IMPROVED ESTIMATION OF FINITE POPULATION VARIANCE USING AUXILIARY INFORMATION IN PRESENCE OF MEASUREMENT ERRORS

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ABSTRACT
This paper discusses the problem of estimating the finite population variance using auxiliary information in presence of measurement errors. We have suggested a class of estimators and its properties are studied under large sample approximation. It has been shown that the usual unbiased estimator and the estimators due to Sharma and Singh [*A generalized class of estimators for finite population variance in presence of measurement errors*, Journal of Modern Applied Statistical Methods, (2013), 12(2), 231-241.] are members of the proposed class of estimators. An alternative expression of the mean squared error of one the estimator due to Sharma and Singh [*A generalized class of estimators for finite population variance in presence of measurement errors*, Journal of Modern Applied Statistical Methods, (2013), 12(2), 231-241.] is also provided. The relative performance of various estimators has been examined through an empirical study.

KEYWORDS: Population mean, Study variate, Auxiliary variates, Measurement errors, Mean squared error, Efficiency.

MSC: 62D05

RESUMEN
En este trabajo se discute el problema de estimar la varianza de una población finita usando información auxiliar en presencia de errores de medición. Sugerimos una clase de estimadores y sus propiedades son estudiadas bajo una aproximación para muestras grandes. Se demuestra que el usual estimador insesgado debido a Sharma & Singh [*A generalized class of estimators for finite population variance in presence of measurement errors*, Journal of Modern Applied Statistical Methods, (2013), 12(2), 231-241.] son miembros de la clase propuesta. Una alternativa expresión del error cuadrático medio de uno de los estimadores debido a Sharma & Singh [*A generalized class of estimators for finite population variance in presence of measurement errors*, Journal of Modern Applied Statistical Methods, (2013), 12(2), 231-241.] Tambien se deriva. El comportamiento relativo de varios estimadores han sido examinados a través de estudio empírico.

1. INTRODUCTION

The statisticians are often interested in the precision of survey estimators. It is well established fact that in survey sampling auxiliary information is traditional used to improve the performance of an estimator of a parameter interest. In survey sampling, the properties of the estimators based on data generally presupposed that the observations are the correct measurements on characteristics being studied. Unfortunately this idea is not met in practice for a variety of reasons, such as non-response errors, reporting errors and computing errors. These sources of variability/errors usually affect a survey. In particular, in this paper we have focused on the problem of estimating population variance when measurement errors are present in the study and auxiliary variate. Various authors including Shalabh (1997), Manisha and Singh (2001), Maneesha and Singh (2002), Allen et al. (2003), Singh and Karpe

(2007, 2008, 2009a, 2009b, 2010a, 2010b), Kumar et al (2011a, 2011b), Diana and Giordan (2012), Sharma and Singh (2013) and Singh et al. (2014) and others have paid their attention towards the estimation of parameters such as population mean μ_Y , variance σ_Y^2 and coefficient of variation C_Y of the study variable Y and ratio and product of two population means in presence of measurement errors.

A finite population $\Omega = \{\Omega_1, \Omega_2, \Omega_3, ..., \Omega_N\}$ of N objects is considered. Let us denote with Y and X the variable of interest and the auxiliary one , respectively , assumed to continuous , for instance, see, Diana and Giordan (2012,p.4303). We also assume that the population mean μ_X of the auxiliary variable X is known. A sample of n objects is drawn from the population Ω using simple random sampling without replacement (*SRSWOR*). We consider a situation where each variable may be observed with error. We assume that x_i and y_i for the sampling units are observed with measurement error as opposed to their true values (X_i, Y_i). For a *SRSWOR* sampling scheme, let (x_i, y_i) be observed values instead of the true values (X_i, Y_i) for the i^{th} (*i*=1, 2,..., *n*) unit, as $U_i = y_i - Y_i$ and $V_i = x_i - X_i$, where U_i and V_i are associated measurement errors which are stochastic in nature with mean 'zero' and variances σ_U^2 and σ_V^2 , respectively. Similarly to Singh and Karpe (2009a), we assume that the error variables U and V are uncorrelated each other and also uncorrelated with X and Y [independence was assumed in Singh and Karpe (2009a)]. That implies $Cov(X,Y) \neq 0$ and $Cov(Y,U) = Cov(Y,V) = Cov(X,U) = Cov(X,V) = Cov(U,V) = 0$. Let (μ_X, μ_Y) and (σ_X^2 , σ_Y^2) be the population means and variances of the variables (X , Y) respectively and ρ be the correlation coefficient between the study variable Y and auxiliary variable X . Let $\bar{x} = n^{-1}\sum_i^n x_i$, $\bar{y} = n^{-1}\sum_i^n y_i$ be the unbiased estimators of population means μ_X and μ_Y respectively. We note that $s_x^2 = (n-1)^{-1}\sum_i^n (x_i - \bar{x})^2$ and $s_y^2 = (n-1)^{-1}\sum_i^n (y_i - \bar{y})^2$ are not unbiased estimators of the population variances σ_X^2 and σ_Y^2 respectively. The expected values of s_x^2 and s_y^2 in presence of measurement errors are respectively given by $E(s_x^2) = \sigma_X^2 + \sigma_V^2$ and $E(s_y^2) = \sigma_Y^2 + \sigma_U^2$.

As mentioned in Singh and Karpe (2009a) and Diana and Giordan (2012) we assume that error variance σ_U^2 and σ_V^2 associated with study variable Y and auxiliary variable X respectively are known. In such situations, the unbiased estimators of σ_Y^2 and σ_X^2 are respectively given by $\hat{\sigma}_Y^2 = s_y^2 - \sigma_U^2 > 0$ and $\hat{\sigma}_X^2 = s_x^2 - \sigma_V^2 > 0$.

Further, we define $\hat{\sigma}_Y^2 = \sigma_Y^2(1+e_0)$ and $\bar{x} = \mu_X(1+e_1)$ such that $E(e_0) = E(e_1) = 0$. Ignoring finite population correction (*fpc*) term, we have the following results:

(i) $E(e_1^2) = \dfrac{C_X^2}{n\theta_X}$,

(ii) $E(e_0 e_1) = \dfrac{\lambda C_X}{n}$, and to the first degree of approximation (ignoring *fpc* terms):

(iii) $E(e_0^2) = \dfrac{A_Y}{n}$,

where $A_Y = [\gamma_{2Y} + \gamma_{2U}(\sigma_U^2/\sigma_X^2)^2 + (2/\theta_Y^2)]$, $\lambda = \mu_{12}(X,Y)/(\sigma_X \sigma_Y^2)$, $C_X = \sigma_X/\mu_X$,
$\gamma_{2Y} = (\beta_2(Y) - 3)$, $\gamma_{2U} = (\beta_2(U) - 3)$, $\beta_2(Y) = \mu_4(Y)/\mu_2^2(Y)$, $\beta_2(U) = \mu_4(U)/\mu_2^2(U)$,

$\mu_2(Y) = E(Y_i - \mu_Y)^2$, $\mu_2(U) = E(U_i - E(U_i))^2 = E(U_i^2)$, $\mu_4(Y) = E(Y_i - \mu_Y)^4$
$\mu_4(U) = E(U_i - E(U_i))^4 = E(U_i^4)$, $\mu_{12}(X,Y) = E\{(X_i - \mu_X)(Y_i - \mu_Y)^2\}$, and
$\theta_X = \sigma_X^2 /(\sigma_X^2 + \sigma_V^2)$ and $\theta_Y = \sigma_Y^2 /(\sigma_Y^2 + \sigma_U^2)$, are reliability ratios of X and Y, respectively, lying between 0 and 1.

