A NOTE ON THE FUZZY EXTENSION PRINCIPLE FOR LP PROBLEMS WITH FUZZY COEFFICIENT MATRIX

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ABSTRACT
In this paper we analyze a special case of Fuzzy Linear Programming (FLP) in which its constraints are composed by fuzzy numbers contained into an interval. The binary relation between crisp constraints and fuzzy coefficients is analyzed, and the membership degree of a random realization of fuzzy parameters is defined. An application example is presented and solved.

KEYWORDS: Fuzzy Linear Programming; α-cuts; Operation points; Fuzzy extension principle.

MSC: 90C70; 06D72.

RESUMEN
En este artículo analizamos un caso especial de Programación Lineal Difusa en el cual sus restricciones se componen por números difusos contenidos en un intervalo. Se analiza la relación binaria entre restricciones clásicas y coeficientes difusos, y se define el grado de pertenencia de una realización aleatoria de parámetros difusos. Un ejemplo de aplicación es presentado y resuelto.

1. INTRODUCTION

Linear Programming (LP) is among the most used optimization models due to its efficiency, simplicity and reliability. In the last decades, fuzzy sets have been incorporated into LP models to represent imprecision coming from human being perceptions with successful results. Different approaches to Fuzzy Linear Programming (FLP) problems have been presented in bibliography, Rommelfanger [18], [19], [16], [17], Zimmermann [20], Zimmermann & Fullér [21], Fiedler et al. [3], Ramík [15], [14], Ramík and Řimánek [13], Gasimov & Yenilmez [6]. Černý & Hladík [2], and Hladík [8] has defined two similar families of fuzzy LPs: problems with fuzzy parameters and fuzzy constraints *(FLP)*, and problems with fuzzy parameters and crisp constraints.

Based on the works of Hladík [8], we analyze how fuzzy coefficients operate over crisp constraints in two ways: a constructivist (a priori) and a practical (a posteriori) approaches. The constructivist approach is based on interval computations using α-cuts, and the a posteriori approach is based on possible choices of the coefficients a_{ij} enclosed into fuzzy parameters which is a common case in real applications. The main idea is to use fuzzy sets to see the behavior of the problem in advance and how a set of observed coefficients are enclosed into the expected results.

The paper is organized into 6 sections; Section 1 introduces the main problem; Section 2 presents some basic definitions of fuzzy sets; in Section 3 we present the fuzzy/crisp LP model; Section 4 presents a discussion about some theoretical aspects of the extension principle for fuzzy sets; In section 5, an example is presented and solved, and finally Section 6 presents some concluding remarks of the study.

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2. BASIC NOTATIONS

In this paper, we consider X as the powerset whose elements $x \in X$ are real numbers $\mathbb{P}(X) \in \mathbb{R}$, and $\mathcal{P}(X)$ is the class of all crisp sets. In a *crisp* set $A \in X$, an element x is either a member of the set or not. The indicator function of A, χ_A is defined as follows:

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \tag{1}$$

A set S is called singleton $\{S\}$ if has a single element $x \in \mathbb{R}$. In the real numbers $\mathbb{R}$, S is a constant. This implies that $\chi_S(x) = 1$, and $\chi_S(\cdot) = 0$ for every $x \notin S$.

$$S := \{x : x = S\} \tag{2}$$

$$S : X \to \{0, 1\} \tag{3}$$

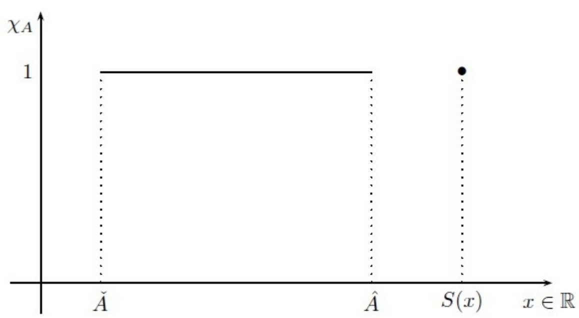

Figure 1: Crisp set A and singleton $S(x)$

A fuzzy set $\tilde{A}$ is a generalization of a *crisp* or boolean set. It is defined on an universe of discourse X and is characterized by a *Membership Function* namely $\mu_{\tilde{A}}(x)$ that takes values in the interval [0,1], $\tilde{A}: X \to [0, 1]$. A fuzzy set $\tilde{A}$ may be represented as a set of ordered pairs of an element x and its membership degree, $\mu_{\tilde{A}}(x)$, i.e.,

