GOODNESS-OF-FIT TESTS FOR LAPLACE DISTRIBUTION USING RANKED SET SAMPLING

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ABSTRACT
The ranked set sampling (RSS) method is more efficient than the commonly used simple random sampling (SRS) method. In this paper, some new goodness-of-fit tests based on the sample entropy and empirical distribution function for the Laplace distribution using ranked set sampling (RSS) are suggested. The newly suggested tests based on RSS are compared with their simple random sampling (SRS) competitors. The critical values of the corresponding test statistic are computed for each method and the powers of the tests are obtained based on different alternatives. Simulation results indicate that RSS tests are more powerful than their SRS counterparts.

KEYWORDS: Goodness-of-fit test; Ranked set sampling; Laplace distribution; Critical value.

MSC: 62G30; 62G20

RESUMEN
El método del muestreo por rangos ordenados (RSS) es más eficiente que el método comúnmente usado, muestreo simple aleatorio (SRS). En este trabajo algunas pruebas de bondad de ajuste, basados en la entropía muestral y la distribución empírica para la distribución de Laplace, usando RSS son sugeridos. Los nuevos tests basados en RSS son comparados con sus competidores basados en SRS. Los valores críticos correspondientes para las pruebas estadísticas son computados para cada método así como sus potencias, estos son obtenidos basándose en diferentes alternativas. Los resultados de las simulaciones indican que las pruebas RSS son más potentes que sus contrapartes de SRS.

1. INTRODUCTION

Assume that *X* is a continuous random variable with a probability density function (PDF) $f(x)$ and a cumulative distribution function (CDF) $F(x)$. The entropy is known as a measure of uncertainty and dispersion and it is defined by Shanon (1948) for the random variable *X* as $H(f) = -\int_{-\infty}^{\infty} f(x)\log f(x)dx$, which is written by Vasicek (1976) as $H(f) = \int_0^1 \log\left(\frac{\mathrm{d}}{\mathrm{d}p}F^{-1}(p)\right)dp.$

Let $X_1, X_2, ..., X_N$ is a simple random sample of size *n* with CDF $F(x)$, and let $X_{(1)}, X_{(2)}, ..., X_{(N)}$ be the order statistics of this sample. Vasicek (1976) estimator of entropy is defined as

$$HV_m = \frac{1}{N}\sum_{i=1}^{N}\log\left[\frac{N}{2m}\left(X_{(i+m)} - X_{(i-m)}\right)\right], \quad (1)$$

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where m is a positive integer less than $N/2$ known as the window size, $X_{(i)} = X_{(N)}$ if $i > N$ and $X_{(i)} = X_{(1)}$ if $i < 1$. The HV_m estimator converges in probability to $H(f)$ as $N, m \to \infty$, and $m/N \to 0$.

Laplace (1774) suggested a symmetric distribution for describing the errors of measurement and long tails data. This distribution is known as a Laplace distribution and its probability density function is given by

$$f(x;\mu,\sigma) = \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}}, \quad -\infty < x, \mu < \infty, \sigma > 0, \tag{2}$$

where σ is an unknown scale parameter and μ is the location parameter. The corresponding cumulative distribution function is given by

$$F(x;\mu,\sigma) = \begin{cases} \frac{1}{2} e^{\frac{x-\mu}{\sigma}}, & if\, x \le \mu, \\ 1 - \frac{1}{2} e^{-\frac{x-\mu}{\sigma}}, & if\, x > \mu. \end{cases} \tag{3}$$

The Laplace distribution is also known as the double exponential distribution. Goodness-of-fit tests for the Laplace distribution is considered by Chen (2002). The Laplace distribution is described in terms of entropy by Choi and Kim (2006) as in the following corollary.

Corollary: Assume that the random variable *X* satisfying the following condition

$$E_{f(x;\mu,\sigma)}(|x|) = \int_{-\infty}^{\infty} |x|\, f(x;\mu,\sigma) dx \equiv \sigma. \tag{4}$$

Then, the distribution of *X* which maximizing Shannon's entropy is the Laplace distribution with $\mu = 0$ and σ, and its entropy is given by

$$H\left[f(x;\mu,\sigma) \right] = \log(2\sigma) + 1. \tag{5}$$

The rest of this paper is organized as follows. Section 2 is devoted for developing the suggested goodness-of-fit tests. In Section 3, a simulation study is conducted to investigate the suggested tests. In Section 4, an illustrative example is presented. Finally, our conclusions are presented in Section 5.

2. THE SUGGESTED GOODNESS-OF-FIT TESTS

The ranked set sampling (RSS) method was first suggested by McIntyre (1952) for estimating the mean of pasture and forage yields. The RSS can be carried out as follows:

1. Randomly selecting k^2 units from the target population.
2. Allocate the k^2 selected units as randomly as possible into *k* sets, each of size *k*, and rank the units within each set with respect to variable of interest.
3. Identify the RSS units for actual measurement by selecting the *i*th smallest ranked unit from the *i*th sample $\left(i = 1, 2, ..., k\right)$.
4. The Steps 1-3 can be repeated *n* times (cycles) to obtain a sample of size *N*=*nk* for actual measurement.