Proof of (i): We have

$$e_1 = \frac{\bar{x} - \mu_X}{\mu_X}$$

Squaring both sides of the above expression we have $e_1^2 = (1/\mu_X^2)(\bar{x} - \mu_X)^2$, since $\bar{x} = \bar{V} + \bar{X}$, therefore $e_1^2 = (1/\mu_X^2)(\bar{V} + \bar{X} - \mu_X)^2$ or $e_1^2 = (1/\mu_X^2)[\bar{V}^2 + (\bar{X} - \mu_X)^2 + 2\bar{V}(\bar{X} - \mu_X)]$, where $\bar{V} = n^{-1}\sum_i^n V_i$.

Taking expectation of both sides of the above expression, we have $E(e_1^2) = (1/\mu_x^2)[E(\bar{V}^2) + E(\bar{X} - \mu_X)^2]$.

Thus (ignoring *fpc* term), we get

$$E(e_1^2) = (1/\mu_x^2)[(\sigma_V^2/n) + (\sigma_X^2/n)] = \left(\frac{\sigma_X^2}{n\mu_X^2}\right)[1 + (\sigma_V^2/\sigma_X^2)]$$

$$= (C_X^2/n)[(\sigma_X^2 + \sigma_V^2)/\sigma_X^2)] = \frac{C_X^2}{n\theta_X}$$

which proves the part(i).

Proof of (ii): We have

$$e_0 e_1 = \frac{(\hat{\sigma}_Y^2 - \sigma_Y^2)}{\sigma_Y^2}\frac{(\bar{x} - \mu_X)}{\mu_X} = \frac{1}{\sigma_Y^2\mu_X}[(s_y^2 - \sigma_U^2 - \sigma_Y^2)(\bar{x} - \mu_X)]$$

Also

$$(n-1)s_y^2 = \sum_i^n (y_i - \bar{y})^2 = \sum_i^n (U_i + Y_i - \bar{U} - \bar{Y})^2$$

$$= \sum_i^n \{(U_i - \bar{U})^2 + (Y_i - \bar{Y})^2 + 2(U_i - \bar{U})(Y_i - \bar{Y})\}$$

or $s_y^2 = (s_U^2 + s_Y^2 + 2s_{UY})$, where

$$s_U^2 = (n-1)^{-1}\sum_i^n (U_i - \bar{U})^2,\ s_Y^2 = (n-1)^{-1}\sum_i^n (Y_i - \bar{Y})^2,$$

$$s_{UY} = (n-1)^{-1}\sum_i^n (U_i - \bar{U})(Y_i - \bar{Y}),\ \bar{U} = n^{-1}\sum_i^n U_i,\ \bar{Y} = n^{-1}\sum_i^n Y_i.$$

Thus

$$e_0 e_1 = \frac{1}{\sigma_Y^2\mu_X}[\{(s_U^2 - \sigma_U^2) + (s_Y^2 - \sigma_Y^2) + 2s_{UY}\}\{\bar{V} + (\bar{X} - \mu_X)\}]$$

$$= \frac{1}{\sigma_Y^2\mu_X}[\bar{V}(s_U^2 - \sigma_U^2) + \bar{V}(s_Y^2 - \sigma_Y^2) + 2\bar{V}s_{UY}$$

$$+ (\bar{X} - \mu_X)(s_U^2 - \sigma_U^2) + (\bar{X} - \mu_X)(s_Y^2 - \sigma_Y^2) + 2s_{UY}(\bar{X} - \mu_X)].$$

Taking expectation of the both sides of above expression, we have

$$E(e_0e_1)=\frac{E[(s_Y^2-\sigma_Y^2)(\overline{X}-\mu_X)]}{\sigma_Y^2\mu_X}$$

Thus (ignoring the *fpc* term), we get

$$E(e_0e_1)=\frac{1}{\sigma_Y^2\mu_X}\frac{\mu_{12}(X,Y)}{n}=\left(\frac{1}{n}\right)\frac{\mu_{12}(X,Y)}{\sigma_X\sigma_Y^2}\frac{\sigma_X}{\mu_X}=\frac{\lambda C_X}{n}$$

which proves the part(ii).

Proof of (iii): We have

$$e_0=\frac{\hat{\sigma}_Y^2-\sigma_Y^2}{\sigma_Y^2}$$

Since $\hat{\sigma}_Y^2=s_y^2-\sigma_U^2$, therefore $e_0=\dfrac{(s_y^2-\sigma_U^2-\sigma_Y^2)}{\sigma_Y^2}$ or

$$e_0=(1/\sigma_Y^2)(s_U^2+s_Y^2+2s_{UY}-\sigma_U^2-\sigma_Y^2)$$

or

$$e_0=(1/\sigma_Y^2)\{(s_U^2-\sigma_U^2)+(s_Y^2-\sigma_Y^2)+2s_{UY}\}.$$

Squaring both sides of the above expression, we have

$$e_0^2=(1/\sigma_Y^4)\{(s_U^2-\sigma_U^2)^2+(s_Y^2-\sigma_Y^2)^2+4s_{UY}^2$$
$$+2(s_U^2-\sigma_U^2)(s_Y^2-\sigma_Y^2)+4s_{UY}(s_U^2-\sigma_U^2)+4s_{UY}(s_Y^2-\sigma_Y^2)\}.$$

Taking expectation of both sides of the above expression, we have

$$E(e_0^2)=(1/\sigma_Y^4)[E(s_U^2-\sigma_U^2)^2+E(s_Y^2-\sigma_Y^2)^2+4n(n-1)^{-2}E(s_U-\sigma_U)^2(s_Y-\sigma_Y)^2]$$

Thus to the first degree of approximation (ignoring *fpc* term), we have

$$E(e_0^2)\cong\left(\frac{1}{\sigma_Y^4}\right)\left[\frac{\sigma_U^4}{n}(\beta_2(U)-1)+\frac{\sigma_Y^4}{n}(\beta_2(Y)-1)+4\frac{\sigma_U^2\sigma_Y^2}{n}\right]$$

$$=\left(\frac{1}{n}\right)\left[(\beta_2(U)-1)+\left(\frac{\sigma_U^4}{\sigma_Y^4}\right)(\beta_2(Y)-1)+4\left(\frac{\sigma_U^2}{\sigma_Y^2}\right)\right]$$

$$=\left(\frac{1}{n}\right)\left[(\gamma_{2Y}+2)+(\gamma_{2U}+2)\left(\frac{\sigma_U^4}{\sigma_Y^4}\right)+4\left(\frac{\sigma_U^2}{\sigma_Y^2}\right)\right]$$

$$=\left(\frac{1}{n}\right)\left[\gamma_{2Y}+\gamma_{2U}\left(\frac{\sigma_U^4}{\sigma_Y^4}\right)+2\left(1+2\frac{\sigma_U^2}{\sigma_Y^2}+\frac{\sigma_U^4}{\sigma_Y^4}\right)\right]$$

$$=\left(\frac{1}{n}\right)\left[\gamma_{2Y}+\gamma_{2U}\left(\frac{\sigma_U^4}{\sigma_Y^4}\right)+2\left(1+\frac{\sigma_U^2}{\sigma_Y^2}\right)^2\right]=\left(\frac{1}{n}\right)\left[\gamma_{2Y}+\gamma_{2U}\left(\frac{\sigma_U^4}{\sigma_Y^4}\right)+\frac{2}{\theta_Y^2}\right]$$

This completes the proof of the part (iii).

- **Singh and Karpe (2009a) class of estimators**

Singh and Karpe (2009a) suggested two interesting classes of estimators of σ_Y^2 in the presence of measurement errors when the population mean μ_X of auxiliary variable X is known, the first one is

$$t_d=\hat{\sigma}_Y^2d(b), \tag{1.1}$$

where $d(b)$ is a function of b $(b = \bar{x} / \mu_X)$ such that $d(1) = 1$ satisfying some regularity conditions . The second one is

$$t_D = D(\hat{\sigma}_Y^2, b), \tag{1.2}$$

where $D(\sigma_Y^2, 1) = \sigma_Y^2$.