$$\tilde{A} = \{(x, \mu_A(x)) \,|\, x \in X\} \tag{4}$$

where $\mathcal{F}(\mathbb{R})$ is the class of all fuzzy sets.
Now, $\tilde{A}$ is contained into a family of fuzzy sets $\mathcal{F} = \{\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_m\}$, each one with a membership function $\{\mu_{\tilde{A}_1}(x), \mu_{\tilde{A}_2}(x); \cdots, \mu_{\tilde{A}_m}(x)\}$. The *support* of $\tilde{A}$, *supp*$(\tilde{A})$, is composed by all the elements of X that have nonzero membership in $\tilde{A}$, this means

$$supp(\tilde{A}) = \{x \,|\, \mu_{\tilde{A}}(x) > 0\} \; \forall \; x \in X \tag{5}$$

The α-*cut* of $\mu_{\tilde{A}}(x)$ namely ${}^{\alpha}\tilde{A}$ represents the interval of all values of x which has a membership degree equal or greatest than α, this means:

$${}^{\alpha}\tilde{A} = \{x \,|\, \mu_{\tilde{A}}(x) \geqslant \alpha\} \; \forall \; x \in X \tag{6}$$

where the interval of values which satisfies ${}^{\alpha}\tilde{A}$ is defined by

$$ {}^{\alpha}\tilde{A} \in \left[\inf_{x} {}^{\alpha}\mu_{\tilde{A}}(x), \, \sup_{x} {}^{\alpha}\mu_{\tilde{A}}(x) \right] = \left[\check{A}_{\alpha}, \, \hat{A}_{\alpha} \right] \tag{7} $$

A graphical display of a triangular fuzzy set is given in Figure 2.

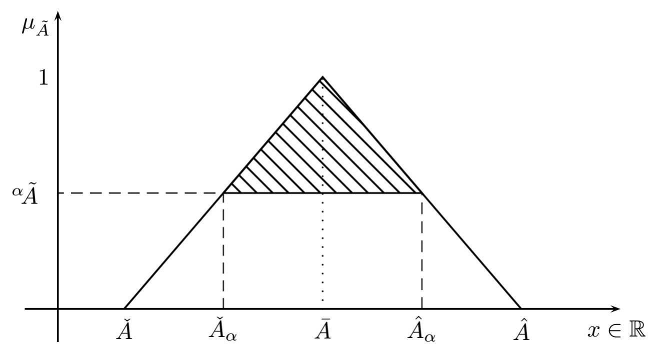

Figure 2: Type-1 Fuzzy set A

Here, $\tilde{A}$ is a Type-1 fuzzy set, its universe of discourse is the set of all values $x \in \mathbb{R}$, the *support* of $\tilde{A}$, $supp(\tilde{A})$ is the interval $x \in [\check{A}, \hat{A}]$ and $\mu_{\tilde{A}}$ is a triangular function with parameters $\check{A}, \bar{A}$ and $\hat{A}$. α is the degree of membership that an specific value x has regarding A and the dashed region is an α-*cut* done over $\tilde{A}$. A *fuzzy number* is then a convex fuzzy set defined over $\mathbb{R}^n$, defined as follows

Definition 1 (Fuzzy Number)**.** *Let $\tilde{A} \in \mathcal{F}(\mathbb{R})$. Then, $\tilde{A}$ is a Fuzzy Number (FN) iff there exists a closed interval $[a, b] \neq 0$ such that*

$$ \mu_{\tilde{A}}(x) = \begin{cases} 1 & \text{for } x \in [a, b], \\ l(x) & \text{for } x \in [-\infty, a], \\ r(x) & \text{for } x \in [b, \infty] \end{cases} \tag{8} $$

where $l : (-\infty, a) \to [0, 1]$ is monotonic non-decreasing, continuous from the right, and $l(x) = 0$ for $x < \omega_1$, and $r : (b, \infty) \to [0, 1]$ is monotonic non-increasing, continuous from the left, and $r(x) = 0$ for $x > \omega_2$.

