# INVITED PAPER
# A TUTORIAL NOTE ON A CONVEXIFICATION PROCEDURE IN NON-CONVEX SEMI-INFINITE OPTIMIZATION

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**ABSTRACT**
In this tutorial note we consider the class of non-convex semi-infinite optimization problems which are defined by (one or) finitely many objective functions as well as infinitely many constraints in a finite-dimensional space. We present an overview of recent results on the so-called p-power transformation which changes the original problem equivalently locally around a solution point. This transformation is a convexification procedure for the Lagrangian where the functions in the original problem are substituted by their p-th powers. As a consequence, the convexity of the so-transformed Lagrangian allows the application of local duality theory and corresponding solution methods locally around this solution point of the original problem.

**KEYWORDS:** Multiobjective) semi-infinite optimization; p-power transformation; convexification procedure; Lagrangian;duality; locally (properly) efficient point

**MSC:** 90C34; 90C46; 65K05

**RESUMEN**
En este artìculo de resumen consideramos la clase de problemas no convexos semi-infinitos los que estàn definidos por (una o) un nùmero finito de funciones objetivo y un nùmero infinito de restricciones en un espacio de dimensiòn finita. Presentamos una panoràmica de resultados recientes sobre la llamada "p-power transformation" la que cambia el problema original equivalente localmente alrededor de un punto de soluciòn. Esta transformaciòn es un procedimiento de convexificaciòn para el Lagrangiano donde las funciones en el problema original son sustituidos por su potencia de orden p. Como consecuencia, la convexidad del Lagrangiano asì transformado permite la aplicaciòn de la teorìa de dualidad local y los mètodos correspondientes de soluciòn localmente alrededor de este punto de soluciòn del problema original.

## 1. INTRODUCTION

In this tutorial note we consider a convexification procedure for the Lagrangian of a, in general, non-convex semi-infinite optimization problem locally around a local minimizer. This approach was originally presented in [30, 31, 32, 40] for a standard optimization problem

$$\min \quad f(x) \quad \text{s.t.} \quad g_j(x) \geq 0, \quad j \in J$$

with finitely many constraints ($J$ is a finite index set) where all appearing functions are assumed to be twice continuously differentiable. For the application of local duality theory (saddle point of the Lagrangian, duality gap) and corresponding solution methods it is often assumed that locally around a certain local minimizer $\bar{x}$ the Lagrangian

$$L(\bar{x}, \lambda) = f(\bar{x}) - \sum_{j \in J} \lambda_j g_j(\bar{x})$$

is convex for all Lagrange multipliers $\lambda_j \geq 0$, $j \in J$ satisfying a corresponding Karush-Kuhn-Tucker condition. However, this local convexity property of the Lagrangian is not fulfilled in general. It is shown in [30, 31, 32, 40] that the original problem can be transformed equivalently by substituting the functions $f,\ g_j,\ j \in J$

by their p-th powers $f^p$, $g_j^p$, $j \in J$ in such a way that for sufficiently large powers $p > 0$ the Lagrangian of the transformed problem is locally convex and, therefore, corresponding local duality methods can be applied. This appreoach is called *p-power transformation* and it was recently generalized to a broader class of problems:
– to non-convex semi-infinite optimization problems (which have infinitely many constraints) in [12, 15] and
– to multiobjective semi-infinite optimization problems in [13].
In this paper we will give an overview in a tutorial form about these generalizations where we mainly present results from [12, 13, 15]. Both semi-infinite and multiobjective semi-infinite optimization became recently an active research topic in theory [3, 8, 11, 14, 17, 20, 34, 35, 37] and applications [2, 4, 5, 9, 10, 21, 24, 38].

The goal of this tutorial note is therefore twofold. In Section 2, we present a resumé of results from [12, 15] by describing the p-power transformation of a non-convex semi-infinite optimization problem. Here, the original approach is generalized to the case with
– infinitely many constraints;
– constraints with Lipschitz continuous gradients under certain conditions and;
– assuming a weaker constraint qualification of Mangasarian-Fromovitz type.
Then, in Section 3 these results are applied to the case of a non-convex *multiobjective* semi-infinite optimization problem. Here, we have finitely many objective functions, infinitely many constraints and we consider locally (properly) efficient points. Besides the convexity of the Lagrangian locally around a local minimizer of the original problem we also present some duality results. This section recalls mainly results from [13].