It has been shown that both the classes of estimators t_d and t_D have the same minimum *MSE* as

$$\min .MSE(t_d \text{ or } t_D) = \frac{\sigma_Y^4}{n}(A_Y - \lambda^2 \theta_X) \tag{1.3}$$

which is equal to the minimum *MSE* of difference-type estimator

$$t_w = \hat{\sigma}_Y^2 + w_2(\mu_x - \bar{x}), \tag{1.4}$$

where w_2 being a suitable chosen constant.

The *MSE* /variance of $\hat{\sigma}_Y^2$ to the first degree of approximation (ignoring *fpc* term) is given by

$$MSE(\hat{\sigma}_Y^2) = Var(\hat{\sigma}_Y^2) = \frac{\sigma_Y^2}{n} A_Y . \tag{1.5}$$

It is observed from (1.3) and (1.5) that the classes of estimators t_d and t_D have smaller minimum *MSE* than the conventional unbiased estimator $\hat{\sigma}_Y^2$.

Sharma and Singh (2013) class of estimators

Sharma and Singh (2013) have proposed the following classes of estimators of σ_Y^2 in the presence of measurement errors:

$$t_1 = w_1 \hat{\sigma}_Y^2 + w_2(\mu_X - \bar{x}), \tag{1.6}$$

$$t_2 = \hat{\sigma}_Y^2 \left\{ 2 - \left(\frac{\bar{x}}{\mu_X} \right)^{\alpha} \exp\left\{ \frac{\beta(\bar{x} - \mu_X)}{(\bar{x} + \mu_X)} \right\} \right\}, \tag{1.7}$$

and

$$t_3 = \left[m_1 \hat{\sigma}_Y^2 + m_2(\mu_X - \bar{x}) \right] \left\{ 2 - \left(\frac{\bar{x}}{\mu_X} \right)^{\alpha} \exp\left\{ \frac{\beta(\bar{x} - \mu_X)}{(\bar{x} + \mu_X)} \right\} \right\}, \tag{1.8}$$

where $(w_1, w_2, \alpha, \beta, m_1, m_2)$ are suitable chosen constants. We note that the estimators t_1 and t_2 are respectively defined on the lines of Singh et al. (1988) and Solanki et al. (2012). The minimum *MSE* of the estimator t_1 due to Sharma and Singh (2013) is given by

$$\min .MSE(t_1) = \left(\frac{\sigma_Y^4}{C^2 - AB} \right)^2 \left[\left(\frac{C^2 - AB}{\sigma_Y^4} \right)^2 + 3BC^2 - AB^2 - 2BC^2 \right], \tag{1.9}$$

[see, Sharma and Singh (2013,equation(12), p.235] where $A = \left(\frac{A_Y}{n} + 1 \right) \sigma_Y^4$, $B = \left(\frac{\mu_X^2 C_X^2}{n\theta_X} \right)$ and

$C = \left(\frac{\sigma_Y^2 \mu_X C_X \lambda}{n} \right)$.

We are observed some typos on minimum *MSE* of t_1 in (1.9) obtained by Sharma and Singh (2013). The correct proof of the minimum *MSE* of t_1 is given in the following theorem 1.1.

Theorem 1.1: The *MSE* of the estimator t_1 to the first degree of approximation is given by

$$MSE(t_1)=\left[\sigma_Y^4+w_1^2\sigma_Y^4\left(1+\frac{A_Y}{n}\right)+w_2^2\left(\frac{\mu_X^2C_X^2}{n\theta_X}\right)-2w_1w_2\left(\frac{\mu_X\lambda C_X\sigma_Y^2}{n}\right)-2w_1\sigma_Y^4\right]$$

$$=[\sigma_Y^4+w_1^2A+w_2^2B-2w_1w_2C-2w_1\sigma_Y^4] \tag{1.10}$$

The optimum values of w_1 and w_2 along with correct minimum *MSE* of the estimator t_1 are respectively given by

$$\left.\begin{aligned} w_{10}&=\frac{B\sigma_Y^4}{\left(AB-C^2\right)}=\frac{1}{\theta_X}\left[\frac{1}{\theta_X}\left(1+\frac{A_Y}{n}\right)-\frac{\lambda^2}{n}\right]^{-1}\\ w_{20}&=\frac{C\sigma_Y^4}{\left(AB-C^2\right)}=\left(\frac{\lambda\sigma_Y^2}{\mu_XC_X}\right)\left[\frac{1}{\theta_X}\left(1+\frac{A_Y}{n}\right)-\frac{\lambda^2}{n}\right]^{-1}\end{aligned}\right\} \tag{1.11}$$

and

$$\min .MSE(t_1)=\left(\frac{\sigma_Y^4}{C^2-AB}\right)^2\left[\frac{(C^2-AB)^2}{\sigma_Y^4}-AB^2+BC^2\right]$$

$$=\sigma_Y^4\left(1-\frac{B\sigma_Y^4}{(AB-C^2)}\right)=\sigma_Y^4\left[1-\frac{1}{\theta_X}\left\{\frac{1}{\theta_X}\left(1+\frac{A_Y}{n}\right)-\frac{\lambda^2}{n}\right\}^{-1}\right]$$

$$=\frac{\min .MSE(t_w)}{1+\left\{\dfrac{\min .MSE(t_w)}{\sigma_Y^4}\right\}} \tag{1.12}$$

Where

$$\min .MSE(t_w)=\frac{\sigma_Y^4}{n}(A_Y-\lambda^2\theta_X) \tag{1.13}$$

Proof is simple so omitted.

The minimum *MSE* of t_2 is same as that of the difference estimator t_w is given by

$$\min .MSE(t_2)=\min .MSE(t_w)=\frac{\sigma_Y^4}{n}(A_Y-\lambda^2\theta_X) \tag{1.14}$$

which is obtained by Sharma and Singh (2013).

The *MSE* of the estimator t_3 to the first degree of approximation (ignoring *fpc* term) obtained by Sharma and Singh (2013) is given by

$$MSE(t_3)=[(1-2m_1)\sigma_Y^4+m_1^2P+m_2^2Q-2m_1m_2R], \tag{1.15}$$

where $P=\left(1+\frac{A_Y}{n}+\frac{k^2C_X^2}{4n\theta_X}-\frac{k}{n}\lambda C_X\right), Q=\left(\frac{\mu_X^2C_X^2}{n\theta_X}\right), R=\sigma_Y^2\left(k\frac{C_X^2}{\theta_X}+2\lambda C_X\right)\frac{\mu_X}{n}$ and $k=(2\alpha+\beta)$.

We are observed some typos on *MSE* of t_3 in (1.15) obtained by Sharma and Singh (2013) and hence the minimum $MSE(t_3)$ obtained by Sharma and Singh [2013, equation (29), p.239] is incorrect. Thus the conclusion based on erroneous result is also not valid. The correct expressions of *MSE* of t_3, optimum values of m_1, m_2 and minimum $MSE(t_3)$ are given in Theorem 1.2.