Note that, based on RSS, even that k^2 units are selected from the population, but only *k* of them are actually measured to compare them with a SRS of size *k*. Thus, the measured RSS units are $X_{[1]j}$, $X_{[2]j}$, …, $X_{[k]j}$. The RSS estimator of the population mean is defined as $\hat{\mu}_{RSS} = \frac{1}{nk}\sum_{j=1}^{n}\sum_{i=1}^{k} X_{[i]j}$, with variance given by

$$\mathrm{Var}\left(\hat{\mu}_{RSS}\right) = \frac{1}{nk}\sum_{j=1}^{n}\sum_{i=1}^{k} \mathrm{Var}\left(X_{[i]j}\right) = \mathrm{Var}\left(\hat{\mu}_{SRS}\right) - \frac{1}{k^2 n}\sum_{i=1}^{k}\left(\mu_{[i]} - \mu\right)^2.$$

Takahasi and Wakimoto (1968) independently conducted the same method and suggested its mathematical properties. They showed that $f(x)=\frac{1}{k}\sum_{i=1}^{k} f_{[i]}(x)$, and $\mu=\frac{1}{k}\sum_{i=1}^{k}\mu_{[i]}$, where $f_{[i]}(x)$ is the pdf of the *i*th order statistic, $X_{[i]}$ of a random sample of size *k* defined as

$$f_{[i]}(x)=k\binom{k-1}{i-1}F^{i-1}(x)\left[1-F(x)\right]^{k-i} f(x),\ -\infty<x<\infty.$$

For more about RSS, Al-Omari (2011) suggested double robust extreme RSS for estimating the population mean. Al-Omari (2014) proposed new estimators of entropy using RSS and double RSS methods. Mahdizadeh (2012) used the RSS in entropy based test of fit for the Laplace distribution. Mahdizadeh and Arghami (2010) considered the RSS in entropy estimation and goodness-of-fit testing for the inverse Gaussian law. Al-Omari and Haq (2012) suggested goodness-of-fit testing for the inverse Gaussian distribution using ranked set sampling and double ranked set sampling methods.

Let $X_{[i]j}$, $i=1,2,\cdots,k$ and $j=1,2,\cdots n,$ be a ranked set sample of size $N=nk$ from the population of interest and $Z_{(1)}\le Z_{(2)}\le\cdots\le Z_{(N)}$ its corresponding ordered values (i.e. Ordered ranked set sample), where $X_{[i]j}$ is the *i*th measured sample unit in the *j*th cycle. Our task is to develop some goodness-of-fit tests of Laplace distribution based on Reyni distance (Reyni, 1961), Kullback-Leibler distance (Kullback and Leibler, 1951), and some other empirical distribution function (EDF) based goodness-of-fit tests using ranked set sampling and compare them with their competitors goodness-of-fit tests in simple random sampling.

Suppose that we are interested in testing the following hypothesis

$$H_0: f(x)=f_0\left(x,\mu,\sigma\right) \text{ vs. } H_1: f(x)\neq f_0\left(x,\mu,\sigma\right)$$

where $f_0\left(x,\mu,\sigma\right)=\frac{1}{2\sigma}e^{-\frac{1}{\sigma}|x-\mu|}, -\infty<x<\infty$, $-\infty<\mu<\infty,$ and $\sigma>0$.

Renyi (1961) showed that the distance between $f(x)$ and $f_0\left(x;\mu,\sigma\right)$ can be obtained as

$$\begin{aligned} D_r\left(f,f_0\right)&=\frac{1}{r-1}\ln\int_{-\infty}^{\infty}\left[\frac{f(x)}{f_0\left(x;\mu,\sigma\right)}\right]^{r-1} f(x)dx \\ &=\frac{1}{r-1}\ln\int_{-\infty}^{\infty}\left[f(x)e^{\frac{1}{\sigma}|x-\mu|}\right]^{r-1} f(x)dx+\ln\left(2\sigma\right). \end{aligned} \tag{6}$$

By using $F(x)=p$, then $D_r\left(f,f_0\right)$ can be rewritten as

$$D_r\left(f,f_0\right)=\frac{1}{r-1}\ln\int_0^1\left[\frac{dF(p)}{dp}e^{\frac{1}{\sigma}\left|F^{-1}(p)-\mu\right|}\right]^{r-1} dp+\ln\left(2\sigma\right). \tag{7}$$

Then by following the lines of Vasicek (1976), the $D_r\left(f,f_0\right)$ can be estimated using a ranked set sample as

$$\hat{D}_{r,m}=\frac{1}{r-1}\ln\left\{\frac{1}{N}\sum_{i=1}^{N}\left[\left(\frac{F_n\left(Z_{(i+m)}\right)-F_n\left(Z_{(i-m)}\right)}{Z_{(i+m)}-Z_{(i-m)}}\right)e^{\left(\frac{\left|Z_{(i)}-\hat{\mu}_{RSS}\right|}{\hat{\sigma}_{RSS}}\right)}\right]^{r-1}\right\}+\ln\left(2\hat{\sigma}_{RSS}\right), \tag{8}$$

where $\hat{\mu}_{RSS}$ is the sample median of $X_{[i]j}$, $i=1,2,\cdots,k$, $j=1,2,\cdots,n,$ and $\hat{\sigma}_{RSS}=\frac{1}{N}\sum_{i=1}^{N}\left|Z_i-\hat{\mu}_{RSS}\right|$, $Z_{(i)}=Z_{(1)}$ for $i<1$ and $Z_{(i)}=Z_{(N)}$ for $i>N$.

However, the power of the test statistic based on $\hat{D}_{r,m}$ strongly depends m. The optimum value of m which causes maximum power depends on alternative distribution and sample size. Since the alternative distribution is often unknown in practice, therefore it is not possible to determine the optimum value for m. To tackle this problem, and by following the lines of Vexler and Gurevich (2010), we propose to use the following test statistic

$$\hat{D}_r = \underset{1 \le m \le N^{0.5}}{Min} \hat{D}_{r,m}.$$

For more information about this method of selecting the value of *m*, see Vexler et al. (2011), Vexler et al. (2012), and Yu et al. (2011). Large enough values of $\hat{D}_r$ can be regarded as a symptom of violations of H_0, and therefore we reject H_0 for large enough values of $\hat{D}_r$. Also, see Wieczorkowski and Grzegorzewsky (1999) for optimal choices of the window size.

