HYPERBOLIC EQUATIONS WITH MEMORY

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ABSTRACT
The existence of a solution to the equation governing the evolution of a displacement vector in an elastic body with non-local time and spatial memory is considered. A global weak solution to an associated initial-boundary value problem is established by constructing Galerkin approximations. Lebesgue or Sobolev spaces can be generalized for all real numbers and can be defined also on Banach spaces. They are equipped with several equivalent norms based on Fourier or Laplace transform and function expansion. These spaces help to derive suitable energy estimates.

KEYWORDS: Hyperbolic differential equations, singular viscoelasticity, materials with memory, generalized Sobolev spaces.

MSC: 35L20, 46E35, 74D10

RESUMEN
La existencia de una solución a la ecuación que gobierna la evolución de un vector de desplazamiento en un cuerpo elástico con el tiempo no local y la memoria espacial se considera. Una solución débil global para un problema de valores iniciales y de contorno asociado se establece mediante la construcción de aproximaciones de Galerkin. espacios de Lebesgue y de Sobolev se pueden generalizar para todos los números reales y se pueden definir también en espacios de Banach. Están equipadas con varias normas equivalentes sobre la base de Fourier o la transformada de Laplace y expansión de funciones. Estos espacios ayudan a obtener estimaciones de energía adecuados.

1. INTRODUCTION

The evolution of a displacement vector in an elastic body is governed by so called wave equation.
The body is assumed to occupy a reference configuration $\Omega \subset \mathbb{R}^N$ at an initial time and to have unit density. The vector $u = (u_1, \dots, u_N)$ represents the displacement and from the Newton law we obtain the wave equation

$$\ddot{u}_i - \frac{\partial}{\partial x_j}\sigma_{ij} = f_i \quad \text{on } \Omega \quad i = 1, \dots, N,$$

where σ_{ij} is the Cauchy stress tensor and $f = (f_1, \dots, f_N)$ is the external body force per unit mass.
The stress tensor (elastic case) is usually given by the constitutive law

$$\sigma^I_{ij}(x,t) = \frac{\partial W}{\partial e_{ij}}(eu(x,t)) \ ,$$

where $W = W(e_{ij})$ is the function of free energy and $e_{ij}u = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is the infinitesimal strain tensor.
It is known that a weak solution exists only in the case of one space dimension (see [2]) and we have some partial results in some special cases for dimension $N \geq 2$ (see [1], [3] or [4]).
There are certain materials with memory; it means that the stress depends not only on the strain at the present time t at given x, but also on the entire history of the strain from zero to time t at all points located around x. In this case, the stress $\sigma = \sigma^I + \sigma^M$ is extended by the memory part

$$\sigma^M_{ij} = -\lambda \int_0^t \int_\Omega (e_{ij}u(\xi,\tau) - e_{ij}u(\xi,t)) \frac{h(t-\tau)}{|x-\xi|^\alpha} d\xi \ d\tau \ .$$

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Let us consider boundary conditions $u(x,\cdot)=0$ for $x\in\partial\Omega$ and initial conditions $u(\cdot,0)=g_0$ and $\dot u(\cdot,0)=g_1$. We suppose that the function $W:\mathbb{R}^{2N}\to\mathbb{R}$ is continuous, has bounded second derivatives, $W(0)=\frac{\partial W}{\partial e_{ij}}(0)=0$ and the condition of ellipticity holds (with some real number $\kappa>0$). Moreover, let us suppose $\lambda>0$, $N-1<\alpha<N$, $h(t)=e^{-t}t^{-\nu}$, where $0<\nu<\frac{1}{2}$ and $\nu>N-\alpha$; we also assume some other properties of f, g_0, g_1

$$f\in W^{\frac{\nu}{2},2}((0;\infty);L^2(\Omega;\mathbb{R}^N))\cap L^2((0;\infty);W^{-1,2}(\Omega;\mathbb{R}^N))\cap L^\infty((0;\infty);L^2(\Omega;\mathbb{R}^N))\ ,$$
$$g_0\in W_o^{1,2}(\Omega;\mathbb{R}^N),\qquad g_1\in L^2(\Omega;\mathbb{R}^N)\ .$$

The aim of this text is not to present the whole proof of weak solution existence, only its main points and features are mentioned here.