Theorem 1.2: The correct *MSE* of the estimator t_3 to the first degree of approximation [ignoring fpc term] is given by

$$MSE(t_3)=\sigma_Y^4[1+m_1^2a_1+m_2^2a_2-2m_1m_2a_3-2m_1a_4+2m_2a_5] \tag{1.16}$$

The optimum values of m_1 and m_2 along with minimum *MSE* of t_3 are respectively given by

$$\left.\begin{aligned} m_{10} &= \frac{(a_2a_4-a_3a_5)}{(a_1a_2-a_3^2)} \\ m_{20} &= \frac{(a_3a_4-a_1a_5)}{(a_1a_2-a_3^2)} \end{aligned}\right\} \tag{1.17}$$

and

$$\min .MSE(t_3)=\sigma_Y^4\left[1-\frac{(a_2a_4^2-2a_3a_4a_5+a_1a_5^2)}{(a_1a_2-a_3^2)}\right], \tag{1.18}$$

where $a_1=\left[1+\frac{1}{n}\left(A_Y+\frac{kC_X^2}{2\theta_X}-2k\lambda C_X\right)\right], a_2=\left(\frac{r^2C_x^2}{n\theta_X}\right), a_3=\left[\frac{rC_X}{n}\left(\lambda-\frac{kC_X}{\theta_X}\right)\right],$

$a_4=\left[1+\frac{kC_X}{2n}\left(\lambda+\frac{(k-2)C_X}{4\theta_X}\right)\right],\ a_5=\left(\frac{rkC_X^2}{2n\theta_X}\right)$ and $r=\frac{\mu_X}{\sigma_Y^2}$.

Proof: Expressing the estimator t_3 at (1.8) in terms e's we have

$$\begin{aligned} t_3 &= [w_1\sigma_Y^2(1+e_0)-m_2\mu_X e_1]\left\{2-(1+e_1)^\alpha \exp\left(\frac{\beta e_1}{2+e_1}\right)\right\}, \\ &= [w_1\sigma_Y^2(1+e_0)-m_2\mu_X e_1]\left\{2-(1+e_1)^\alpha \exp\left[\frac{\beta e_1}{2}\left(1+\frac{e_1}{2}\right)^{-1}\right]\right\}. \end{aligned} \tag{1.19}$$

We assume that $|e_1|<1$ so that $(1+e_1)^\alpha$ is expandable. Expanding the right hand side of (1.19), multiplying out and neglecting terms of e's having power greater than two we have

$$t_3 \cong \sigma_Y^4\left[m_1\left\{1+e_0-\frac{k}{2}e_1-\frac{k(k-2)}{8}e_1^2-\frac{k}{2}e_0e_1\right\}-m_2r\left(e_1-\frac{k}{2}e_1^2\right)\right]$$

or

$$(t_3-\sigma_Y^4)\cong\sigma_Y^4\left[m_1\left\{1+e_0-\frac{k}{2}e_1-\frac{k(k-2)}{8}e_1^2-\frac{k}{2}e_0e_1\right\}-m_2r\left(e_1-\frac{k}{2}e_1^2\right)-1\right], \tag{1.20}$$

where $k=(2\alpha+\beta)$.

Taking expectation of both sides of (1.20) we get the bias of t_3 to the first degree of approximation (ignoring *fpc* term) we have

$$B(t_3)\cong\sigma_Y^2\left[m_1\left\{1+e_0-\frac{k}{2}e_1-\frac{k(k-2)}{8}\frac{C_X^2}{n\theta_X}-\frac{k}{2}\frac{\lambda C_X}{n}\right\}+m_2r\frac{k}{2}\frac{C_X^2}{n\theta_X}-1\right]. \tag{1.21}$$

Squaring both sides of (1.20) and neglecting terms of e's having power greater than two we have

$$\begin{aligned}(t_3-\sigma_Y^2)^2 \cong \sigma_Y^4\Bigg[&1+m_1^2\left\{1+2e_0-ke_1+e_0^2+\frac{k}{2}e_1^2-2ke_0e_1\right\}+m_2^2r^2e_1^2-2m_1m_2r(e_1+e_0e_1-ke_1^2) \\ &-2m_1\left\{1+e_0-\frac{k}{2}e_1-\frac{k(k-2)}{8}e_1^2-\frac{k}{2}e_0e_1\right\}+2m_2r\left(e_1-\frac{k}{2}e_1^2\right).\end{aligned} \tag{1.22}$$

Taking expectation on both sides of (1.22) we get the *MSE* of t_3 to the first degree of approximation (ignoring fpc term) as

$$MSE(t_3) = \sigma_Y^4[1 + m_1^2 a_1 + m_2^2 a_2 - 2m_1 m_2 a_3 - 2m_1 a_4 - 2m_2 a_5]. \qquad (1.23)$$

Minimizing (1.23) with respect to m_1 and m_2 we get the optimum values of m_1 and m_2 as given in (1.17). Substituting the optimum values m_{10} and m_{20} [as given in (1.17)] in place of m_1 and m_2 respectively, we get the minimum *MSE* as given in (1.18).Thus, the theorem is proved.

In this paper we have suggested a more general class of estimators of the population variance σ_Y^2 of the study variable Y when the population mean μ_X of the auxiliary variable X is known, in presence of measurement errors. It is identified that the usual unbiased estimator $\hat{\sigma}_Y^2$, and the ratio-type estimator $t_R = \hat{\sigma}_Y^2(\mu_X / \bar{x})$ due to Singh and Karpe (2009a) and the estimators proposed by Sharma and Singh (2013) are member of suggested class of estimators. Properties of suggested class of estimators are studied under large sample approximation. An empirical study is carried out to demonstrate the performance of the proposed class of estimators with other existing estimators.

2. THE PROPOSED CLASS OF ESTIMATORS

We define the class of estimators for the population variance σ_Y^2 of the study variable Y as

$$t = \left[m_1 \hat{\sigma}_Y^2 \left(\frac{\bar{x}}{\mu_X} \right)^{\eta} + m_2(\mu_X - \bar{x}) \right] \left[\delta + (1-\delta) \left(\frac{\bar{x}}{\mu_X} \right)^{\alpha} \exp\left\{ \frac{\beta(\bar{x} - \mu_X)}{(\bar{x} + \mu_X)} \right\} \right], \qquad (2.1)$$

where $(m_1, m_2, \alpha, \beta, \eta, \delta)$ are suitably chosen constants. It is to be mentioned that the class of estimators t reduces to the following set of known estimators of the population variance σ_Y^2 as

(i) $t_0 = \hat{\sigma}_Y^2$ for $(m_1, m_2, \alpha, \beta, \eta, \delta) = (1,0,0,0,0,1)$, [Usual unbiased estimator]

(ii) $t_R = \hat{\sigma}_Y^2 \left(\dfrac{\mu_X}{\bar{x}} \right)$ for $(m_1, m_2, \alpha, \beta, \eta, \delta) = (1,0,0,0,-1,1)$, [Singh and Karpe (2009a) estimator]

(iii) $t_1 = w_1 \hat{\sigma}_Y^2 + w_2(\mu_X - \bar{x})$ for $(m_1, m_2, \alpha, \beta, \eta, \delta) = (w_1, w_2, 0,0,0,1)$, [Sharma and Singh (2013) estimator]

(iv) $t_2 = \hat{\sigma}_Y^2 \left\{ 2 - \left(\dfrac{\bar{x}}{\mu_X} \right)^{\alpha} \exp\left[\dfrac{\beta(\bar{x} - \mu_X)}{(\bar{x} + \mu_X)} \right] \right\}$ for , $(m_1, m_2, \alpha, \beta, \eta, \delta) = (1,0,\alpha,\beta,0,2)$,

[Sharma and Singh (2013) estimator]

(v) $t_3 = [m_1 \hat{\sigma}_Y^2 + m_2(\mu_X - \bar{x})] \left\{ 2 - \left(\dfrac{\bar{x}}{\mu_X} \right)^{\alpha} \exp\left[\dfrac{\beta(\bar{x} - \mu_X)}{(\bar{x} + \mu_X)} \right] \right\}$ for $(m_1, m_2, \alpha, \beta, \eta, \delta) = (m_1, m_2, \alpha, \beta, 0, 2)$,

[Sharma and Singh (2013) estimator]

Many more acceptable estimators can be generated from the class of estimators t defined by (2.1).