The Kullback-Leibler distance (Kullback and Leibler, 1951) between $f(x)$ and $f_0(x;\mu,\sigma)$ is defined as

$$\begin{aligned} KL(f, f_0) &= \int_{-\infty}^{\infty} f(x)\log\left[\frac{f(x)}{f_0(x;\mu,\sigma)}\right]dx \\ &= -H(f) - \int_{-\infty}^{\infty} f(x)\log\left[f_0(x;\mu,\sigma)\right]dx, \end{aligned} \tag{9}$$

and it can be estimated using RSS by

$$KL_m^{RSS} = -HV_m^{RSS} - \frac{1}{N}\sum_{i=1}^{N}\log\left[f_0\left(X_{[i]};\hat{\mu}_{RSS},\hat{\sigma}_{RSS}\right)\right], \tag{10}$$

where $\hat{\mu}_{RSS}$, $\hat{\sigma}_{RSS}$ are as defined in above and HV_m^{RSS} is Vasicek's (1976) counterpart of the entropy estimator for RSS which has the form

$$HV_m^{RSS} = \frac{1}{N}\sum_{i=1}^{N}\log\left[\frac{N}{2m}\left(Z_{(i+m)} - Z_{(i-m)}\right)\right], \tag{11}$$

where $Z_{(i)} = Z_{(1)}$ for $i < 1$ and $Z_{(i)} = Z_{(N)}$ for $i > N$.

Again, the power of the test statistic based on KL_m depends on the parameter *m*. The optimal choice of parameter m depends on the sample size and the alternative distribution. But since the alternative distribution is often unknown in practice, it is not possible to determine the optimum value for *m*. Our solution here is to apply Vexler and Gurevich's (2010) method and use the following test statistic

$$KL = \underset{1 \le m \le N^{0.5}}{Min} KL_m.$$

The Kolmogrov-Smirnov statistic (*KS*) (Kolmogorov (1933) and Smirnov (1933)) , Anderson-Darling statistic (Anderson and Darling, 1954).) (A^2), Cramer-von Mises statistic (W^2), and Zhang statistics (Zhang, 2002) (Z_K, Z_A, Z_C) counterparts test statistics in RSS are, respectively,

$$KS_{RSS} = Max\left\{\underset{1\le i\le n}{Max}\left[\frac{i}{N} - F_0\left(Z_{(i)};\hat{\mu}_{RSS},\hat{\sigma}_{RSS}\right)\right], \underset{1\le i\le n}{Max} F_0\left(Z_{(i)};\hat{\mu}_{RSS},\hat{\sigma}_{RSS}\right) - \frac{i-1}{N}\right\}, \tag{12}$$

$$A_{RSS}^2 = -\frac{2}{N}\sum_{i=1}^{N}\left\{\left(i-\frac{1}{2}\right)\log\left[F_0\left(Z_{(i)};\hat{\mu}_{RSS},\hat{\sigma}_{RSS}\right)\right] + \left(N-i+\frac{1}{2}\right)\log\left[1-F_0\left(Z_{(i)};\hat{\mu}_{RSS},\hat{\sigma}_{RSS}\right)\right]\right\} - N, \tag{13}$$

$$W_{RSS}^2 = \sum_{i=1}^{N}\left[F_0\left(Z_{(i)};\hat{\mu}_{RSS},\hat{\sigma}_{RSS}\right)-\frac{2i-1}{2N}\right]^2+\frac{1}{12N}, \tag{14}$$

$$Z_{RSS}^K = \underset{1\le i\le N}{Max}\left\{\left(i-\frac{1}{2}\right)\log\left(\frac{2i-1}{2NF_0\left(Z_{(i)},\hat{\mu}_{RSS},\hat{\sigma}_{RSS}\right)}\right)+\left(N-i+\frac{1}{2}\right)\log\left(\frac{2N-2i+1}{2N\left[1-F_0\left(Z_{(i)},\hat{\mu}_{RSS},\hat{\sigma}_{RSS}\right)\right]}\right)\right\}, \tag{15}$$

$$Z_{RSS}^A = -\sum_{i=1}^{N}\left\{\frac{2\log\left[F_0\left(Z_{(i)};\hat{\mu}_{RSS},\hat{\sigma}_{RSS}\right)\right]}{2N-2i+1}+\frac{2\log\left[1-F_0\left(Z_{(i)};\hat{\mu}_{RSS},\hat{\sigma}_{RSS}\right)\right]}{2i-1}\right\}, \tag{16}$$

$$Z_{RSS}^C = \sum_{i=1}^{N}\left[\log\left(\frac{F_0\left(Z_{(i)};\hat{\mu}_{RSS},\hat{\sigma}_{RSS}\right)^{-1}-1}{\left(N-\frac{1}{2}\right)\big/\left(i-\frac{3}{4}\right)-1}\right)\right], \tag{17}$$

where $F_0\left(\cdot;\mu,\sigma\right)$ is distribution function of the Laplace distribution and $\hat{\mu}_{RSS}$, $\hat{\sigma}_{RSS}$ are as defined above.

3. SIMULATION STUDY

In this section, a simulation study is conducted to investigate the performance of the power of the proposed goodness-of-fit tests. The entropy tests considered reject the null hypothesis if the test statistics are less than the corresponding critical values at a pre-assigned significance level. Under SRS and RSS methods, 100000 samples are generated from the Laplace distribution with mean of $\mu = 0$ and a variance of $\sigma = 1$ with different sample size $N = 10, 20,$ and 50 for $\alpha = 0.01$, 0.05, 0.1.

In order to compare the powers of goodness-of-fit tests in SRS and RSS designs, we have considered the eleven following distributions as alternative distribution.