Note that α-cuts done over fuzzy numbers are monotonically increasing/decreasing which means that given $\alpha_1 < \alpha_2, \alpha \in [0, 1]$, then ${}^{\alpha_2}\tilde{A} \subseteq {}^{\alpha_1}\tilde{A}$, so it is clear that ${}^{\alpha}\tilde{A} \subseteq supp(\tilde{A})$, $\forall \, \alpha \in [0, 1]$.

3. FUZZY/CRISP LP MODELS

Crisp LPs i.e $z = \text{Max}_x\{z = c'x \,|\, Ax \leqslant b, \, x \geqslant 0\}$ are a special cases of Fuzzy LPs (FLPs for short) in which all parameters are defined as singletons (see (10)) and the inequality $\leqslant b_i$ is a set whose indicator function is $\chi_{b_i}(x)$. In crisp LPs, the membership degree a solution has is always one, so no uncertainty is involved into the problem which means that global optimization is possible in that kind of problems.
On the other hand, FLPs involve fuzzy uncertainty $\text{Max}_x\{\tilde{z} = \tilde{c}'x \,|\, \tilde{A}x \precsim \tilde{b}, \, x \geqslant 0\}$. In this paper, all its parameters are considered as finite-domain fuzzy numbers which means that $supp(\tilde{A})$ is a closed interval. The binary relation $\precsim$ has been defined by Ramík and Řimánek [13] which description is as follows:

Definition 2. *Let $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R})$ be two fuzzy numbers. Then $\tilde{A} \precsim \tilde{B}$ if and only if* $\sup {}^{\alpha}\tilde{A} \leqslant \sup {}^{\alpha}\tilde{B}$ *and* $\inf {}^{\alpha}\tilde{A} \leqslant \inf {}^{\alpha}\tilde{B}$ *for each $\alpha \in [0, 1]$, where ${}^{\alpha}\tilde{A}$ and ${}^{\alpha}\tilde{B}$ are α-cuts of $\tilde{A}$ and $\tilde{B}$ respectively, and* ${}^{\alpha}\tilde{A} := [\inf {}^{\alpha}\tilde{A}, \sup {}^{\alpha}\tilde{A}]$ *and* ${}^{\alpha}\tilde{B} := [\inf {}^{\alpha}\tilde{B}, \sup {}^{\alpha}\tilde{B}]$. *This binary relation satisfies the axioms of a partial order relation on $\mathcal{F}(\mathbb{R})$ and is called the fuzzy max order.*

This binary relation (fuzzy max order) has been extended to Interval-valued fuzzy numbers by Figueroa-García et al. [4], and Figueroa-García [5], and it can be extended to fuzzy/crisp sets as follows:

Proposition 1. *Let $\tilde{A} \in \mathcal{F}(\mathbb{R})$ be a fuzzy number, and $b \in \mathcal{P}(\mathbb{R})$ be a crisp set. Then $\tilde{A} \lesssim b$ if and only if $\sup {}^\alpha\tilde{A} \leqslant b$ and $\inf {}^\alpha\tilde{A} \leqslant b$ for each $\alpha \in [0,1]$, where ${}^\alpha\tilde{A} := [\inf {}^\alpha\tilde{A}, \sup {}^\alpha\tilde{A}] = [\check{A}_\alpha, \hat{A}_\alpha]$. This binary relation satisfies the axioms of a partial order relation on $\mathcal{F}(\mathbb{R})$ and is called the fuzzy/crisp max order.*

Proof. To show how this binary relation relates a convex fuzzy number to a crisp constraint, we will prove the three main partial ordering conditions: **i) (reflexivity)**: it is clear that ${}^\alpha\tilde{A} \leqslant {}^\alpha\tilde{A} \,\forall\, \alpha \in [0,1]$ and $b \leqslant b$. If $\tilde{A} \leqslant b$ then ${}^\alpha\tilde{A} \leqslant b \,\forall\, \alpha \in [0,1]$ which means $\hat{A}_\alpha \leqslant b \,\forall\, \alpha \in [0,1]$. **ii) (antisymmetry)**: the only case in which $\tilde{A} \leqslant b$ and $\tilde{A} \geqslant b$ lead to $\tilde{A} = b$ is the case $\mu_A(x) = \chi_b(x)$ which is equivalent to say that $\tilde{A}$ and b are crisp sets (i.e sets with the same indicator function). **iii) (transitivity)**: suppose that $\tilde{A} \leqslant b$ and we introduce a new crisp set $\{c : b \leqslant c\}$. If ${}^\alpha\tilde{A} \leqslant b, \forall\, \alpha \in [0,1]$ and $b \leqslant c$ then ${}^\alpha\tilde{A} \leqslant c, \forall\, \alpha \in [0,1]$. On the other hand, we can consider the case in which two fuzzy numbers $\tilde{A}$ and $\tilde{C}$ where $\tilde{C} \leqslant \tilde{A}$ which implies ${}^\alpha\tilde{C} \leqslant {}^\alpha\tilde{A}, \forall\, \alpha \in [0,1]$, so if $\tilde{A} \leqslant b$ and $\tilde{C} \leqslant \tilde{A}$ then we have $\tilde{C} \leqslant \tilde{A} \leqslant b$, which concludes the proof. □