In the remainder of this paper we will use the following notations. If $x \in \mathbb{R}^n$, then $x_i$, $i = 1, \ldots, n$ represent its components. By $C^k(U, \mathbb{R})$, $k \geq 1$ we denote the space of $k$-times continuously differentiable real-valued functions defined on an open set $U \subset \mathbb{R}^n$. Analogously, $C^{1.1}(U, \mathbb{R})$ denotes the corresponding set of functions with Lipschitz continuous gradients. If $g \in C^1(U, \mathbb{R})$, then the *row vector* $Dg(x)$ $(D_{x^1}g(x))$ represents the gradient (partial derivative with respect to the subvector $x^1$ of $x$) of $g$ at $x$. Second derivatives are analogously defined. Moreover, let $M(n, m)$ denote the set of real matrices with $n$ rows and $m$ columns. If $R$ is an $m$-dimensional subspace of $\mathbb{R}^n$ $(m < n)$, then a matrix $Z \in M(n, m)$ is called a *basis matrix of* $R$ if its columns form a basis of $R$. Let $\|\cdot\|$ denote the Euclidean norm, $0_n$ the origin in $\mathbb{R}^n$ and for $\overline{x} \in \mathbb{R}^n$, $\varepsilon > 0$ let $B(\overline{x}, \varepsilon) = \{x \in \mathbb{R}^n \mid \|x - \overline{x}\| < \varepsilon\}$.

## 2. THE CONVEXIFICATION PROCEDURE FOR SEMI-INFINITE PROBLEMS

### 2.1. Preliminary results

In this section we will describe the p-power transformation as a convexification procedure for the Lagrangian of a semi-infinite problem SIP under certain conditions. The problem under consideration is defined as

$$\text{SIP} \qquad \min \ f(x) \ \text{ s.t. } \ x \in M$$

with the feasible set

$$M = \{x \in \mathbb{R}^n \mid G(x, y) \geq 0, \ y \in Y\},$$

where $f$ and $G$ are twice continuously differentiable real-valued functions and $Y \subset \mathbb{R}^r$ is an infinite compact index set. Note that *semi-infinite* means here that there exist finitely many decision variables $x \in \mathbb{R}^n$ and infinitely many inequality constraints. Each $\bar{y} \in Y$ represents a corresponding constraint $G(\cdot, \bar{y}) \geq 0$. Suppose that the compact index set $Y$ is given as

$$Y = \{y \in \mathbb{R}^r \mid u_l(y) = 0, \ l \in A, \ v_k(y) \leq 0, \ k \in B\}$$

with $A = \{1, \ldots, a\}$, $a < r$, $B = \{1, \ldots, b\}$ and $u_l, v_k \in C^2(\mathbb{R}^r, \mathbb{R})$, $l \in A$, $k \in B$. Furthermore, assume that at each $\bar{y} \in Y$ the active gradients $Du_l(\bar{y})$, $Dv_k(\bar{y})$, $l \in A$, $k \in B_0(\bar{y}) := \{k \in B \mid v_k(\bar{y}) = 0\}$ are linearly independent. This latter property is generically fulfilled and it implies that $Y$ is a topological manifold with boundary (see e.g. [22]).

In the remainder of this paper let $\bar{x} \in M$ our (fixed) point under consideration with the corresponding set of active indices

$$Y_0(\bar{x}) = \{y \in Y \mid G(\bar{x}, y) = 0\}.$$

Since $\bar{x} \in M$ each index $\bar{y} \in Y_0(\bar{x})$ is a global minimizer of the associated parametric so-called *lower level problem*

$$\text{Lo(x)} \qquad \min \; G(x,y) \quad \text{s.t.} \quad y \in Y$$

at $x = \bar{x}$ (where $x$ is the parameter vector). Moreover, by the linear independence of the active gradients, there exist uniquely determined multipliers $\bar{\alpha}_l$, $l \in A$, $\bar{\beta}_k \geq 0$, $k \in B_0(\bar{y})$ such that

$$D_y G(\bar{x}, \bar{y}) + \sum_{l \in A} \bar{\alpha}_l Du_l(\bar{y}) + \sum_{k \in B_0(\bar{y})} \bar{\beta}_k Dv_k(\bar{y}) = 0;$$

for $\bar{x} \in M$ and $\bar{y} \in Y_0(\bar{x})$ define $B_+(\bar{y}) = \{k \in B_0(\bar{y}) \mid \bar{\beta}_k > 0\}$.