- Standard normal distribution denoted by N(0,1),
- Student's *T* distribution with 3 degrees of freedom, denoted by T(3),
- Student's *T* distribution with 5 degrees of freedom, denoted by T(5),
- Standard exponential distribution, denoted by Exp(1),
- Uniform distribution on (0,1), denoted by U(0,1),
- Beta distribution with parameters 0.5 and 0.5, denoted by B(0.5,0.5),
- Beta distribution with parameters 2 and 1, denoted by B(2,1),
- Gamma distribution with scale parameter 1 and shape parameter 0.5, denoted by G(0.5),
- Gamma distribution with scale parameter 1 and shape parameter 2, denoted by G(2),
- Weibull distribution with scale parameter 1 and shape parameter 0.8, denoted by W(0.8),
- Weibull distribution with scale parameter 1 and shape parameter 1.4, denoted by W(1.4).

It is important to note that all the above test statistics are location and scale invariant, and therefore, the critical values and the powers of these tests do not depend on the unknown parameters of Laplace distribution. For power comparison, we have generated 100,000 random samples of sizes $N = 10, 20,$ and 50 in SRS and RSS designs. Tables 1-3 present the critical values of the tests for the Laplace distribution for $\alpha = 0.01$, 0.05, 0.1, respectively, while Tables 4-6 present the estimated powers, respectively, for $N = 10, 20,$ and 50 for $\alpha = 0.05$ using SRS and RSS methods. In RSS scheme, the value of *k* (set size) is taken to be 2 and 5, so we can observe the effect of increasing sample size while set size is fixed, and the effect of increasing set size while sample is fixed.

Remark 1. We have estimated the powers of the tests based on $\hat{D}_r$ for $r \in \{0.2, 0.5, 0.9, 1.5, 2\}$, and we observed that the highest powers of tests occur with $r = 0.5$ or $r = 1.5$ depending on the alternative distribution. Thus, we only report the power of the tests based on $\hat{D}_{0.5}$ and $\hat{D}_{1.5}$.
Based on Tables 3-6 we can conclude that:

- The suggested RSS goodness-of-fit tests are more powerful than their SRS counterparts regardless of the alternative distribution. For illustration, consider the case of $N = 20$, $k = 5$, $\alpha = 0.05$ the standard exponential distribution as an alternative the power values of the tests *KS* and A^2 based on SRS are 0.485, 0.543 compared to 0.538, 0.622 using RSS, respectively.

- Also, we can see that as the k values increase, an improvement in the power of the suggested goodness-of-fit tests is attained. To explain this, consider that $N = 10$, the power values of the Cramer-von Mises statistic based on RSS are 0.403 and 0.439 for $k = 2, 5$, respectively for W(0.8) alternative distribution.

- In general, the large power values are when the alternatives are Beta(0.5,0.5), Gamma(0.5) and Weibull(0.8) distributions. However, the power values of the suggested goodness-of-fit tests depend on the distribution parameters for the same test and sample size. For example, when $k = 5$, $\alpha = 0.05$ and $N = 20$, the power values of the Z_C are 0.914 and 0.330, for $W(0.8)$ and $W(1.4)$, respectively.

- The power increases in the sample size and Table 6 shows that the maximum powers are for $N = 50$. As an example, assume that the Weibull distribution W(1.4) is the alternative distribution, then based on the Anderson and Darling test with $\alpha = 0.05$ and $k = 2$, the power values are 0.123, 0.268, 0.732 for $N = 10$, 20, 50, respectively.

Table 1. Critical values of different goodness-of-fit tests of Laplace distribution for different values of (*n*,*k*) in RSS design, at significance level $\alpha = 0.01$

n	k	$D_{0.5}^{RSS}$	$D_{1.5}^{RSS}$	KL_{RSS}	KS_{RSS}	A_{RSS}^2	W_{RSS}^2	Z_{RSS}^K	Z_{RSS}^A	Z_{RSS}^C
2	2	0.469	1.015	1.025	0.417	15.947	0.135	0.819	1.011	2.133
	3	0.589	1.058	1.091	0.359	36.667	0.152	1.411	0.669	1.894
	4	0.521	1.005	0.966	0.319	64.941	0.154	1.675	0.491	1.630
	5	0.393	0.746	0.772	0.290	101.256	0.157	1.875	0.384	1.390
3	2	0.599	1.066	1.099	0.359	36.585	0.150	1.390	0.668	1.899
	3	0.378	0.799	0.815	0.326	84.129	0.198	2.006	0.441	1.635
	4	0.353	0.723	0.695	0.274	146.35	0.169	2.110	0.318	1.311
	5	0.275	0.654	0.579	0.257	229.382	0.192	2.352	0.250	1.135
4	2	0.542	1.04	0.988	0.322	65.401	0.162	1.745	0.495	1.684
	3	0.360	0.734	0.706	0.276	146.733	0.173	2.164	0.320	1.331
	4	0.267	0.596	0.556	0.243	259.761	0.176	2.366	0.233	1.080
	5	0.219	0.525	0.465	0.220	404.227	0.174	2.513	0.183	0.907
5	2	0.418	0.782	0.798	0.297	102.082	0.169	1.970	0.389	1.491
	3	0.288	0.669	0.592	0.260	230.449	0.201	2.436	0.252	1.177
	4	0.223	0.534	0.469	0.222	405.315	0.182	2.583	0.184	0.934
	5	0.164	0.452	0.383	0.203	632.179	0.192	2.716	0.144	0.779
6	2	0.369	0.747	0.715	0.277	146.855	0.176	2.164	0.320	1.350
	3	0.245	0.573	0.511	0.233	328.640	0.180	2.514	0.206	1.015
	4	0.188	0.485	0.405	0.204	582.132	0.182	2.701	0.151	0.812
	5	0.144	0.414	0.334	0.184	906.892	0.180	2.847	0.119	0.677
7	2	0.325	0.700	0.637	0.261	199.642	0.179	2.321	0.271	1.229
	3	0.209	0.529	0.450	0.223	448.595	0.202	2.706	0.175	0.932
	4	0.156	0.434	0.356	0.190	791.497	0.182	2.850	0.128	0.726
	5	0.123	0.385	0.295	0.173	1234.588	0.190	2.958	0.101	0.611
8	2	0.277	0.611	0.567	0.246	260.331	0.181	2.404	0.234	1.111
	3	0.192	0.498	0.410	0.205	583.286	0.187	2.742	0.152	0.837
	4	0.138	0.413	0.321	0.179	1032.772	0.184	2.909	0.111	0.665
	5	0.102	0.349	0.264	0.161	1609.419	0.181	3.060	0.088	0.544