2. GALERKIN METHOD

We shall use Galerkin approximation which gives us partial solutions u^n.
Let us denote $v_1,v_2,\dots$ the base in space $W_o^{1,2}(\Omega;\mathbb{R}^N)$ orthonormal in $L^2(\Omega;\mathbb{R}^N)$ composed from eigenfunctions of equations

$$\frac{\partial}{\partial x_j}\left(\frac{\partial^2 W}{\partial e_{ij}\partial e_{kl}}(0)\frac{\partial v_k}{\partial x_l}\right)+\lambda v_i=0\qquad\text{on }\Omega$$

for $i=1,2,\dots,N$, with condition $v=0$ on $\partial\Omega$; eigenvalues of such problem are denoted by λ_1, λ_2,... .
Approximants u^n are partial solutions of the equation in finite-dimensional space generated by base $v_1,\dots,v_n$. They have the form $u_i^n=\sum_{j=1}^n c_{ij}v_i^j$ for $i=1,\dots,N$ and it holds

$$\int_\Omega \ddot u_i^n\cdot v_i^k\,dx+\int_\Omega\frac{\partial W}{\partial e_{ij}}(e(u^n))\cdot e_{ij}(v^k)\,dx-$$
$$-\int_\Omega\int_0^t\int_\Omega\frac{h(t-\tau)}{|x-\xi|^\alpha}\left(e_{ij}u^n(\xi,\tau)-e_{ij}u^n(\xi,t)\right)\cdot e_{ij}(v^k)\,d\xi d\tau dx=\int_\Omega f_i v_i^k\,dx$$

for $k=1,2,\dots,n$.

3. GENERALIZED SOBOLEV SPACES

The following estimates for u^n can be got using the equivalent norms in generalized Sobolev spaces.
These spaces are introduced by norms

$$||f||^2_{W^{s,2}(\Omega)}=||f||^2_{L^2(\Omega)}+\int_{\Omega\times\Omega}\frac{|f(x)-f(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\quad\text{for }0<s<1,$$

$$||f||^2_{W^{s,2}(\Omega)}=||f||^2_{W^{1,2}(\Omega)}+\sum_{i=1}^N\int_{\Omega\times\Omega}\frac{|\frac{\partial f}{\partial x_i}(x)-\frac{\partial f}{\partial x_i}(y)|^2}{|x-y|^{N+2(s-1)}}\,dx\,dy\quad\text{for }1<s<2$$

and so on for $s\in(2,3)\cup(3,4)\cup\dots$. We put $W^{0,2}(\Omega)=L^2(\Omega)$.
Spaces $W^{s,2}(\Omega)$ are composed by functions f for which $||f||_{W^{s,2}(\Omega)}<\infty$, $s\ge 0$ and they are equipped by the presented norms.
Subspaces $W_o^{s,2}(\Omega)$ are defined as the closure of $\mathcal{D}(\Omega)$ in $W^{s,2}(\Omega)$ (where $\mathcal{D}(\Omega)$ contains all functions with compact support in interior of Ω which have all derivatives). Norms consisting only from the latter parts

$$||f||^2_{W_o^{s,2}(\Omega)}=\int_{\Omega\times\Omega}\frac{|f(x)-f(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\ ,$$

$$||f||^2_{W_o^{s,2}(\Omega)}=\sum_{i=1}^N\int_{\Omega\times\Omega}\frac{|\frac{\partial f}{\partial x_i}(x)-\frac{\partial f}{\partial x_i}(y)|^2}{|x-y|^{N+2(s-1)}}\,dx\,dy$$

respectively, induce the same topology on these spaces.

Spaces $W^{-s,2}(\Omega)$ for $s \geq 0$ are defined as duals of spaces $W_o^{s,2}(\Omega)$ equipped with strong topology, i.e. the norm

$$||f||_{W^{-s,2}(\Omega)} = \sup_{||\varphi||_{W_o^{s,2}(\Omega)} \leq 1} \int_\Omega f\varphi \, dx \ .$$

Thereby we have spaces $W^{s,2}(\Omega)$ for all real s.
Topology on this spaces (of functions defined on $\mathbb{R}^N$) can be equipped with an equivalent norm expressed by means of the Fourier transform or expansions

$$||f||^2_{W^{s,2}(\Omega)} \approx \int_{\mathbb{R}^N} (1+|\xi|^s)^2 |\hat{f}(\xi)|^2 \quad \approx \sum_{i=1}^{\infty} (1+\lambda_i)^s c_i^2 \, d\xi$$

$$||f||^2_{W_o^{s,2}(\Omega)} \approx \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{f}(\xi)|^2 \, d\xi \quad \approx \sum_{i=1}^{\infty} \lambda_i^s c_i^2 \ ,$$

respectively, where

$$c_i = \int_\Omega f v_i \, dx$$

are coefficients of expansion of a function $f \in W^{s,2}(\Omega)$ into a series

$$f = \sum_{i=1}^{\infty} c_i v_i \ .$$

Generalized Sobolev spaces can be defined for functions with values in some Banach space, too; the corresponding integrals must be considered in Bochner sense. Let $1 < \mu < \frac{3}{2}$ and $-\frac{3}{2} < \beta < \frac{3}{2}$. For instance we can consider spaces $W^{\mu,2}((0,T); W_o^{\beta,2}(\Omega))$. Any function $f \in W^{\mu,2}((0,T); W_o^{\beta,2}(\Omega))$ may be expanded into the double series

$$f = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} c_{ij} \cdot h_i(t) \cdot v^j(x) \ ,$$

where

$$h_o(t) = \frac{1}{\sqrt{T}}, \qquad h_i(t) = \sqrt{\frac{2}{T}} \cdot \cos\frac{i\pi}{T} \cdot t, \quad i = 1, 2, \dots \ .$$

The equivalent norm

$$\|v\|^2_{W^{\mu,2}((0,T); W_o^{\beta,2}(\Omega))} \approx \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} c_{ij}^2 \cdot (1+i^2)^\mu \cdot \lambda_j^\beta$$

can be used.
For $0 < \mu$ and $0 < \beta < 1$, the space $W^{\mu,2}((0,T); W^{-\beta,2}(\Omega))$ is compactly imbeded into space $W^{1,2}((0,T); W^{-1,2}(\Omega))$.