In this section we present the main results from [15]. In the following we recall the constraint qualification EMFCQ and the second order sufficient optimality condition SSOSC which will be used for the characterization of the p-power transformation.

The *extended Mangasarian Fromovitz constraint qualification EMFCQ* is said to hold at $\bar{x} \in M$ if there exists a vector $\xi \in \mathbb{R}^n$ such that $D_x G(\bar{x}, y)\xi > 0$ for all $y \in Y_0(\bar{x})$ ([19, 33]). If $\bar{x}$ is a local minimizer of (SIP) and EMFCQ holds at $\bar{x}$, then there exist finitely many $y^j \in Y_0(\bar{x})$, $j = 1, \ldots, s$, and corresponding multipliers $\lambda_j \geq 0$, $j = 1, \ldots, s$, such that a combination of Karush-Kuhn-Tucker type is fulfilled ([17]):

$$Df(\bar{x}) - \sum_{j=1}^{s} \lambda_j D_x G(\bar{x}, y^j) = 0.$$

The *strong second order sufficient condition SSOSC* holds at (the local minimizer of $\text{Lo}(\bar{x})$) $\bar{y} \in Y_0(\bar{x})$ if the matrix $V^\top H V$ is positive definite where

$$H = D^2_{yy} G(\bar{x}, \bar{y}) + \sum_{l \in A} \bar{\alpha}_l D^2 u_l(\bar{y}) + \sum_{k \in B_0(\bar{y})} \bar{\beta}_k D^2 v_k(\bar{y})$$

and $V$ is a basis matrix of the subspace

$$\{y \in \mathbb{R}^r \mid Du_l(\bar{y})y = 0, \; Dv_k(\bar{y})y = 0, \; l \in A, \; k \in B_+(\bar{y})\}.$$

If SSOSC holds at $\bar{y}$, then for any index set $\bar{B}$ with $B_+(\bar{y}) \subseteq \bar{B} \subseteq B_0(\bar{y})$, the matrix $V(\bar{B})^\top H V(\bar{B})$ is also positive definite, where $V(\bar{B})$ is a basis matrix of the subspace

$$\{y \in \mathbb{R}^r \mid Du_l(\bar{y})y = 0, \; Dv_k(\bar{y})y = 0, \; l \in A, \; k \in \bar{B}\}.$$

Moreover, if SSOSC holds at $\bar{y}$, then $\bar{y}$ is a strongly stable local minimizer of $\text{Lo}(\bar{x})$ in the sense of Kojima ([28]). We recall the corresponding lemma stating the existence of an associated implicit function defined locally around $\bar{x}$.

**Lemma 2.1.** *([28]) Assume that SSOSC holds at $\bar{y} \in Y_0(\bar{x})$. Then, there exists an open neighborhood $U$ of $\bar{x}$ and a uniquely determined function*

$$\bar{y} : x \in U \mapsto \bar{y}(x) \in \mathbb{R}^r$$

*with the following properties:*

**(a)** $\bar{y}(\bar{x}) = \bar{y}$.

**(b)** *$\bar{y}(x)$ is a local minimizer of $Lo(x)$ and SSOSC holds at $\bar{y}(x)$ for each $x \in U$.*

**(c)** *$\bar{y}(\cdot)$ is Lipschitz continuous on $U$ and $G(\cdot, \bar{y}(\cdot)) \in C^{1.1}(U, \mathbb{R})$.*

**(e)** *If $B_+(\bar{y}) = B_0(\bar{y})$, then $G(\cdot, \bar{y}(\cdot)) \in C^2(U, \mathbb{R})$.*

We conclude this subsection by recalling the reduction approach which implies that the feasible set $M$ can be described by finitely many $C^{1.1}$-constraints locally around $\bar{x}$.
The *reduction approach RA* holds at $\bar{x} \in M$ if SSOSC holds at all $y \in Y_0(\bar{x})$.