9	2	0.250	0.584	0.518	0.235	329.403	0.187	2.561	0.207	1.036
	3	0.161	0.451	0.370	0.198	738.719	0.201	2.901	0.134	0.766
	4	0.117	0.375	0.291	0.169	1305.800	0.185	3.048	0.098	0.604
	5	0.092	0.327	0.242	0.154	2037.642	0.192	3.156	0.078	0.501
10	2	0.229	0.552	0.476	0.224	405.705	0.184	2.637	0.184	0.954
	3	0.149	0.427	0.340	0.185	908.451	0.187	2.911	0.119	0.709
	4	0.107	0.356	0.268	0.161	1611.094	0.186	3.097	0.088	0.559
	5	0.077	0.307	0.222	0.145	2512.322	0.184	3.218	0.070	0.457

Table 2. Critical values of different goodness-of-fit tests of Laplace distribution for different values of (*n*,*k*) in RSS design, at significance level $\alpha = 0.05$.

n	*k*	$D_{0.5}^{RSS}$	$D_{1.5}^{RSS}$	KL_{RSS}	KS_{RSS}	A_{RSS}^2	W_{RSS}^2	Z_{RSS}^K	Z_{RSS}^A	Z_{RSS}^C
2	2	0.426	0.697	0.940	0.342	14.835	0.090	0.584	0.936	1.454
	3	0.445	0.774	0.905	0.298	34.752	0.104	1.010	0.623	1.298
	4	0.376	0.733	0.771	0.273	62.452	0.109	1.218	0.460	1.106
	5	0.278	0.570	0.629	0.251	98.030	0.110	1.356	0.363	0.945
3	2	0.461	0.786	0.917	0.298	34.703	0.104	0.996	0.623	1.295
	3	0.278	0.609	0.685	0.281	80.549	0.135	1.461	0.413	1.140
	4	0.244	0.542	0.558	0.235	142.077	0.116	1.524	0.301	0.874
	5	0.188	0.473	0.459	0.222	223.533	0.131	1.715	0.238	0.760
4	2	0.391	0.763	0.794	0.276	62.656	0.113	1.249	0.463	1.143
	3	0.251	0.548	0.566	0.237	142.422	0.119	1.557	0.302	0.888
	4	0.179	0.439	0.444	0.210	253.731	0.120	1.717	0.223	0.721
	5	0.141	0.380	0.369	0.189	396.951	0.120	1.833	0.176	0.602
5	2	0.300	0.598	0.658	0.256	98.554	0.117	1.432	0.366	1.006
	3	0.197	0.488	0.470	0.224	224.181	0.136	1.770	0.240	0.795
	4	0.144	0.383	0.372	0.191	397.494	0.124	1.872	0.176	0.618
	5	0.100	0.316	0.306	0.174	622.212	0.130	1.986	0.140	0.519
6	2	0.256	0.555	0.572	0.238	142.437	0.120	1.561	0.302	0.900
	3	0.164	0.415	0.409	0.200	321.940	0.124	1.831	0.197	0.679
	4	0.124	0.344	0.325	0.175	573.055	0.125	1.963	0.146	0.540
	5	0.086	0.284	0.268	0.158	895.888	0.124	2.069	0.116	0.451
7	2	0.227	0.514	0.509	0.224	194.400	0.123	1.672	0.257	0.810
	3	0.138	0.378	0.361	0.192	439.879	0.138	1.975	0.168	0.622
	4	0.097	0.301	0.287	0.163	780.758	0.126	2.073	0.124	0.483
	5	0.075	0.261	0.239	0.149	1221.101	0.130	2.177	0.099	0.406
8	2	0.188	0.450	0.454	0.212	254.308	0.125	1.759	0.224	0.742
	3	0.126	0.352	0.328	0.176	573.548	0.127	2.011	0.146	0.555
	4	0.085	0.280	0.259	0.154	1020.297	0.127	2.145	0.108	0.443
	5	0.056	0.229	0.213	0.138	1594.509	0.124	2.241	0.086	0.363
9	2	0.169	0.419	0.414	0.202	322.168	0.126	1.856	0.198	0.690
	3	0.099	0.312	0.295	0.170	727.411	0.137	2.113	0.129	0.517
	4	0.065	0.248	0.235	0.146	1292.053	0.128	2.218	0.096	0.403
	5	0.051	0.214	0.196	0.132	2020.292	0.131	2.315	0.076	0.334
10	2	0.153	0.395	0.381	0.193	398.071	0.128	1.912	0.177	0.639
	3	0.091	0.293	0.273	0.160	897.153	0.129	2.134	0.116	0.470
	4	0.059	0.233	0.216	0.139	1595.583	0.128	2.275	0.086	0.371
	5	0.037	0.192	0.180	0.124	2493.599	0.127	2.355	0.068	0.307