To obtain the bias and *MSE* of t in terms e's we have

$$t = [m_1 \sigma_Y^2 (1+e_0)(1+e_1)^{\eta} - m_2 \mu_x e_1] \left\{ \delta + (1-\delta)(1+e_1)^{\alpha} \exp\left(\frac{\beta e_1}{2 + e_1} \right) \right\}$$

$$= \sigma_Y^2 [m_1(1+e_0)(1+e_1)^{\eta} - m_2 r e_1] \left\{ \delta + (1-\delta)(1+e_1)^{\alpha} \exp\left[\frac{\beta e_1}{2} \left(1 + \frac{e_1}{2} \right)^{-1} \right] \right\} \qquad (2.2)$$

Expanding the right hand side of (2.2), multiplying out and neglecting terms of e's having power greater than two, we have

$$t \cong \sigma_Y^2\left[m_1\left\{1+e_0+\theta e_1+\theta e_0 e_1+\frac{\theta^*}{8}e_1^2\right\}-m_2 r\left(e_1+\frac{k(1-\delta)}{2}e_1^2\right)\right]$$

or

$$(t-\sigma_Y^2) \cong \sigma_Y^2\left[m_1\left\{1+e_0+\theta e_1+\theta e_0 e_1+\frac{\theta^*}{8}e_1^2\right\}-m_2 r\left(e_1+\frac{k(1-\delta)}{2}e_1^2\right)-1\right], \tag{2.3}$$

where $\theta=\left[\eta+\frac{k(1-\delta)}{2}\right]$ and $\theta^*=\left[4\eta(\eta-1)+k(1-\delta)(4\eta+k-2)\right]$.

Taking expectation of both sides of (2.3) we get the bias of the class of estimators t to the first degree of approximation (ignoring *fpc* term) as

$$B(t) \cong \sigma_Y^2\left[m_1\left\{1+\frac{\theta\lambda C_X}{n}+\frac{\theta^* C_X^2}{8n\theta_X}\right\}-m_2 r\frac{k(1-\delta)}{2}\frac{C_X^2}{n\theta_X}-1\right]. \tag{2.4}$$

Squaring both sides of (2.3) and neglecting terms of e's having power greater than two we have

$$\begin{aligned}(t-\sigma_Y^2)^2 \cong \sigma_Y^4\Bigg[&1+m_1^2\left\{1+2e_0+2\theta e_1+e_0^2+4\theta e_0 e_1+\left(\frac{\theta^*}{4}+\theta^2\right)e_1^2\right\}+m_2^2 r^2 e_1^2\\ &-2m_1 m_2 r\left\{e_1+e_0 e_1+\left(\theta+\frac{k(1-\delta)}{2}\right)e_1^2\right\}\\ &-2m_1\left\{1+e_0+\theta e_1+\theta e_0 e_1+\frac{\theta^*}{8}e_1^2\right\}+2m_2 r\left(e_1+\frac{k(1-\delta)}{2}e_1^2\right)\Bigg].\end{aligned} \tag{2.5}$$

Taking expectation of both sides of (2.5) we get the *MSE* of t to the first degree of approximation (ignoring the *fpc* term*)* as

$$MSE(t)=\sigma_Y^4[1+m_1^2 A_1+m_2^2 A_2-2m_1 m_2 A_3-2m_1 A_4+2m_2 A_5] \tag{2.6}$$

$$A_1=\left[1+\frac{1}{n}\left(A_Y+4\theta\lambda C_X+\left(\frac{\theta^*}{4}+\theta^2\right)\frac{C_X^2}{\theta_X}\right)\right],\ A_2=\left(\frac{r^2 C_X^2}{n\theta_X}\right)$$

$$A_3=\frac{rC_X}{n}\left(\lambda+\left\{\theta+\frac{k(1-\delta)}{2}\right\}\frac{C_X}{\theta_X}\right),\ A_4=\left[1+\frac{C_X}{n}\left(\theta\lambda+\frac{\theta^* C_X}{8\theta_X}\right)\right]$$

$$A_5=\left(\frac{rk(1-\delta)}{2n}\right)\frac{C_X^2}{\theta_X},\ r=\frac{\mu_X}{\sigma_Y^2} \text{ and } k=(\beta+2\alpha).$$

Differentiating (2.6) partially with respect to m_1 and m_2 and equating them to zero, we have

$$\begin{bmatrix}A_1 & -A_3\\ -A_3 & A_2\end{bmatrix}\begin{bmatrix}m_2\\ m_2\end{bmatrix}=\begin{bmatrix}A_4\\ -A_5\end{bmatrix}. \tag{2.7}$$

After simplification of (2.7) we get the optimum values of m_1 and m_2 as

$$\left.\begin{aligned}m_{1(0)}&=\frac{(A_2A_4-A_3A_5)}{(A_1A_2-A_3^2)}\\ m_{2(0)}&=\frac{(A_3A_4-A_1A_5)}{(A_1A_2-A_3^2)}\end{aligned}\right\}. \tag{2.8}$$

Thus the resulting minimum *MSE* of the proposed class of estimators t is given by

$$\min .MSE(t) = \sigma_Y^4\left[1 - \frac{(A_2 A_4^2 - 2A_3 A_4 A_5 + A_1 A_5^2)}{(A_1 A_2 - A_3^2)}\right]. \tag{2.9}$$

Thus we establish the following theorem.

Theorem 2.1: To the first degree of approximation,

$$MSE(t) \geq \sigma_Y^4\left[1 - \frac{(A_2 A_4^2 - 2A_3 A_4 A_5 + A_1 A_5^2)}{(A_1 A_2 - A_3^2)}\right]$$

with equality holding if

$$\left.\begin{array}{l} m_1 = m_{1(0)} \\ m_2 = m_{2(0)} \end{array}\right\}$$

where $m_{i(0)}$'s , i=1,2; are given by (2.8).

Special Case: For m_1=1, the class of estimators t at (2.1) reduces to the class of estimators:

$$t_{(1)} = \left[\hat{\sigma}_Y^2\left(\frac{\bar{x}}{\mu_X}\right)^\eta + m_2(\mu_X - \bar{x})\right]\left[\delta + (1-\delta)\left(\frac{\bar{x}}{\mu_X}\right)^\alpha \exp\left\{\frac{\beta(\bar{x} - \mu_X)}{(\bar{x} + \mu_X)}\right\}\right], \tag{2.10}$$

Inserting m_1=1, in (2.4) and (2.6) we get the bias and *MSE* of the estimator $t_{(1)}$ to the first degree of approximation, respectively as

$$B(t_{(1)}) \cong \frac{\sigma_Y^2}{n}\left[\left\{\theta\lambda + \frac{\theta^* C_X}{8n_X}\right\}C_X - m_2\frac{rk(1-\delta)C_X^2}{\theta_X}\right] \tag{2.11}$$

and

$$MSE(t_{(1)}) = \sigma_Y^4[1 + A_1 - 2A_4 + m_2^2 A_2 - 2m_2(A_3 - A_5)]. \tag{2.12}$$

The $MSE(t_{(1)})$ is minimized for

$$m_2 = \frac{(A_3 - A_5)}{A_2} = m_{2(0)}^*(say) \tag{2.13}$$

Thus the resulting minimum $MSE(t_{(1)})$ is given by

$$\min .MSE(t_{(1)}) = \sigma_Y^2\left[1 + A_1 - 2A_4 - \frac{(A_3 - A_5)^2}{A_2}\right] = \frac{\sigma_Y^2}{n}(A_Y - \lambda^2\theta_X)$$

$$= \min .MSE(t_w \text{ or } t_d \text{ or } t_D) \tag{2.14}$$

Thus we established the following corollary.