**Lemma 2.2.** *(For more details on the reduction approach see e.g. [16, 18, 23, 25, 26, 39]).*
*Assume that RA holds at $\bar{x} \in M$. Then, we have:*

**(a)** *$Y_0(\bar{x})$ is a finite set, say $Y_0(\bar{x}) = \{y^1, \dots, y^s\}$.*

**(b)** *There exist an open neighborhood $U$ of $\bar{x}$ and uniquely determined Lipschitz continuous functions*

$$\bar{y}^j : x \in U \mapsto \bar{y}^j(x) \in \mathbb{R}^r, \ \ j = 1, \dots, s$$

*such that $\bar{y}^j(\bar{x}) = y^j$, $G(\cdot, \bar{y}^j(\cdot)) \in C^{1.1}(U, \mathbb{R})$, $j = 1, \dots, s$ and $Y_0(x) \subseteq \{\bar{y}^j(x) \mid j = 1, \dots, s\}$ for all $x \in U$.*

**(c)** *The feasible set $M$ can be described by finitely many $C^{1.1}$-constraints locally around $\bar{x}$:*

$$M \cap U = \{x \in U \mid G(x, \bar{y}^j(x)) \geq 0, \ \ j = 1, \dots, s\}.$$

## 2.2. The p-power transformation of SIP

In this subsection we summarize the main results from [15] on the p-power transformation of (SIP) locally around our point $\bar{x} \in M$. Suppose the following four properties in the remainder of this section:

- RA holds at $\bar{x}$ and $Y_0(\bar{x}) = \{y^1, \dots, y^s\}$ (further notations will be used as introduced in Lemma 2.2).
- EMFCQ holds at $\bar{x}$.
- There exist multipliers $\lambda_j \geq 0$, $j = 1, \dots, s$, satisfying

$$Df(\bar{x}) - \sum_{j=1}^{s} \lambda_j D_x G(\bar{x}, y^j) = 0. \tag{2-1}$$

  Then, by EMFCQ, the set
  $$\Lambda = \{\lambda \in \mathbb{R}^s \mid \lambda \text{ is a solution of (2-1)}\}$$
  is nonempty and compact ([6, 27]).

- An *extended strong second order sufficient condition ESSOSC* (see Condition (C2) in [36]) holds at $\bar{x}$. Since this is a tutorial note trying to avoid any unnecessary technicalities we do not present the very technical definition of ESSOSC; all details can be found in [15, 36]. However, note that ESSOSC coincides with SSOSC in case of a finite problem which is defined by $C^2$-functions.

By Lemma 2.2, the latter properties imply that the problem (SIP) can be described locally on an appropriate neighborhood $U$ of $\bar{x}$ as

$$\min_{x \in U} \ f(x) \quad \text{s.t.} \quad G(x, \bar{y}^j(x)) \geq 0, \ \ j = 1, \dots, s.$$

Moreover, throughout this subsection we assume that $f(x) > 0$ for all $x \in U$ and that $G(x, y)$ can be written as the difference of a positive constant $\bar{r}$ and a $C^2$-function $g(x, y)$ as

$$G(x, y) = \bar{r} - g(x, y)$$

$$\text{with} \quad \bar{r} > 0, \ \ g(x, y) \geq 0 \text{ for all } (x, y) \in U \times Y.$$

Note that this latter condition can be fulfilled without loss of generality by an appropriate equivalent transformation; e.g. by an exponential transformation $\left(f(x) \to e^{f(x)}\right)$ or by adding a sufficiently large constant ($f(x) \to f(x) + r^1$, $r^1 > 0$). Hence, we obtain the following equivalent problem for $x \in U$:

$$\min_{x \in U} \ f(x) \quad \text{s.t.} \quad g(x, \bar{y}^j(x)) \leq \bar{r}, \ \ j = 1, \dots, s. \tag{2-2}$$

As introduced in [30, 31, 32], for a power $p \geq 1$, the $p$-power transformation of (2-2) is the equivalent problem

$$\min_{x \in U} \quad (f(x))^p \quad \text{s.t.} \quad \left(g(x, \bar{y}^j(x))\right)^p \leq \bar{r}^p, \;\; j = 1, \ldots, s$$

with its Lagrangian

$$L_p(x, \mu) = (f(x))^p + \sum_{j=1}^{s} \mu_j \left[ \left(g(x, \bar{y}^j(x))\right)^p - \bar{r}^p \right]. \tag{2-3}$$

Note that the feasible set remains the same when substituting the constraints by their p-th powers. A short calculation shows that the solution set of

$$D_x L_p(\bar{x}, \mu) = 0, \;\; \mu_j \geq 0, \;\; j = 1, \ldots, s$$

is the following:

$$\mathcal{M}^p = \{\mu \in \mathbb{R}^s \mid \mu_j = \lambda_j {\frac{(f(\bar{x}))^{p-1}}{\bar{r}}}^{p-1}, \;\; j = 1, \ldots, s, \;\; \lambda \in \Lambda\}.$$

According to our goal we state now that there exists a sufficiently large power $p$ such that $(\bar{x}, \mu)$ is for all $\mu \in \mathcal{M}^p$ a saddle point for the corresponding Lagrangian in (2-3).