Table 3. Critical values of different goodness-of-fit tests of Laplace distribution for different values of (*n*,*k*) in RSS design at significance level $\alpha = 0.1$

n	*k*	$D_{0.5}^{RSS}$	$D_{1.5}^{RSS}$	KL_{RSS}	KS_{RSS}	A_{RSS}^2	W_{RSS}^2	Z_{RSS}^K	Z_{RSS}^A	Z_{RSS}^C
2	2	0.389	0.568	0.908	0.303	14.280	0.078	0.470	0.897	1.082
	3	0.368	0.656	0.813	0.275	33.909	0.088	0.812	0.601	1.022
	4	0.302	0.619	0.680	0.251	61.340	0.090	1.000	0.447	0.878
	5	0.219	0.491	0.557	0.231	96.635	0.091	1.116	0.354	0.754
3	2	0.382	0.667	0.825	0.275	33.883	0.088	0.805	0.601	1.017
	3	0.223	0.530	0.609	0.258	78.991	0.109	1.208	0.400	0.908
	4	0.192	0.459	0.491	0.216	140.266	0.095	1.257	0.294	0.697
	5	0.144	0.400	0.404	0.204	221.091	0.106	1.424	0.234	0.608
4	2	0.317	0.642	0.698	0.253	61.494	0.093	1.023	0.449	0.907
	3	0.197	0.466	0.498	0.218	140.529	0.097	1.291	0.295	0.711
	4	0.133	0.371	0.390	0.192	251.310	0.098	1.431	0.218	0.580
	5	0.102	0.32	0.325	0.174	393.953	0.098	1.525	0.173	0.488
5	2	0.239	0.516	0.582	0.235	97.046	0.096	1.175	0.356	0.803

	3	0.154	0.411	0.415	0.206	221.586	0.110	1.470	0.235	0.639
	4	0.106	0.323	0.329	0.175	394.280	0.100	1.560	0.174	0.500
	5	0.067	0.266	0.271	0.160	618.318	0.106	1.660	0.138	0.421
6	2	0.203	0.471	0.503	0.219	140.587	0.098	1.295	0.295	0.719
	3	0.122	0.305	0.360	0.184	319.071	0.101	1.519	0.194	0.546
	4	0.091	0.291	0.287	0.161	569.326	0.102	1.647	0.144	0.438
	5	0.058	0.238	0.238	0.145	891.295	0.101	1.729	0.114	0.368
7	2	0.178	0.434	0.448	0.206	192.122	0.100	1.385	0.252	0.652
	3	0.102	0.319	0.319	0.176	436.205	0.110	1.645	0.166	0.504
	4	0.065	0.252	0.254	0.150	776.277	0.102	1.724	0.123	0.394
	5	0.051	0.219	0.212	0.137	1215.58	0.105	1.832	0.098	0.333
8	2	0.142	0.381	0.399	0.195	251.720	0.102	1.467	0.219	0.598
	3	0.093	0.295	0.290	0.162	569.618	0.103	1.678	0.144	0.450
	4	0.057	0.234	0.229	0.142	1015.255	0.103	1.796	0.107	0.361
	5	0.033	0.190	0.190	0.127	1588.413	0.101	1.873	0.085	0.298
9	2	0.126	0.354	0.364	0.185	319.276	0.103	1.537	0.194	0.555
	3	0.066	0.261	0.261	0.156	722.748	0.110	1.762	0.128	0.419
	4	0.038	0.207	0.208	0.134	1286.383	0.104	1.860	0.095	0.329
	5	0.030	0.178	0.174	0.121	2012.957	0.106	1.944	0.075	0.275
10	2	0.113	0.332	0.335	0.177	394.817	0.104	1.593	0.174	0.514
	3	0.062	0.244	0.242	0.147	892.159	0.105	1.791	0.114	0.383
	4	0.035	0.193	0.192	0.128	1589.264	0.104	1.912	0.085	0.305
	5	0.016	0.159	0.159	0.114	2485.827	0.103	1.981	0.068	0.253

Table 4. Power estimates of different goodness-of-fit tests for Laplace distribution in SRS and RSS designs for $N=10$ and $\alpha=0.05$.

Sampling scheme	Alternative distribution	Test Statistics								
		$D_{0.5}$	$D_{1.5}$	KL	KS	A^2	W^2	Z_K	Z_A	Z_C
SRS	N(0,1)	0.113	0.057	0.094	0.047	0.046	0.051	0.049	0.027	0.024
	T(3)	0.059	0.077	0.061	0.057	0.062	0.058	0.061	0.071	0.075
	T(5)	0.077	0.055	0.068	0.046	0.045	0.048	0.046	0.04	0.039
	Exp(1)	0.237	0.367	0.341	0.236	0.267	0.241	0.276	0.326	0.290
	U(0,1)	0.317	0.201	0.300	0.101	0.107	0.118	0.115	0.063	0.065
	B(0.5,0.5)	0.585	0.509	0.622	0.239	0.273	0.289	0.261	0.197	0.208
	B(2,1)	0.261	0.185	0.259	0.104	0.122	0.123	0.122	0.098	0.087
	G(0.5)	0.405	0.730	0.685	0.563	0.550	0.519	0.599	0.642	0.590
	G(2)	0.163	0.159	0.176	0.102	0.128	0.118	0.124	0.139	0.124
	W(0.8)	0.298	0.573	0.517	0.412	0.425	0.388	0.456	0.514	0.467
	W(1.4)	0.176	0.155	0.185	0.097	0.121	0.114	0.117	0.122	0.108
RSS ($k=2$)	N(0,1)	0.125	0.064	0.105	0.051	0.048	0.054	0.051	0.028	0.026
	T(3)	0.061	0.079	0.063	0.059	0.062	0.06	0.063	0.072	0.075
	T(5)	0.083	0.059	0.074	0.048	0.047	0.051	0.047	0.042	0.040
	Exp(1)	0.259	0.387	0.366	0.243	0.271	0.249	0.28	0.335	0.294
	U(0,1)	0.338	0.212	0.320	0.103	0.104	0.119	0.111	0.053	0.059
	B(0.5,0.5)	0.603	0.512	0.632	0.224	0.256	0.282	0.238	0.165	0.188
	B(2,1)	0.283	0.197	0.278	0.104	0.123	0.129	0.119	0.092	0.084
	G(0.5)	0.430	0.761	0.715	0.585	0.570	0.542	0.618	0.666	0.611
	G(2)	0.179	0.170	0.193	0.106	0.133	0.126	0.126	0.142	0.125
	W(0.8)	0.323	0.604	0.549	0.426	0.436	0.403	0.468	0.533	0.480
	W(1.4)	0.196	0.168	0.205	0.100	0.123	0.119	0.119	0.124	0.108
RSS ($k=5$)	N(0,1)	0.133	0.069	0.11	0.053	0.050	0.056	0.052	0.029	0.027
	T(3)	0.068	0.088	0.070	0.063	0.070	0.065	0.07	0.08	0.083
	T(5)	0.089	0.064	0.079	0.051	0.051	0.053	0.052	0.045	0.044
	Exp(1)	0.283	0.430	0.404	0.249	0.306	0.265	0.308	0.381	0.339
	U(0,1)	0.370	0.219	0.338	0.104	0.101	0.115	0.112	0.043	0.054
	B(0.5,0.5)	0.660	0.541	0.680	0.211	0.246	0.275	0.226	0.128	0.176
	B(2,1)	0.308	0.201	0.293	0.099	0.123	0.126	0.120	0.087	0.082
	G(0.5)	0.477	0.831	0.791	0.636	0.633	0.592	0.681	0.735	0.682
	G(2)	0.194	0.182	0.205	0.105	0.145	0.131	0.136	0.158	0.141
	W(0.8)	0.354	0.674	0.616	0.461	0.491	0.439	0.521	0.597	0.545
	W(1.4)	0.211	0.174	0.213	0.097	0.131	0.122	0.123	0.133	0.117