Corollary 2.1: To the first degree of approximation,

$$\min .MSE(t_{(1)}) \geq \sigma_Y^2\left[1 + A_1 - 2A_4 - \frac{(A_3 - A_5)^2}{A_2}\right] = \frac{\sigma_Y^2}{n}(A_Y - \lambda^2\theta_X)$$

with equality holding if $m_2 = m_{2(0)}^* = \dfrac{(A_3 - A_5)}{A_2} = \dfrac{\theta_X\left(\lambda + \{(\theta C_X)/\theta_X\}\right)}{rC_X}$.

Remark 2.1: Suppose that the observations for both the variables *X* and *Y* are recorded without error. The *MSE* of the proposed class of estimators't' to the first degree of approximation is given by

$$MSE(t)_t = \sigma_Y^4[1 + m_1^2 A_{1t} + m_2^2 A_{2t} - 2m_1 m_2 A_{3t} - 2m_1 A_{4t} + 2m_2 A_{5t}], \qquad (2.15)$$

where $A_{1t} = \left[1 + \frac{1}{n}(\beta_2(y) - 1)\right], A_{2t} = \left(\frac{r^2 C_x^2}{n\theta_X}\right), A_{3t} = \frac{rC_X}{n}\left[\lambda + \left(\theta + \frac{k(1-\delta)}{2}\right)C_X\right],$

$$A_{4t} = \left[1 + \frac{C_X}{n}\left(\theta\lambda + \frac{\theta^* C_X}{8}\right)\right], A_5 = \left(\frac{rk(1-\delta)C_X^2}{2n}\right).$$

The $MSE(t)_t$ at (2.10) is minimum when

$$\left.\begin{aligned} m_1 &= \frac{(A_{2t}A_{4t} - A_{3t}A_{5t})}{(A_{1t}A_{2t} - A_{3t}^2)} = m_{10}^* \\ m_2 &= \frac{(A_{3t}A_{4t} - A_{1t}A_{5t})}{(A_{1t}A_{2t} - A_{3t}^2)} = m_{20}^* \end{aligned}\right\}. \qquad (2.16)$$

Thus the resulting minimum value of $MSE(t)_t$ is given by

$$\min .MSE(t)_t = \sigma_Y^4\left[1 - \frac{(A_{2t}A_{4t}^2 - 2A_{3t}A_{4t}A_{5t} + A_{1t}A_{5t}^2)}{(A_{1t}A_{2t} - A_{3t}^2)}\right]. \qquad (2.17)$$

Corollary 2.2: To the first degree of approximation,

$$MSE(t)_t \geq \sigma_Y^4\left[1 - \frac{(A_{2t}A_{4t}^2 - 2A_{3t}A_{4t}A_{5t} + A_{1t}A_{5t}^2)}{(A_{1t}A_{2t} - A_{3t}^2)}\right].$$

with equality holding if

$$\left.\begin{aligned} m_1 &= m_{10}^* \\ m_2 &= m_{20}^* \end{aligned}\right\},$$

where m_{i0}^*'s (i=1, 2); are given by (2.16).

Remark 2.2: Let the observations on both the variables *X* and *Y* be recorded without error. Then *MSE* of the of the estimators' $t_{(1)}$ ' to the first degree of approximation is given by

$$MSE(t_{(1)})_t = \sigma_Y^4[1 + A_{1t} - 2A_{4t} + m_2^2 A_{2t} - 2m_2(A_{3t} - A_{5t})] \qquad (2.18)$$

which is minimum when

$$m_2 = \frac{(A_{3t} - A_{5t})}{A_{2t}} = m_{2(0)t}^* \qquad (2.19)$$

Thus the resulting minimum *MSE* of $t_{(1)}$ is given by

$$\min .MSE(t_{(1)})_t = \sigma_Y^4\left[1 + A_{1t} - 2A_{4t} - \frac{(A_{3t} - A_{5t})^2}{A_{2t}}\right] = \frac{\sigma_Y^2}{n}(\beta_2(Y) - 1 - \lambda^2). \qquad (2.20)$$

Thus we arrived at the following corollary.

Corollary 2.3: To the first degree of approximation,

$$\min .MSE(t_{(1)})_t \geq \sigma_Y^2\left[1 + A_{1t} - 2A_{4t} - \frac{(A_{3t} - A_{5t})^2}{A_{2t}}\right] = \frac{\sigma_Y^2}{n}(\beta_2(y) - 1 - \lambda^2)$$

with equality holding if $m_2 = m^*_{2(0)t} = \frac{(A_{3t} - A_{5t})}{A_{2t}} = \frac{(\lambda + \theta C_X)}{rC_X}$.

From (2.17) and (2.20) we have

$$\min .MSE(t_{(1)})_t - \min .MSE(t)_t = \frac{\sigma_Y^4[A_{2t}(A_{1t} - A_{4t}) + A_{3t}(A_{5t} - A_{3t})]^2}{A_{2t}(A_{1t}A_{2t} - A_{3t}^2)} \tag{2.21}$$

which is always positive. It follows that the proposed class of estimators is more efficient than the $t_{(1)}$ family of estimators when both the variables (Y, X) are measured without error.

3. EFFICIENCY COMPARISONS

To the first degree of approximation (ignoring *fpc* term), the *MSE* of the ratio estimator $t_R = \hat{\sigma}_Y^2(\mu_X / \bar{x})$ is given by

$$MSE(t_R) = \frac{\sigma_Y^4}{n}\left[A_Y - 2\lambda C_X + (C_X^2 / \theta_X)\right] \tag{3.1}$$

From (1.3), (1.5) and (3.1) we have

$$MSE(\hat{\sigma}_Y^2) - \min .MSE(t_d \text{ or } t_D \text{ or } t_w) = \frac{\sigma_Y^4 \lambda^2 \theta_X}{n} \geq 0 \tag{3.2}$$

$$MSE(t_R) - \min .MSE(t_d \text{ or } t_D \text{ or } t_w) = \frac{\sigma_Y^2(\lambda\theta_X - C_X)^2}{\theta_X} \geq 0 \tag{3.3}$$

From (3.2) and (3.3) we have the inequalities:

$$\min .MSE(t_d \text{ or } t_D \text{ or } t_w) \leq MSE(\hat{\sigma}_Y^2) \tag{3.4}$$

and

$$\min .MSE(t_d \text{ or } t_D \text{ or } t_w) \leq MSE(t_R) \tag{3.5}$$

It follows from (3.4) and (3.5) that the difference estimator $t_w(t_d \text{ or } t_D)$ is more efficient than the usual unbiased estimator $\hat{\sigma}_Y^2$ and the ratio estimator t_R due to Singh and Karpe (2009a).

From (2.14) and (2.9) we have

$$\min .MSE(t_{(1)} \text{ or } t_d \text{ or } t_w \text{ or } t_D) - \min .MSE(t) = \sigma_Y^4\left[A_1 - 2A_4 - \frac{(A_3 - A_5)^2}{A_2} + \frac{(A_2A_4^2 - 2A_3A_4A_5 + A_1A_5^2)}{(A_1A_2 - A_3^2)}\right]$$

$$= \sigma_Y^4 \frac{[A_2(A_1 - A_4) + A_3(A_5 - A_3)]^2}{A_2(A_1A_2 - A_3^2)} \geq 0 \tag{3.6}$$

Thus we have the inequality:

$$\min .MSE(t) \leq \min .MSE(t_{(1)} \text{ or } t_d \text{ or } t_w \text{ or } t_D) \tag{3.7}$$

Combining (3.4), (3.5) and (3.7) we have the following inequalities:

$$\min .MSE(t) \leq \min .MSE(t_{(1)} \text{ or } t_d \text{ or } t_w \text{ or } t_D) \leq MSE(\hat{\sigma}_Y^2) \tag{3.8}$$

$$\min .MSE(t) \leq \min .MSE(t_{(1)} \text{ or } t_d \text{ or } t_w \text{ or } t_D) \leq MSE(t_R) \tag{3.9}$$

It follows from (3.8) and (3.9) that the proposed class of estimators 't' is more efficient than the usual unbiased estimator $\hat{\sigma}_Y^2$, ratio estimator t_R due to Singh and Karpe(2009a), the difference-type estimator t_w or t_2 due to Sharma and Singh (2013) , the classes of estimator(t_d, t_D) due to Singh and Karpe(2009a) and the proposed family of estimators $t_{(1)}$.