**Theorem 2.1.** *([15], Theorem 3.4) There exist a power $p^*$ and a neighborhood $U^* \subset U$ of $\bar{x}$ such that for all $\mu \in \mathcal{M}^{p^*}$ and all $x \in U^*$ we have*

$$L_{p^*}(\bar{x}, \mu) \leq L_{p^*}(x, \mu).$$

The latter theorem yields

$$L_{p^*}(\bar{x}, \mu) \leq L_{p^*}(\bar{x}, \bar{\mu}) \leq L_{p^*}(x, \bar{\mu})$$

for all $\mu \geq 0$, $\bar{\mu} \in \mathcal{M}^{p^*}$ and $x \in \bar{U}$. For the corresponding dual function

$$\phi_{p^*}(\mu) = \min_{x \in \bar{U}} L_{p^*}(x, \mu).$$

we obtain the following duality result (see also [1, Theorem 6.2.5]).

**Corollary 2.1.** *(Saddle point optimality and absence of a duality gap). Let $p^*$ be chosen as in the previous theorem. Then:*

**(a)** *$(\bar{x}, \bar{\mu})$ is a saddle point of the Lagrangian $L_{p^*}(x, \mu)$ for each $\bar{\mu} \in \mathcal{M}^{p^*}$.*

**(b)** *Each $\bar{\mu} \in \mathcal{M}^{p^*}$ is a solution of the dual problem*

$$\max_{\mu \geq 0} \phi_{p^*}(\mu) \;\; \textit{with} \;\; L_{p^*}(\bar{x}, \bar{\mu}) = (f(\bar{x}))^{p^*} = \phi_{p^*}(\bar{\mu}).$$

We conclude this subsection with the so-called *partial p-power transformation* (see [40]) of (2-2) which is defined as the equivalent problem

$$\min_{x \in U} \quad f(x) \quad \text{s.t.} \quad \left(g(x, \bar{y}^j(x))\right)^p \leq \bar{r}^p, \;\; j = 1, \ldots, s.$$

The difference to the $p$-power transformation of (2-2) described above is that now only the constraints are substituted by their p-th powers. An analogous calculation as for the $p$-power transformation provides a saddle point result for its Lagrangian

$$\mathcal{L}_p(x, \gamma) = f(x) + \sum_{j=1}^{s} \gamma_j \left[ \left(g(x, \bar{y}^j(x))\right)^p - \bar{r}^p \right]$$

where the solution set of

$$D_x \mathcal{L}_p(\bar{x}, \gamma) = 0, \;\; \gamma_j \geq 0, \;\; j = 1, \ldots, s$$

is the following:

$$\Gamma^p = \{\gamma \in \mathbb{R}^s \mid \gamma_j = \frac{\lambda_j}{p\bar{r}^{p-1}},\ j = 1, \ldots, s,\ \lambda \in \Lambda\}.$$

The corresponding dual function is

$$\psi_p(\gamma) = \min_{x \in \hat{U}} \mathcal{L}_p(x, \gamma)$$

and we obtain the following result.

**Theorem 2.2.** *([15], Theorem 3.6) There exist a power $\hat{p}$ and a neighborhood $\hat{U} \subset U$ of $\bar{x}$ such that*

**(a)** *$(\bar{x}, \bar{\gamma})$ is a saddle point of the Lagrangian $\mathcal{L}_{\hat{p}}(x, \gamma)$ for each $\bar{\gamma} \in \Gamma^{\hat{p}}$. In particular, we have*

$$\mathcal{L}_{\hat{p}}(\bar{x}, \gamma) \leq \mathcal{L}_{\hat{p}}(\bar{x}, \bar{\gamma}) \leq \mathcal{L}_{\hat{p}}(x, \bar{\gamma})$$

*for all $\gamma \geq 0$, $\bar{\gamma} \in \Gamma^{\hat{p}}$ and $x \in \hat{U}$.*