Table 5. Power estimates of different goodness-of-fit tests for Laplace distribution in SRS and RSS designs for $N = 20$ and $\alpha = 0.05$.

Sampling scheme	Alternative distribution	Test Statistics								
		$D_{0.5}$	$D_{1.5}$	KL	KS	A^2	W^2	Z_K	Z_A	Z_C
SRS	N(0,1)	0.225	0.101	0.183	0.087	0.069	0.076	0.06	0.037	0.034
	T(3)	0.065	0.117	0.069	0.068	0.070	0.066	0.082	0.099	0.11
	T(5)	0.110	0.073	0.094	0.063	0.055	0.058	0.051	0.051	0.049
	Exp(1)	0.659	0.812	0.827	0.485	0.543	0.443	0.698	0.753	0.647
	U(0,1)	0.794	0.629	0.770	0.249	0.254	0.255	0.215	0.208	0.235
	B(0.5,0.5)	0.979	0.958	0.983	0.511	0.625	0.602	0.523	0.655	0.688
	B(2,1)	0.699	0.550	0.695	0.223	0.269	0.247	0.281	0.298	0.254
	G(0.5)	0.877	0.990	0.989	0.896	0.879	0.815	0.963	0.968	0.931
	G(2)	0.438	0.418	0.491	0.187	0.268	0.214	0.314	0.394	0.311
	W(0.8)	0.764	0.954	0.952	0.762	0.767	0.673	0.896	0.917	0.855
	W(1.4)	0.478	0.421	0.517	0.182	0.252	0.209	0.294	0.359	0.278
RSS ($k = 2$)	N(0,1)	0.239	0.112	0.198	0.095	0.075	0.083	0.064	0.040	0.040
	T(3)	0.068	0.119	0.072	0.071	0.075	0.070	0.084	0.103	0.113
	T(5)	0.116	0.076	0.101	0.063	0.058	0.062	0.053	0.053	0.051
	Exp(1)	0.678	0.837	0.847	0.502	0.568	0.464	0.714	0.773	0.674
	U(0,1)	0.81	0.654	0.787	0.260	0.264	0.271	0.210	0.207	0.252
	B(0.5,0.5)	0.981	0.962	0.984	0.513	0.639	0.618	0.501	0.664	0.712
	B(2,1)	0.719	0.579	0.715	0.227	0.287	0.265	0.284	0.311	0.274
	G(0.5)	0.888	0.993	0.993	0.912	0.901	0.840	0.971	0.977	0.946
	G(2)	0.457	0.441	0.513	0.193	0.283	0.226	0.324	0.413	0.332
	W(0.8)	0.785	0.964	0.962	0.782	0.790	0.697	0.910	0.930	0.873
	W(1.4)	0.504	0.449	0.542	0.189	0.268	0.223	0.304	0.378	0.296
RSS ($k = 5$)	N(0,1)	0.257	0.117	0.207	0.100	0.082	0.094	0.070	0.043	0.042
	T(3)	0.073	0.124	0.075	0.074	0.081	0.076	0.091	0.110	0.119
	T(5)	0.125	0.081	0.105	0.067	0.064	0.069	0.059	0.059	0.056
	Exp(1)	0.723	0.889	0.899	0.538	0.622	0.503	0.778	0.831	0.733
	U(0,1)	0.858	0.701	0.834	0.281	0.285	0.295	0.216	0.209	0.279
	B(0.5,0.5)	0.991	0.980	0.993	0.531	0.687	0.661	0.493	0.715	0.781
	B(2,1)	0.772	0.620	0.767	0.232	0.308	0.284	0.294	0.335	0.300
	G(0.5)	0.917	0.998	0.998	0.945	0.940	0.888	0.987	0.990	0.971
	G(2)	0.495	0.477	0.556	0.205	0.317	0.252	0.357	0.460	0.371
	W(0.8)	0.823	0.984	0.983	0.831	0.845	0.750	0.945	0.958	0.914
	W(1.4)	0.541	0.481	0.586	0.192	0.295	0.240	0.332	0.420	0.330

Table 6. Power estimates of different goodness-of-fit tests for Laplace distribution in SRS and RSS designs for $N = 50$ and $\alpha = 0.05$.