From (1.12) and (2.9) we have that

$$\frac{1}{\theta_X}\left[\frac{1}{\theta_X}\left(1+\frac{A_Y}{n}\right)-\frac{\lambda^2}{n}\right]^{-1} < \left[\frac{(A_2A_4^2-2A_3A_4A_5+A_1A_5^2)}{(A_1A_2-A_3^2)}\right] \tag{3.9a}$$

It follows that the proposed class of estimators't' is more efficient than the class of estimator t_1 due to Sharma and Singh (2013) as long as the condition (3.9) is satisfied. Thus to obtain the estimators better than Sharma and Singh (2013) estimator t_1 from the proposed class of estimators 't' one should select the values of scalars $(\alpha,\beta,\eta,\delta)$ in such a way that the condition (3.9) is satisfied.

Further from (1.8) and (2.9) it can be shown that the proposed class of estimators't' is better than the class of estimators 't_3' due to Sharma and Singh (2013) if the following inequality:

$$\left[\frac{(a_2a_4^2-2a_3a_4a_5+a_1a_5^2)}{(a_1a_2-a_3^2)}\right] < \left[\frac{(A_2A_4^2-2A_3A_4A_5+A_1A_5^2)}{(A_1A_2-A_3^2)}\right] \tag{3.10}$$

holds true.

We not that a_i's and A_i's (i= 1 to 5) depend on the values of scalars $(\alpha,\beta,\eta,\delta)$, therefore to obtain the estimators better than Sharma and Singh (2013) estimator 't_3' one should select the values of the scalars $(\alpha,\beta,\eta,\delta)$ in such a manner that the condition (3.10) is satisfied.

From (2.14) and (2.20) we have

$$\min .MSE(t_{(1)})-\min .MSE(t_{(1)})_t = \frac{\sigma_Y^4}{n}\left[\gamma_{2u}\left(\frac{\sigma_Y}{\sigma_X}\right)^2+2\left(\frac{\sigma_U}{\sigma_Y}\right)^4+\frac{\lambda^2\sigma_U^2}{(\sigma_X^2+\sigma_V^2)}\right] \tag{3.11}$$

which is always positive. Thus the proposed $t_{(1)}$ family of estimators has larger *MSE* in presence of measurement errors than in the error free case.

Again from (2.6) and (2.17) we note that

$$\min .MSE(t)-\min .MSE(t)_t = \sigma_y^4\left[\frac{(A_{2t}A_{4t}^2-2A_{3t}A_{4t}A_{5t}+A_{1t}A_{5t}^2)}{(A_{1t}A_{2t}-A_{3t}^2)}-\frac{(A_2A_4^2-2A_3A_4A_5+A_1A_5^2)}{(A_1A_2-A_3^2)}\right] \tag{3.12}$$

It can be shown that the difference $[\min .MSE(t)-\min .MSE(t)_t]>0$ which follows that t family of estimators has larger *MSE* in presence of measurement errors than in the error free case.

We note from (3.11) and (3.12) that the presence of measurement errors associated with both the variables(Y, X) increases the *MSEs* of proposed class of estimators $(t, t_{(1)})$. Thus the presence of measurement errors disturb the optimal properties of suggested class of estimators $(t, t_{(1)})$.

4. EMPIRICAL STUDY

We have considered the hypothetical data given in Gujarati and Sangeetha (2007, Table13.2, p.539) as hypothetical population in our study. The variables are:

Table 4.1: *Parameter values from empirical data*

N	λ	μ_Y	μ_X	σ_Y^2	σ_X^2	ρ	σ_U^2	σ_V^2	$\beta_2(Y)$	$\beta_2(U)$
10	0.0112	127	170	1277.9998	3300.0000	0.9641	32.4001	32.3998	1.9026	17.1860

Y_i =True consumption expenditure,

X_i =True income,

y_i =Measured consumption expenditure,

x_i =Measured income,

To illustrate our results we have taken sample size $n = 4$. To judge the merits and inflation in the *MSE* /min.*MSE* of the different estimators of the population variance σ_Y^2, we have computed the relative *MSE*/ min.*MSE* of the estimators when both the variables are measured (i) without error and (ii) with error by using the formulae:

(i)

- Relative $MSE(\hat{\sigma}_Y^2) = \dfrac{MSE(\hat{\sigma}_Y^2)}{\sigma_Y^4} = \dfrac{1}{n}\left(\beta_2(Y) - 1\right),$ (4.1)
- Relative $MSE(t_R) = \dfrac{MSE(t_R)}{\sigma_Y^4} = \dfrac{1}{n}\left[\left(\beta_2(Y) - 1\right) + C_X^2 - 2\lambda C_X\right],$ (4.2)
- Relative min. $MSE(t_2 ort_w) = \dfrac{\min .MSE(t_2 ort_w)}{\sigma_Y^4} = \dfrac{1}{n}\left(\beta_2(Y) - \lambda^2 - 1\right)$ (4.3)
- Relative min. $MSE(t)_t = \dfrac{\min .MSE(t)_t}{\sigma_Y^4} = \left[1 - \dfrac{(A_{2t}A_{4t}^2 - 2A_{3t}A_{4t}A_{5t} + A_{1t}A_{5t}^2)}{(A_{1t}A_{2t} - A_{3t}^2)}\right]$ (4.4)

(ii)

- Relative $MSE(\hat{\sigma}_Y^2) = \dfrac{MSE(\hat{\sigma}_Y^2)}{\sigma_Y^4} = \dfrac{1}{n}A_Y$ (4.5)
- Relative $MSE(t_R) = \dfrac{MSE(t_R)}{\sigma_Y^4} = \dfrac{1}{n}\left(A_Y + \dfrac{C_X^2}{\theta_X} - 2\lambda C_X\right)$ (4.6)
- Relative min. $MSE(t_2 ort_w) = \dfrac{\min .MSE(t_2 ort_w)}{\sigma_Y^4} = \dfrac{1}{n}\left(A_Y - \lambda^2\theta_X\right)$ (4.7)
- Relative min. $MSE(t) = \dfrac{\min .MSE(t)}{\sigma_y^4} = \left[1 - \dfrac{(A_2A_4^2 - 2A_3A_4A_5 + A_1A_5^2)}{(A_1A_2 - A_3^2)}\right]$ (4.8)

for different values of the scalars $(\alpha, \beta, \eta, \delta)$.
Finds are given in Table 4.2 to 4.5.
It is observed from Table 4.2 to 4.5 that the proposed class of estimators 't' has smaller MSE/ min.MSE for suitable values of scalars $(\alpha, \beta, \eta, \delta)$ considered in the Table 4.2 to 4.5, than the usual unbiased estimator $\hat{\sigma}_Y^2$, ratio-type estimator t_R due to Singh and Karpe(2009a) , the difference-type estimator t_w, the estimators t_1, t_2 and t_3 due to Sharma and Singh (2013)and the classes of estimator t_d and t_D envisaged by Singh and Karpe (2009a). Thus there is enough scope of selecting the values of the scalars $(\alpha, \beta, \eta, \delta)$ in order to obtain estimators better than $\hat{\sigma}_Y^2$, t_R, t_1, t_2, t_3 and t_w from the proposed class of estimators t. We have traced from Table 4.2 to 4.5 that the proposed class of estimators has least $\min .MSE(t)$ =78190.8652(without error) and $\min .MSE(t)$ = 912406.5384 for $(\alpha, \beta, \eta, \delta)$ = (2,-1, 1/2, 0). It follows that the measurement error inflates the $\min .MSE(t)$ considerably. So our recommendation is that the practitioner should be very cautious while measuring the units.

