well Covered Graphs. **J. Algebraic Comb.**, 11, 3, 197-210.

[5] CARBALLOSA, W. (2013): Exact Defensive Alliances in Graphs. **Applied Mathematical Sciences**, 7, 74, 3673-3679.

[6] CARBALLOSA, W., RODRÍGUEZ, J.M., SIGARRETA, J.M. and TORRES-NUÑEZ, Y. (2014): Computing the alliance polynomial of a graph. To appear in **Ars Comb.** http://arxiv.org/abs/1410.2940.

[7] FARRELL, E.J. (1979): An introduction to matching polynomials. **J. Comb. Theory Ser. B**, 27, 75-86.

[8] FRICKE, G.H., LAWSON, L.M., HAYNES, T.W., HEDETNIEMI, S.M. and HEDETNIEMI, S.T. (2003): A Note on Defensive Alliances in Graphs. **Bull. Inst. Comb. Appl.**, 38, 37-41.

[9] GODSIL, C.D. and GUTMAN, I. (1981): On the theory of the matching polynomial. *J. Graph Theory*, 5, 137-144.

[10] HAYNES, T.W., HEDETNIEMI, S.T. and HENNING, M.A. (2003): Global defensive alliances in graphs. **Electr. J. Comb.**, 10, 139-146.

[11] HOEDE, C. and LI, X. (1994): Clique polynomials and independent set polynomials of graphs. **Discrete Math.**, 125, 219-228.

[12] KRISTIANSEN, P., HEDETNIEMI, S.M. and HEDETNIEMI, S.T. (2004): Alliances in graphs. **J. Comb. Math. Comb. Comput.**, 48, 157-177.

[13] MENON, K.V. (1969): On the convolution of logarithmically concave sequences. **Proc. Amer. Math. Soc.**, 23, 439-441.

[14] READ, R.C. (1968): An introduction to chromatic polynomials. **J. Comb. Theory**, 4, 1, 52-71.

[15] RODRÍGUEZ, J.A. and SIGARRETA, J.M. (2007): Spectral study of alliances in graphs. **Discuss. Math. Graph Theory**, 27, 1, 143-157.

[16] RODRÍGUEZ, J.A., GONZALEZ, I. and SIGARRETA, J.M. (2009): Defensive k-alliance graphs. **Appl. Math. Letters**, 22, 96-100.

[17] SHAFIQUE, K.H. and DUTTON, R.D. (2003): Maximum alliance-free and minimum alliance-cover sets. **Congr. Numer.**, 162, 139-146.

[18] SIGARRETA, J.M. (2011): Upper k-alliances in graphs. **Int. J. Contemp. Math. Sciences**, 6(43), 2121-2128.

[19] SIGARRETA, J.M., BERMUDO, S. and FERNAU, H. (2009): On the complement graph and defensive k-alliances. **Discrete Appl. Math.**, 157, 8, 1687-1695.

[20] SIGARRETA, J.M. and RODRÍGUEZ, J.A. (2006): On defensive alliance and line graphs. **Appl. Math. Letters**, 19, 12, 1345-1350.

[21] SIGARRETA, J.M. and RODRÍGUEZ, J.A. (2009): Global defensive k-aliances in graphs. **Discrete Appl. Math.**, 157, 2, 211-218.

[22] TUTTE, W.T. (1954): A contribution to the theory of chromatic polynomials. **Canad. J. Math**, 6, 80-91, 3-4.