Figure 3 shows the binary relation $\lesssim$. Note that $\tilde{A} \lesssim b$ since $supp\tilde{A} \leqslant b$ which is equivalent to say $\hat{A} \leqslant b$.

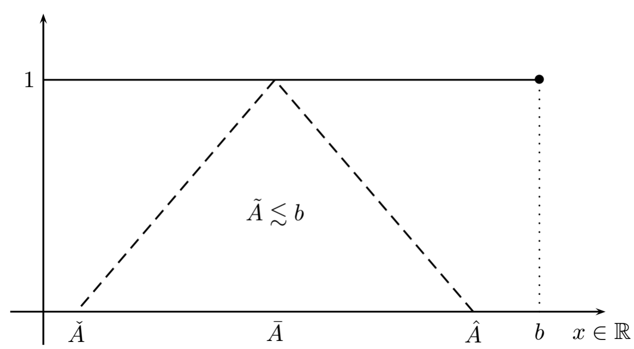

Figure 3: Fuzzy/crisp binary relation $\lesssim$

Some practical optimization applications have well defined cost structures and well bounded constraints, but sometimes the technological coefficient matrix (a.k.a technological matrix) contains non-probabilistic uncertainty coming from different sources. This uncertainty sometimes is measured using human like language through human perception about different variables while well defined costs and constraints can be defined as singletons/crisp sets.

To exemplify a fuzzy/crisp LP problem think on a situation in which a company has a reliable cost structure, a well defined inventory system, and clearly defined human resources policies. On the other side, human time consumption, manufacturing time consumption, and raw materials consumption are not usually constant since they contain some degree of uncertainty. In this case, the system is composed by crisp costs/constraints and fuzzy technological coefficients, so it is a fuzzy/crisp LP problem. An LP formulation for this fuzzy/crisp problem is:

$$\tilde{z}^* = \mathrm{Max}_x\{\tilde{z} = c'x \mid \tilde{A}x \lesssim b,\, x \geqslant 0\} \tag{9}$$

This model can be solved using the extension principle or approximation techniques. For instance, the Zimmermann's soft constraints problem is solved via an auxiliary variable λ that operates as the maximum satisfaction degree between fuzzy constraints and a symmetric fuzzy objective function, and other fuzzy LPs can be solved using a similar reasoning.

An approach to find an appropriate way for modeling fuzzy functions is given by the Zadeh's *Extension principle* (see Bellman & Zadeh [1], Klir & Yuan [9]). Let f be a function $f : X_1, X_2, \cdots, X_n \to z$, and $\tilde{A}_i$

be a fuzzy set in X_i, $i = 1, 2, \cdots, n$ with $x_i \in X_i$, then we have

$$\tilde{z} = f(\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_n)(z) = \sup_{z=f(x_1,x_2,\cdots,x_n)} \min_i [\tilde{A}_1(x_1), \tilde{A}_2(x_2), \cdots, \tilde{A}_n(x_n] \tag{10}$$

Then, the extension principle is useful to project any function e.g $z = f(x_1, x_2, \cdots, x_n)$ into a fuzzy set of solutions $\tilde{z}$ using the memberships of $\tilde{A}_1(x_1), \tilde{A}_2(x_2), \cdots, \tilde{A}_n(x_n)$. Some comments about the use of the extension principle to the model (9) are provided as follows.