Table 4.2: *The relative MSE / min.MSE and the MSE / min.MSE of the usual unbiased estimator* $\hat{\sigma}_Y^2$, *Singh and Karpe's (2009a) estimators and Sharma and Singh's (2013) estimators with and without measurement errors for* $\eta = 0$ *and* $\delta = 2$.

Estimator	The estimators without measurement error		The estimators in presence of measurement error		Contribution of measurement error	
	relative *MSE*/ relative min.*MSE*	*MSE*/ min.*MSE*	relative *MSE*/ relative min.*MSE*	*MSE*/ min.*MSE*	relative *MSE*/ relative min.*MSE*	*MSE*/ min.*MSE*
$\hat{\sigma}_Y^2$	0.2257	368557.3115	1.6118	2632519.4268	1.3861	2263962.1153
t_R	0.2523	412100.8937	1.6387	2676520.7781	1.3864	2264419.8844
t_2, t_w	0.2256	368506.4031	1.6118	2632469.0135	1.3862	2263962.6104
t_1	0.1841	300668.9623	0.6212	1014623.5204	0.4371	713954.5581
$t_{3(\alpha=1,\beta=0)}$	0.1744	284763.2181	0.6194	1011729.8893	0.4450	726966.6712
$t_{3(\alpha=0,\beta=1)}$	0.1766	288430.2323	0.6191	1011157.7120	0.4425	722727.4797
$t_{3(\alpha=1,\beta=1)}$	0.1756	286777.0035	0.6220	1015902.6897	0.4464	729125.6862
$t_{3(\alpha=1,\beta=-1)}$	0.1766	288430.2323	0.6191	1011157.7120	0.4425	722727.4797
$t_{3(\alpha=0,\beta=-1)}$	0.1975	322548.7531	0.6259	1022217.2612	0.4284	699668.5081
$t_{3(\alpha=-1,\beta=0)}$	0.2168	354079.4977	0.6329	1033722.9633	0.4161	679643.4656
$t_{3(\alpha=-1,\beta=2)}$	0.1841	300668.9623	0.6212	1014623.5204	0.4371	713954.5581
$t_{3(\alpha=2,\beta=-1)}$	0.1756	286777.0035	0.6220	1015902.6897	0.4464	729125.6862

Table 4.3: *The relative MSE / min.MSE and the MSE / min.MSE of the suggested class of estimator* t *with and without measurement errors for* $\eta = 0.5$ *and* $\delta = 0$.

Estimator	The estimators without measurement error		The estimators in presence of measurement error		Contribution of measurement error	
	relative *MSE*/ relative min.*MSE*	*MSE*/ min.*MSE*	relative *MSE*/ relative min.*MSE*	*MSE*/ min.*MSE*	relative *MSE*/ relative min.*MSE*	*MSE*/ min.*MSE*
$t_{(\alpha=1,\beta=0)}$	0.1247	203615.3673	0.6158	1005818.6132	0.4911	802203.2459
$t_{(\alpha=0,\beta=1)}$	0.1657	270582.4329	0.6209	1014036.2218	0.4552	743453.7889
$t_{(\alpha=1,\beta=1)}$	0.0479	78190.8652	0.6065	990597.1042	0.5586	912406.2390
$t_{(\alpha=1,\beta=-1)}$	0.1657	270582.4329	0.6209	1014036.2218	0.4552	743453.7889
$t_{(\alpha=0,\beta=-1)}$	0.1837	300045.2222	0.6182	1009706.7709	0.4345	709661.5487
$t_{(\alpha=-1,\beta=0)}$	0.1660	271164.6867	0.6104	997012.5083	0.4444	725847.8216
$t_{(\alpha=-1,\beta=2)}$	0.1838	300206.6878	0.6217	1015339.4798	0.4379	715132.7920
$t_{(\alpha=2,\beta=-1)}$	0.0479	78190.8652	0.6065	990597.1042	0.5586	912406.2390

Table 4.4: *The relative MSE / min.MSE and the MSE / min.MSE of the suggested class of estimator* t *with and without measurement errors for* $\eta = 0.5$ *and* $\delta = 0.5$.

Estimator	The estimators without measurement error		The estimators in presence of measurement error		Contribution of measurement error	
	relative *MSE*/ relative min.*MSE*	*MSE*/ min.*MSE*	relative *MSE*/ relative min.*MSE*	*MSE*/ min.*MSE*	relative *MSE*/ relative min.*MSE*	*MSE*/ min.*MSE*
$t_{(\alpha=1,\beta=0)}$	0.1596	260717.6283	0.6192	1011262.6608	0.4596	750545.0325
$t_{(\alpha=0,\beta=1)}$	0.1757	286983.9528	0.6214	1014857.7883	0.4457	727873.8355
$t_{(\alpha=1,\beta=1)}$	0.1345	219749.1476	0.6151	1004555.9719	0.4806	784806.8243
$t_{(\alpha=1,\beta=-1)}$	0.1757	286983.9528	0.6214	1014857.7883	0.4457	727873.8355
$t_{(\alpha=0,\beta=-1)}$	0.1845	301285.2360	0.6200	1012703.1225	0.4355	711417.8865
$t_{(\alpha=-1,\beta=0)}$	0.1779	290532.9829	0.6165	1006941.3326	0.4386	716408.3497
$t_{(\alpha=-1,\beta=2)}$	0.1838	300206.6878	0.6217	1015339.4798	0.4379	715132.7920
$t_{(\alpha=2,\beta=-1)}$	0.1345	219749.1476	0.6151	1004555.9719	0.4806	784806.8243

Table 4.5: *The relative MSE / min.MSE and the MSE / min.MSE of the suggested class of estimator* t *with and without measurement errors for* $\eta = 1$ *and* $\delta = 0.5$.

Estimator	The estimators without measurement error		The estimators in presence of measurement error		Contribution of measurement error	
	relative *MSE*/ relative min.*MSE*	*MSE*/ min.*MSE*	relative *MSE*/ relative min.*MSE*	*MSE*/ min.*MSE*	relative *MSE*/ relative min.*MSE*	*MSE*/ min.*MSE*
$t_{(\alpha=1,\beta=0)}$	0.1144	186923.7927	0.6119	999475.6790	0.4975	812551.8863
$t_{(\alpha=0,\beta=1)}$	0.1430	233540.7463	0.6152	1004829.8059	0.4722	771289.0596
$t_{(\alpha=1,\beta=1)}$	0.0743	121384.4391	0.6068	991013.5722	0.5325	869629.1331
$t_{(\alpha=1,\beta=-1)}$	0.1430	233540.7463	0.6152	1004829.8059	0.4722	771289.0596
$t_{(\alpha=0,\beta=-1)}$	0.1725	281798.6453	0.6160	1006170.0334	0.4435	724371.3881
$t_{(\alpha=-1,\beta=0)}$	0.1755	286623.0564	0.6136	1002127.7779	0.4381	715504.7215
$t_{(\alpha=-1,\beta=2)}$	0.1619	264496.0006	0.6166	1007065.5390	0.4547	742569.5384
$t_{(\alpha=2,\beta=-1)}$	0.0743	121384.4391	0.6068	991013.5722	0.5325	869629.1331

Acknowledgements: Authors are thankful to Prof. Dr. Carlos Bouza and both the referees for their valuable suggestions regarding improvement of the paper.

RECEIVED: OCTOBER, 2015
REVISED: DECEMBER, 2015

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