**(b)** *Each $\bar{\gamma} \in \Gamma^{\hat{p}}$ is a solution of the dual problem*

$$\max_{\gamma \geq 0} \psi_{\hat{p}}(\gamma) \text{ with } \mathcal{L}_{\hat{p}}(\bar{x}, \bar{\gamma}) = f(\bar{x}) = \psi_{\hat{p}}(\bar{\gamma}).$$

## 3. THE CONVEXIFICATION PROCEDURE FOR MULTIOBJECTIVE SEMI-INFINITE PROBLEMS

### 3.1. Preliminary results

The results in this section are mainly taken from [12, 13]. We will apply and extend the results for the problem class SIP (which has one objective function) from the previous section to the problem class MOSIP of multiobjective semi-infinite problems (which has finitely many objective functions). Such a problem is defined as

$$\text{MOSIP} \quad \text{“}\min\text{”} \ f(x) \ \text{ s.t. } x \in M$$

with the vector $f = (f_1, \ldots, f_q)^\top$ of objective functions $f_i \in C^2(\mathbb{R}^n, \mathbb{R})$, $i = 1, \ldots, q$ and where the feasible set $M$ is defined as for SIP. Analogously to the SIP case suppose the following properties in the remainder of this section at $\overline{x} \in M$:

- RA holds at $\bar{x}$ with $Y_0(\bar{x}) = \{y^1, \ldots, y^s\}$ and (for sake of simplicity) $B_+(y^j) = B_0(y^j)$, $j = 1, \ldots, s$. The latter property and Lemma 2.2.(c) imply that the feasible set can be described on an appropriate neighbourhood $U$ of $\overline{x}$ as

  $$M \cap U = \{x \in U \mid G(x, \overline{y}^j(x)) \geq 0,\ j = 1, \ldots, s\}$$

  where $G(\cdot, \overline{y}^j(\cdot)) \in C^2(U, \mathbb{R})$, $j = 1, \ldots, s$.

- EMFCQ holds at $\bar{x}$.

- There exist (fixed) $\overline{\rho} > 0_q$ as well as multipliers $\lambda_j \geq 0$, $j = 1, \ldots, s$ such that

  $$\sum_{i=1}^{q} \overline{\rho}_i D f_i(\overline{x}) - \sum_{j=1}^{s} \lambda_j D_x G(\overline{x}, y^j) = 0_n. \tag{3-1}$$

  Then, by EMFCQ, the set

  $$\Lambda_1 = \{\lambda \in \mathbb{R}^s \mid \lambda \text{ is a solution of (3-1)}\}$$

  is compact (cf. [6, 27]).

- Condition ESSOSC (cf. [36]) holds at $\overline{x}$. Since $B_+(y^j) = B_0(y^j),\ j = 1,\ldots,s$ this condition is equivalent to SSOSC which means that for each $\lambda \in \Lambda_1$ the matrix

$$\sum_{i=1}^{q} \overline{\rho}_i D^2 f_i(\overline{x}) - \sum_{j=1}^{s} \lambda_j D^2 G(\overline{x}, y^j(\overline{x}))$$

is positive definite on the subspace

$$T(\lambda) = \{w \in \mathbb{R}^n \mid D_x G(\overline{x}, y^j)w = 0,\ j \in \{\nu \in \{1,\ldots s\} \mid \lambda_\nu > 0\}\}.$$

By [36], these latter four assumptions imply that $\overline{x}$ is a local minimizer of the so-called *weighted sum optimization problem*

$$\min_{x\in U} \sum_{i=1}^{q} \overline{\rho}_i f_i(x) \quad \text{s.t.} \quad G(x, \overline{y}^j(x)) \geq 0,\ j = 1,\ldots,s.$$

Next we will recall some terminology for local solutions of MOSIP (since we consider the general non-convex case we will only consider *local* solutions). For $c, e \in \mathbb{R}^n$ let
- $c \leqq e$, if $c_i \leq e_i,\ i = 1,\ldots,n$,
- $c < e$, if $c_i < e_i,\ i = 1,\ldots,n$,
- $c \leq e$, if $c_i \leq e_i,\ i = 1,\ldots,n$ and $c \neq e$.

The following definition for a locally efficient point as well as for a locally properly efficient point (in the sense of *Geoffrion* respectively in the sense of *Kuhn and Tucker*) are straightforward generalizations of the original definitions in [3, 7, 29].