4. COMMENTS ON THE EXTENSION PRINCIPLE FOR FLP PROBLEMS

In general, the problem of finding a global optimal solution of an FLP is more complex than a crisp LP since we have infinite combinations of $\tilde{A}x^* \leqslant \tilde{b}$ which leads to the same value z and lead to an NP-hard problem (see Kreinovich & Tao [11], Heindl, Kreinovich & Lakeyev [7], Lakeyev & Kreinovich [12], and Kreinovich, Lakeyev & Noskov [10]).
There are different ways to obtain the set of optimal solutions of a fuzzy/crisp LP that depend on the point of view of the analyst. In this paper we consider three possible ways to compose the set of optimal solutions of a fuzzy/crisp LP: a first abstract or *analytical* approach based on the use of the extension principle, a second *constructivist* approach based on α-cuts, and a third approach called *operation points* based on possible realizations of $\tilde{A}$.

4.1. ANALYTICAL APPROACH

Formally speaking, the solution of any FLP is given by the Bellman-Zadeh's decision making principle [1] where the idea is to maximize the membership degree of the intersection among all fuzzy/crisp constraints. This way, the fuzzy set of optimal solutions of an LP problem with crisp fuzzy costs/constraints/coefficient matrix is given as follows:

$$\mu_{\tilde{z}}(z) = \sup_{z=\tilde{c}'x^*} \min_k \{\tilde{c}, \tilde{b}_1, \cdots, \tilde{b}_n, \tilde{A}\} \tag{11}$$

where $\tilde{z} = \mu_{\tilde{z}}(z)$ is the fuzzy set of optimal solutions in which decision making is done, $\tilde{b}$ is the set of fuzzy constraints, $\tilde{A}$ is the set of fuzzy technical coefficients, and $\tilde{c}$ is the set of fuzzy costs.
In general, the solution of any set of equations via extension principle leads to an NP-Hard problem since (in theory) every single value $z \in \text{supp}(\tilde{z})$ can be obtained by an infinite amount of combinations of $\tilde{A}x$ and $\tilde{b}$ at an infinite amount of membership degrees, so there is no a polynomial time algorithm that computes $\tilde{z}$ in its pure form.
More specifically, the optimal solution of any fuzzy/crisp problem is only located at a extreme point conformed by a set of $k \in K \supseteq \mathbb{N}_n$ self called *binding* constraints whose solution maximizes $z = c'x$. A crisp optimal solution x^* is then the set of decision variables that solves $\tilde{c}, \tilde{A}x \leqslant b$ while maximizing $z = c'x^*$, and x^* is a function of the values of $\tilde{A}$ associated to x^* i.e $\tilde{A}_{k,*}$ in the $k \in K$ binding constraints since the extreme point that leads to z is defined by $\tilde{A}_{k,*}x \leqslant b_k$, this is $z^* = f(\tilde{A}_{k,*}x^* \leqslant b)$. This is:

$$\mu_{\tilde{z}}(z^*) = \sup_{z^*=\tilde{c}'x^*} \min_k \{\tilde{c}_*, \tilde{b}_1, \cdots, \tilde{b}_k, \cdots, \tilde{b}_K, \tilde{A}_{k,*}\} \tag{12}$$

where $\tilde{z}$ is the fuzzy set of optimal solution in which decision making is done, $\tilde{b}_k, k \in K$ is the k_{th} binding constraint, $\tilde{A}_{K,*}$ is the set of fuzzy technical coefficients that compose $\tilde{A}_{K,*}x^* \lesssim \tilde{b}_k$, and $\tilde{c}_*$ is the set of fuzzy costs associated to the set of optimal variables of the problem x^*.
Eq. (12) opens the door to simplify the computation of the membership of any solution of an FLP since x^* is in fact located at binding constraints, so both non-binding constraints and non-basic decision variables does not affect the membership of $\tilde{z}$.

4.2. CONSTRUCTIVIST α-CUTS APPROACH

To avoid NP-Hard implementations of (12), we propose to compute α-cuts to approximate the set of optimal solutions. In our case, we propose to compute ${}^{\alpha}\tilde{A}$ and obtain a set of optimal solutions through the extension principle, as follows:

$$\hat{z}_{\alpha} = \text{Max}_x\{c'x : \hat{A}_{\alpha}x \lesssim b, \, x \geqslant 0\} \tag{13}$$

$$\check{z}_{\alpha} = \text{Max}_x\{c'x : \check{A}_{\alpha}x \lesssim b, \, x \geqslant 0\} \tag{14}$$

$$\tilde{z} = f(\tilde{A})(z) = \bigcup_{\alpha \in [0,1]} \alpha \cdot f({}^{\alpha}\tilde{A})(z) \tag{15}$$