4. BASIC ESTIMATES

Our starting point is well-known trick, when time derivative $\dot{u}^n$ of the partial solution is put into the equation for it. We shall use $\ddot{u} \cdot \dot{u} = \frac{1}{2}(u^2)^{\cdot}$, $(W(eu))^{\cdot} = \sum \frac{\partial W}{\partial e_{ij}} \cdot e_{ij}\dot{u}$ and by using the equivalent norm of $W^{\frac{\nu}{2},2}((0,T); W^{1-\frac{N-\alpha}{2},2}(\Omega))$ based on Fourier transformation we obtain estimates

$$\|u^n\|^2_{L^\infty((0;\infty); W_o^{1,2}(\Omega;\mathbb{R}^N))} + \|\dot{u}^n\|_{L^\infty((0;\infty); L^2(\Omega;\mathbb{R}^N))} + C_1 \|u^n\|_{W^{\frac{\nu}{2},2}((0;T); W^{1-\frac{N-\alpha}{2},2}(\Omega;\mathbb{R}^N))} \leq$$
$$\leq C_2 \left(\|f\|_{L^2((0;\infty); L^2(\Omega;\mathbb{R}^N))} + \|\dot{g}_0\|_{L^2(\Omega;\mathbb{R}^N)} + \|g_1\|_{W_o^{1,2}(\Omega;\mathbb{R}^N)} \right)$$

for some positive constants C_1 and C_2. We have used rules for transforms of convolution, Parseval equality, the fact that Fourier transform of power $\frac{1}{|\cdot|^\alpha}$ gives the power $|\xi|^{N-\alpha}$, derivative $\frac{\partial}{\partial x_i}$ adds another ξ_i, and Laplace transform of power $\frac{1}{|\cdot|^\nu}$ gives the power $\tau^{\frac{\nu}{2}}$.

Hence

u^n is bounded in $L^\infty((0;\infty);W_o^{1,2}(\Omega;\mathbb{R}^N))$
$\dot{u}^n$ is bounded in $L^\infty((0;\infty);L^2(\Omega;\mathbb{R}^N))$
u^n is bounded in $W^{\frac{\nu}{2},2}((0;T);W^{1-\frac{N-\alpha}{2},2}(\Omega;\mathbb{R}^N))$.

These estimates and the definition of dual norm imply

$\ddot{u}^n$ is bounded in $W^{\frac{\nu}{2},2}((0;T);W^{-1-\frac{N-\alpha}{2},2}(\Omega;\mathbb{R}^N))$.

5. CONCLUSION

By simple calculation with norms using expansions of functions it is possible to get the interpolation inequality for $0<\delta<\frac{1}{2}$, $0<\epsilon<\frac{1}{2}$ and $0\le\gamma\le 1$

$$\|v\|_{W^{(1+\delta)\cdot\gamma,2}((0,T);W^{-(1+\epsilon)\cdot\gamma,2}(\Omega;\mathbb{R}^N))} \le C_3\ \|v\|^{1-\gamma}_{L^2((0,T);L^2(\Omega;\mathbb{R}^N))}\ \|v\|^{\gamma}_{W^{1+\delta,2}((0,T);W^{-1-\epsilon,2}(\Omega;\mathbb{R}^N))}\ .$$

As $\dot{u}^n$ is bounded both in $L^2((0,T);L^2(\Omega;\mathbb{R}^N))$ and
$W^{1+\frac{\nu}{2},2}((0,T);W^{-1-\frac{N-\alpha}{2},2}(\Omega;\mathbb{R}^N))$ and as there exists γ such that

$$0\le\frac{1}{1+\frac{\nu}{2}}<\gamma<\frac{1}{1+\frac{N-\alpha}{2}}\le 1,$$

$\dot{u}^n$ is bounded in $W^{(1+\frac{\nu}{2})\cdot\gamma,2}((0,T);W^{-(1+\frac{N-\alpha}{2})\cdot\gamma,2}(\Omega;\mathbb{R}^N))$ and hence totally bounded in
$W^{1,2}((0,T);W^{-1,2}(\Omega;\mathbb{R}^N))$.
Existence of a subsequence $\dot{u}^{n_k}$ strongly convergent in $W^{1,2}((0,T);W^{-1,2}(\Omega;\mathbb{R}^N))$ leads to the statement about existence of the required solution.

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