**Definition 3.1.** (i) A point $\overline{x} \in M$ is called *locally efficient* on $B(\overline{x}, \varepsilon)$ if there exists a real number $\varepsilon > 0$ and if there does not exist any $x \in B(\overline{x}, \varepsilon) \cap M$ with $f(x) \leq f(\overline{x})$.
(ii) A point $\overline{x} \in M$ is called *locally properly efficient in the sense of Geoffrion* (shortly: *G-locally properly efficient*) if there exists a real number $\varepsilon > 0$ such that
- $\overline{x}$ is locally efficient on $B(\overline{x}, \varepsilon)$ and
- there exists a real number $K > 0$ such that for each index $i \in \{1,\ldots,q\}$ and any $x \in B(\overline{x}, \varepsilon) \cap M$ with $f_i(x) < f_i(\overline{x})$ there exists an index $j \in \{1,\ldots,q\}$ such that $f_j(x) > f_j(\overline{x})$ and

$$\frac{f_i(\overline{x}) - f_i(x)}{f_j(x) - f_j(\overline{x})} \leq K.$$

(iii) A point $\overline{x} \in M$ is called *locally properly efficient in the sense of Kuhn and Tucker* (shortly: *KT-locally properly efficient*) if there exists a real number $\varepsilon > 0$ such that
- $\overline{x}$ is locally efficient on $B(\overline{x}, \varepsilon)$ and
- the following system has no solution $d \in \mathbb{R}^n$ :

$$\begin{aligned} Df_i(\overline{x})d &\leq 0,\ i = 1,\ldots,q, \\ Df_k(\overline{x})d &< 0,\ \text{for some } k \in \{1,\ldots,q\}, \\ D_x G(\overline{x}, y^j)d &\geq 0,\ j = 1,\ldots,s \end{aligned}$$

The following lemma presents a motivation for considering the p-power transformation of MOSIP.

**Lemma 3.3.** *([3])* If the Lagrangian

$$\sum_{i=1}^{q} \overline{\rho}_i f_i(x) - \sum_{j=1}^{s} \overline{\lambda}_j G(x, \overline{y}^j(x))$$

is convex on $B(\overline{x}, \varepsilon)$ for some $\varepsilon > 0$ with $B(\overline{x}, \varepsilon) \subset U$ and some $\overline{\lambda} \in \Lambda_1$, then $\overline{x}$ is G-locally properly efficient for the problem

$$\text{“}\min_{x\in U}\text{”}\ f(x) \qquad \text{s.t.} \qquad G(x, \overline{y}^j(x)) \geq 0,\ j = 1,\ldots,s. \tag{3-2}$$

However, the assumption of convexity in the previous lemma is in general not fulfilled under the condition ESSOSC. In the next subsection we will see how the p-power transformation of MOSIP can overcome this disadvantage.

### 3.2. The p-power transformation of MOSIP

Analogously to Section 2, in the remainder of this subsection we assume without loss of generality for all $x \in U$ that

$$f_i(x) > 0,\ i = 1, \ldots, q$$

and that $G(x, y)$ can be written as the difference of a positive constant and a $C^2$-function:

$$G(x, y) = \bar{r} - g(x, y)$$

with $\bar{r} > 0$ and $g(x, y) > 0$, for all $(x, y) \in U \times Y$. Thus, our problem (3-2) can be written as

$$\text{“}\min_{x \in U}\text{”}\ (f_1(x), \ldots, f_q(x)) \ \text{ s.t } \ g(x, \overline{y}^j(x)) \leq \bar{r}, \quad j = 1, \ldots, s. \tag{3-3}$$

For a real number $p > 0$, we define the *p-power transformation* of (3-3) by substituting the original functions by their p-th powers (we write $f_1^p(x)$ for $(f_1(x))^p$):

$$\text{“}\min_{x \in U}\text{”}\ (f_1^p(x), \ldots, f_q^p(x)) \ \text{ s.t } \ g^p(x, \overline{y}^j(x)) \leq \bar{r}^p, \quad j = 1, \ldots, s. \tag{3-4}$$

The next theorem shows the relationship between solutions of the problems (3-3) and (3-4); in particular, the property of being a locally (properly) efficient point remains invariant.

**Theorem 3.3.** *([12], Theorem 1)* (i) A point $\widetilde{x} \in U$ is locally efficient for (3-3) if and only if $\widetilde{x} \in U$ is locally efficient for (3-4).
(ii) A point $\widetilde{x} \in U$ is G-locally properly efficient for (3-3) if and only if $\widetilde{x} \in U$ is G-locally properly efficient for (3-4).
(iii) A point $\widetilde{x} \in U$ is KT-locally properly efficient for (3-3) if and only if $\widetilde{x} \in U$ is KT-locally properly efficient for (3-4).