As z comes from an optimization method over x and we use ${}^{\alpha}\tilde{A}$ to approximate $\tilde{z}$ then we have saved computations in the min operator of the extension principle since all cuts have $\alpha \in [0,1]$ membership; and as z is also a single optimal value for LP problems, then it is reasonable to think that the Max operator leads to a single solution that comes from only from their binding constraints, this is:

$${}^{\alpha}\tilde{z} = f^{\,\alpha}(\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_n)(z_{\alpha}) = f({}^{\alpha}\tilde{A}_1(x_1), \, {}^{\alpha}\tilde{A}_2(x_2), \cdots, \, {}^{\alpha}\tilde{A}_n(x_n))(z_{\alpha})$$

Both the analytical and constructionist methods allow to know in advance the behavior of the optimal solutions of the problem, so they are *a priori* approaches. Note that the analytical approach can lead to NP-Hard implementations, and the constructionist approach need as much α-cuts as possible to get an appropriate approximation of $\tilde{z}$.

4.3. OPERATION POINTS

In practice, there is no any certainty of reaching a desired set of values $A \in supp(\tilde{A})$. Usually, the analyst should set the system in terms of a desired point (i.e minimal uncertainty) in order to get its best performance, but it does not mean that it will happen.
Now, if the analyst has choices (a.k.a realizations of $\tilde{A}$) or *operation points* then decision making can be improved because the analyst can set the system at different points, so if it does not reach a expected result, the analyst can compare its current performance to a set of possible choices in order to see how good its performance is. To do so, we define an operation point as follows:

Definition 3 (Operation point)**.** *An operation point of $\tilde{A}$ is a set of observed values of A namely A_r contained into $supp(\tilde{A})$, which leads to an optimal solution x^*, and z_r.*

The membership degree that any operation point coming from A_r regards only to a set of $k \in K \in m$ binding constraints namely b_k where $b \subseteq b_k$. Note that the optimal solution of any LP only involves binding constraints and basic optimal variables x^* where non-binding constraints does not allocate an optimal extreme point and non-optimal decision variables does not affect the solution. As z comes from x^* then we can use (12) to compute the membership degree of any operation point. To do so, we provide the following proposition.

Proposition 2. *Let x^* be the optimal decision variables of an LP problem, $z_r = c'x^*$ be its optimal value given $A_r \in supp(\tilde{A})$, $\mu_{\tilde{z}}(z_r)$ be the membership degree of $c'x^*$ projected over $\tilde{z}$, and $A_{r;k,*} = \mu_{A_{r;k,*}}$ be the membership degree of the values of A_r associated to x^* and the k_{th} binding constraint over $\tilde{A}$. Then, the membership degree of z_r given b_k and A_r is:*

$$\mu_{\tilde{z}}(z_r : b_k, A_r) = \min_k\{A_{r;1,*}, \cdots, A_{r;k,*}, \cdots, A_{r;K,*}\}, \tag{16}$$

where $k \in K$ is the set of all binding constraints.

Proof. Let $A_r \in supp(\tilde{A})$ be a random realization of $\tilde{A}$, and χ_{b_k} be the indicator function of the k_{th} binding constraint $k \in K$. We can rewrite the problem as:

$$\mu_{\tilde{z}}(z_r : b_k, A_r) = \sup_{z_r = c'x^*} \min_k \{A_{r;1,*}, \cdots, A_{r;k,*}, \cdots, A_{r;K,*}, \chi_{b_1}, \cdots, \chi_{b_k}, \cdots, \chi_{b_K}\},$$

where it is clear that

$$\min\{A_{r;k,*}, \chi_{b_k}\} = A_{r;k,*} \quad \forall\, x^*, k \in K,$$

so we can rewrite the problem as:

$$\mu_{\tilde{z}}(z_r : b_k, A_r) = \sup_{z_r = c'x^*} \min_k \{A_{r;1,*}, \cdots, A_{r;k,*}, \cdots, A_{r;K,*}\},$$

in which the solution of $z = c'x$ for A_r is supposed to be single optimal $z_r = c'x^*$, so the sup (union) operator operates over the K binding constraints which in turn solve x^* as its global solution. This leads to only solve the intersection among the K binding constraints which compose the vertex in which the K binding constraints reach the optimal value $z_r = c'x^*$ with x^* as its solution, this is (sup $\min_{k \in K} \rightarrow \min_{k \in K}$). This allows us to rewrite the problem as:

$$\mu_{\tilde{z}}(z_r : b_k, A_r) = \min_k \{A_{r;1,*}, \cdots, A_{r;k,*}, \cdots, A_{r;K,*}\}$$

which concludes the proof. □

This means that the membership degree of z_r given A_r is only given by $A_{r;k,*}$ since the value $z_r = c'x^*$ in LPs is single optimal (iff the problem is feasible), so the computation of $\sup_{z_r = c'x^*}$ leads to the same value for all extreme points provided by the K binding constraints at a minimum uncertainty (maximum membership) computed using the sup operator.
Also note that an operation point has a smaller membership degree over $\tilde{A}$ than the fuzzy global optimal solution provided by the extension principle since it provides the maximum satisfaction degree among all possible choices. This way, while an operation point can overpass the value of z provided by the extension principle, it should definitely has a smaller satisfaction degree of either the goal or a binding constraint.
Proposition 2 solves the intersection among fuzzy coefficients, its crisp constraints, and the goal through the inf or min operator, and the union of all possible combinations of $z_r = c'x^*$ through the sup or Max operator. In the case of an optimal LP, we have only two cases: a single optimal solution x^* (the most possible case), or multiple solutions, so the sup operator is only useful in the second case where multiple x^* should be compared.

5. AN EXAMPLE

Suppose that a company has to plan production quantities (in thousands) of two products x_1 and x_2, where a sold unit returns $c_1 = 2$ and $c_2 = 3$ thousand USD per unit, and its manufacturing requires two raw materials which are available by $b_1 = 12$ and $b_2 = 15$ tons, respectively. The material consumption a_{ij} per product x_1, x_2 is uncertain since historical data is absent, so the analyst is encouraged to find a way to plan the best production quantities that maximize profits.
Given hypothetical normal operation conditions, it is supposed that the material consumption per product should be $a_{11} = 1, a_{12} = 4, a_{21} = 3$ and $a_{22} = 2$, but it is a hard supposition since we do not have historical information about the material consumption in order to verify the performance of the company. To have a better idea about the system, we have to ask the people involved into production planning, so we enquire to system's experts (people on manufacturing, engineering, mechanical processes, etc) about their perception

around material consumption of every product. The description of the FLP is given next:

$$
\begin{aligned}
z = \text{Max } & 2x_1 + 3x_2 \\
& s.t. \\
\tilde{1}_{11}x_1 + \tilde{4}_{12}x_2 & \lesssim 12 \qquad (17)\\
\tilde{3}_{21}x_1 + \tilde{2}_{22}x_2 & \lesssim 15 \qquad (18)\\
x_j & \geqslant 0
\end{aligned}
$$

Finally, the experts provide pessimistic and optimistic perceptions around a a_{ij} which are used to construct triangular fuzzy sets due to its simplicity and efficiency for computing purposes. The complete description of every $\tilde{A}_{ij}$ is shown next:

$$\mu_{\tilde{A}_{11}} = T(0,1,3),\ \mu_{\tilde{A}_{12}} = T(2,4,7),$$
$$\mu_{\tilde{A}_{21}} = T(1,3,5),\ \mu_{\tilde{A}_{22}} = T(0,2,5)$$

And the set $\tilde{z}$ of optimal solutions computed using the constructivist approach through 10 α-cuts is shown in Figure 4.