Sampling scheme	Alternative distribution	Test Statistics								
		$D_{0.5}$	$D_{1.5}$	KL	KS	A^2	W^2	Z_K	Z_A	Z_C
SRS	N(0,1)	0.531	0.186	0.451	0.197	0.146	0.172	0.102	0.144	0.133
	T(3)	0.053	0.199	0.059	0.083	0.084	0.079	0.127	0.157	0.189
	T(5)	0.156	0.084	0.134	0.095	0.071	0.083	0.065	0.083	0.076
	Exp(1)	0.978	1.000	1.000	0.960	0.972	0.89	1.000	1.000	0.998
	U(0,1)	1.000	0.996	1.000	0.631	0.816	0.771	0.741	0.980	0.972
	B(0.5,0.5)	1.000	1.000	1.000	0.936	0.997	0.991	0.998	1.000	1.000
	B(2,1)	0.999	0.986	0.999	0.591	0.785	0.704	0.935	0.977	0.937
	G(0.5)	0.998	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000
	G(2)	0.896	0.930	0.967	0.511	0.719	0.549	0.943	0.971	0.895
	W(0.8)	0.988	1.000	1.000	0.999	0.999	0.987	1.000	1.000	1.000
	W(1.4)	0.936	0.942	0.979	0.511	0.703	0.551	0.946	0.971	0.884
RSS ($k = 2$)	N(0,1)	0.542	0.195	0.462	0.206	0.163	0.189	0.114	0.160	0.144
	T(3)	0.056	0.199	0.059	0.084	0.091	0.084	0.134	0.164	0.191
	T(5)	0.156	0.086	0.136	0.099	0.080	0.090	0.070	0.089	0.080
	Exp(1)	0.980	1.000	1.000	0.967	0.979	0.905	1.000	1.000	0.998
	U(0,1)	1.000	0.997	1.000	0.643	0.836	0.790	0.745	0.983	0.976
	B(0.5,0.5)	1.000	1.000	1.000	0.937	0.998	0.992	0.998	1.000	1.000
	B(2,1)	0.999	0.988	0.999	0.599	0.816	0.730	0.955	0.984	0.948
	G(0.5)	0.998	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000
	G(2)	0.899	0.937	0.970	0.525	0.745	0.573	0.953	0.977	0.908

	W(0.8)	0.989	1.000	1.000	0.999	0.999	0.99	1.000	1.000	1.000
	W(1.4)	0.939	0.950	0.981	0.523	0.732	0.576	0.957	0.978	0.900
RSS ($k=5$)	N(0,1)	0.569	0.206	0.482	0.224	0.187	0.214	0.122	0.172	0.165
	T(3)	0.058	0.206	0.061	0.092	0.102	0.094	0.142	0.174	0.200
	T(5)	0.167	0.091	0.144	0.112	0.094	0.106	0.078	0.098	0.089
	Exp(1)	0.986	1.000	1.000	0.985	0.993	0.94	1.000	1.000	1.000
	U(0,1)	1.000	0.999	1.000	0.680	0.891	0.841	0.775	0.994	0.992
	B(0.5,0.5)	1.000	1.000	1.000	0.953	0.999	0.997	1.000	1.000	1.000
	B(2,1)	1.000	0.996	1.000	0.629	0.873	0.78	0.982	0.995	0.977
	G(0.5)	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	G(2)	0.921	0.964	0.984	0.566	0.806	0.622	0.975	0.990	0.944
	W(0.8)	0.992	1.000	1.000	1.000	1.000	0.997	1.000	1.000	1.000
	W(1.4)	0.956	0.975	0.992	0.560	0.799	0.626	0.980	0.992	0.944

4. REAL DATA EXAMPLE

The data set which is used in this example contains the daily closing prices of Switzerland stock index from 1991 to 1998, and is available in the datasets package in R. The data are sampled only in the business time, i.e., the holidays and weekends are omitted. A common way to measure a stock's performance is log ratio of prices which is defined as

$$LR_i = \log\left(\frac{\text{Price in the } i\text{th day}}{\text{Price in } (i\text{-}1)\text{th day}}\right).$$

The salient feature of LR distribution is that it has heavier tails than the normal distribution; therefore Laplace distribution can be regarded as a potential distribution. We draw a ranked set sample of size N=10, using set size $k=5$ from LR values in the population and apply different goodness-of-fit tests on them. The sampling process is drawn with replacement, so the assumption of independence is preserved. The sampled LR values are:

0.0000, 0.0036, 0.0025, 0.0062, 0.0059, -0.0061, 0.0012, 0.0023, 0.0053, 0.0113

The estimated parameters are $\hat{\mu}_{RSS} = 0.0031$ and $\hat{\sigma}_{RSS} = 0.0033$. The values of all the test statistics are computed and given in Table 7. By comparing these values with the corresponding critical values in Table 2, we observe that the null hypothesis that the data follow a Laplace distribution is not rejected by $D_{0.5}^{RSS}$, $D_{1.5}^{RSS}$ and KL_{RSS} at significance level of $\alpha = 0.05$.

Table 7. Observed values of different test statistics in RSS.

Test Statistic	$D_{0.5}^{RSS}$	$D_{1.5}^{RSS}$	KL_{RSS}	KS_{RSS}	A_{RSS}^2	W_{RSS}^2	Z_{RSS}^K	Z_{RSS}^A	Z_{RSS}^C
Value	0.2720	0.3271	0.5001	0.9983	3539.639	398.3974	25685.67	0.0167	14.3607

5. CONCLUSIONS

In this paper, our aim was to develop goodness-of-fit tests for the Laplace distribution using SRS and RSS methods. A simulation study is conducted to evaluate the suggested estimators. It is found that the suggested goodness-of-fit tests based on RSS are more efficient than their counterparts using SRS based on the same test, alternatives, and number of measured units.

RECEIVED: AUGUST 2015
REVISED: JANUARY 2016

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