According to our objective, the next theorem states that for a sufficiently large power $p > 0$ the Lagrangian (which corresponds to the fixed $\overline{\rho}$)

$$L_p^1(x, \overline{\delta}, \gamma) = \sum_{i=1}^{q} \overline{\delta}_i f_i^p(x) + \sum_{j=1}^{s} \gamma_j (g^p(x, \overline{y}^j(x)) - \bar{r}^p)$$

of the problem (3-4) is convex on a neighbourhood of $\overline{x}$; here, $\gamma \in \mathbb{R}^s$ and $\overline{\delta} \in \mathbb{R}^q$ is fixed with

$$\overline{\delta}_i = \frac{\overline{\rho}_i}{f_i^{p-1}(\overline{x})},\ i = 1, \ldots, q. \tag{3-5}$$

Moreover, it is $\overline{\delta} > 0_q$ and $\overline{\delta}$ depends on $p$. Analogously to Section 2, the compact set of solutions $\gamma \in \mathbb{R}^s$ satisfying

$$D_x L_p^1(\overline{x}, \overline{\delta}, \gamma) = 0,\ \gamma \geq 0_s$$

is

$$\Gamma_1^p = \left\{ \gamma \in \mathbb{R}^s \mid \gamma_j = \frac{\lambda_j}{g^{p-1}(\overline{x}, \overline{y}^j(\overline{x}))},\ j = 1, \ldots, s,\ \lambda \in \Lambda_1 \right\}.$$

**Theorem 3.4.** *([12], Theorem 2)*
Let $\overline{\delta} > 0_q$ be chosen as in (3-5). Then, there exists a power $\overline{p} > 0$ such that the Hessian $D_x^2 L_p^1(\overline{x}, \overline{\delta}, \gamma)$ is positive definite for all $\gamma \in \Gamma_1^p$ whenever $p > \overline{p}$.

We conclude this section by presenting two duality results. Let $p > 0$ be chosen so sufficiently large that the Lagrangian $L^1_p(\cdot, \overline{\delta}, \gamma)$ is convex with respect to $x$ on a neighbourhood $\hat{U} \subset U$ of $\overline{x}$ for all $\gamma \in \Gamma^p_1$. Consider the following corresponding dual vector optimization problem:

$$\text{D-MOSIP}^p_{loc} \quad \max z \quad \text{s.t.} \quad (\delta, \gamma, z) \in \text{DM}_p$$

with the feasible set

$$\text{DM}_p = \left\{ (\delta, \gamma, z) \in \text{ int } \mathbb{R}^q_+ \times \Gamma^p_1 \times \mathbb{R}^q \mid \delta^\top z \leq \inf_{x \in \hat{U}} L^1_p(x, \delta, \gamma) \right\}.$$

**Theorem 3.5.** *([13], Theorem 3.6)*
(a) For all $x \in M \cap \hat{U}$ and all $(\delta, \gamma, z) \in DM_p$ we have $\delta^\top z \leq \delta^\top f^p(x)$ (weak duality).
(b) There exists a locally efficient point $(\overline{\delta}, \gamma, \overline{z}) \in DM_p$ *for D-MOSIP*$^p_{loc}$ with $f^p(\overline{x}) = \overline{z}$ (strong duality).

## 4. CONCLUSIONS

In this tutorial note we presented an overview on results from the recent papers [12, 13, 15] on the p-power transformation of a non-convex semi-infinite optimization problem which may have one (standard) or several (multiobjective) objective functions. This approach represents a convexification procedure for the Lagrangian of the original problem locally in a neighbourhood of a local minimizer. The main idea of the p-power transformation is to substitute the functions describing the original problem by their p-th powers. Then, the so-transformed problem is locally equivalent to the original one (e.g. leaving the property of being a locally (properly) efficient point invariant) and if the power $p > 0$ is chosen sufficiently large, then its Lagrangian is convex locally in the neighbourhood of the local minimizer under consideration. Therefore, it fulfills the fundamental assumption for the application of local duality theory and corresponding solution methods (e.g. local dual research methods) which, in general, is not fulfilled for the non-convex original problem.

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