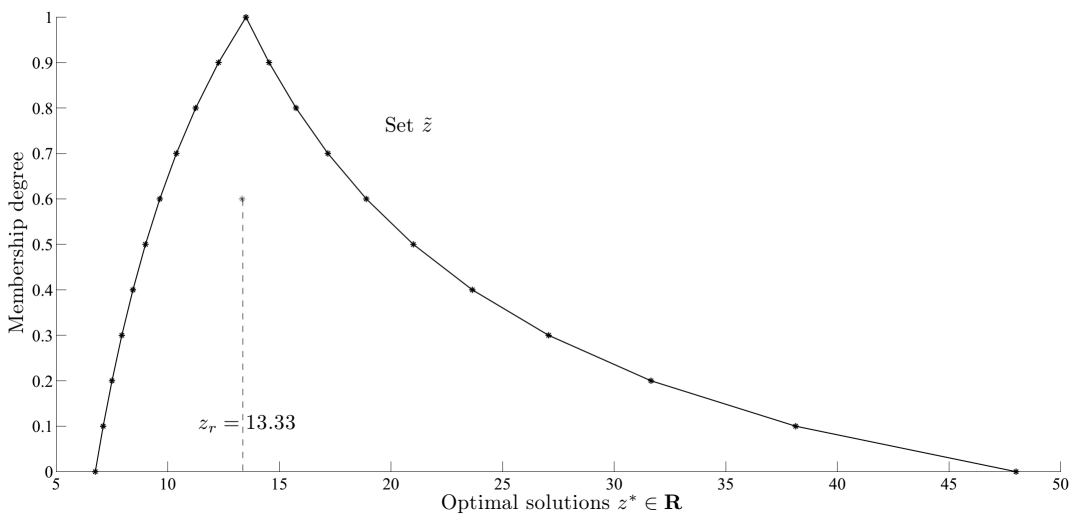

Figure 4: Set of optimal solutions $\tilde{z}$

The set $\tilde{z}$ comprises all possible values of z that can occur in a real scenario, coming from pessimistic to optimistic values passing through the most possible values (like $\bar{z} = 13.5$ with membership one). The worst possible scenario leads to $\check{z}$ and the best possible scenario leads to $\hat{z}$ while the most possible scenario leads to $\bar{z}$ as shown as follows:

$$
\begin{array}{rcc}
\bar{z} = 13.5 & \Leftrightarrow & x_1^* = 3.6, x_2^* = 2.1 \\
\check{z} = 6.75 & \Leftrightarrow & x_1^* = 2.25, x_2^* = 0.75 \\
\hat{z} = 48 & \Leftrightarrow & x_1^* = 15, x_2^* = 6
\end{array}
$$

Now, what if we tried to set the system to go to $\hat{z}$ (best returns), but we simply failed in doing it? Now suppose that our best tryout was to set the system in the following way:

$$z = \text{Max } 2x_1 + 3x_2$$
$$s.t.$$
$$1.8x_1 + 2.8x_2 \lesssim 12 \tag{19}$$
$$1.4x_1 + 2.9x_2 \lesssim 15 \tag{20}$$
$$x_j \geqslant 0$$

In this case we have that $A_r = \{A_{11} = 1.8, A_{12} = 2.8; A_{21} = 1.4, A_{22} = 2.9\}$ whose membership degrees are $\mu_{A_{11}} = 0.6, \mu_{A_{12}} = 0.4; \mu_{A_{21}} = 0.2, \mu_{A_{22}} = 0.7$, then our solution $z_r = 13.33 \Leftrightarrow x_1^* = 6.666, x_2^* = 0$ is pretty much closer to the expected value $\bar{z}$ than the best possible scenario. In this case, we have only one optimal variable x_1^* and the binding constraint is $k = 1$. Its computation comes from Eq. (16):

$$\mu_{\tilde{z}}(z_r = 13.33 : b_1, A_{11} = 1.8) = \min\{0.6\} = 0.6$$

Note that the second constraint is not involved in this computation since it is not a binding constraint having no effect over z_r, as well as the second decision variable x_2 since it is not optimal. The satisfaction degree of z_r is clearly inferior to what is expected if the system was set using an α-cuts approach. On the other hand, the use of the extension principle to get $z = 13.33$ leads to an infinite amount of choices of A_r that can reach $z = 13.33$ which basically untractable.

6. CONCLUDING REMARKS

The set of solutions $\tilde{z}$ can be used as a priori solution of the problem, since it maps all possible scenarios of the problem passing through its expected value.
When using a realization A_r to find an optimal operation point, then it could not be as satisfactory as the solution provided by ${}^{\alpha}\tilde{A}$. This means that random operation points lead to less satisfaction degrees of what the experts expect from the system.
The analyst should keep in mind that there is no any guarantee that desired values of $\tilde{A}$ can be used in real applications, so our proposal helps decision making since we provide a simple way to get a priori solutions in order to see the behavior of the system given certain operation conditions, and a method to compute the satisfaction degree of a posteriori realization A_r which is a common case.
The application example shows that a priori solutions $\tilde{z}$ are different to a posteriori solutions z_r even when it is a small example. It is important to note that simpler methods are desirable in decision making, so we have provided a complementary tool to enrich it.

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