REVISTA INVESTIGACIÓN OPERACIONAL VOL. 39, NO. 3, 326-340, 2018

ON MEASURES, PRICING AND SHARING OF RISK

Sjur Didrik Flåm*
Informatics Department, University of Bergen, Norway.

ABSTRACT

Suppose each member of some syndicate applies a monetary *measure* to *price* risk. Then, how might they reasonably *share* risk? What premiums could apply to insurance policies? More basically: can modestly informed, moderately skilled members eventually allocate risk efficiently and fairly?
These questions are framed here below by convoluting the members' monetary measures. If the resulting inf-convolution admits a global subgradient at the aggregate risk, then any such gradient provides equilibrium pricing in a pure exchange economy.
Most important, it's shown that clearing prices - and efficient sharing - might emerge after repeated bilateral exchanges.

KEYWORDS: convolution, exchange market, equilibrium, risk measures.
MSC: 90C15. 90C25, 91A26.

RESUMEN

Supongamos que cada miembro de un sindicato aplica una medida monetaria de precio de riesgo. Entonces surgen cuestiones como de qué forma se puede compartir el riesgo de forma razonable, qué premiums pueden aplicarse como parte de las poíticas de seguros o si miembros poco informados y moderadamente capacitados pueden afijar el riego de forma eficiente y justa.
Estas preguntas se encuentran en el marco de la convolución de las medidas monetarias de los miembros del sindicato. Si la inf-convolución resultante admite un sub-gradiente global para la función de riesgo agregado, entonces cualquier sub-gradiente provee un precio de equilibrio en una economía de intercambio puro.
Se prueba también que los precios "clearing"-y el reparto eficiente- puede emerger como resultado de intercambios bilaterales repetidos.

PALABRAS CLAVE: convolución, equilibrio, mercados de intercambio, medidas de riesgo.

1. INTRODUCTION

Large parts of actuarial sciences have considered *single agents* - or just two of them - who face individual risks. Focus on a risk-exposed *group of agents,* typically having many members, came first with Arrow (1953) who emphasized the allocative rôle of security markets. Later, Borch (1962)

*sjur.flaam@uib.no

stressed the constructive part played by reinsurance. Thereby, these two pioneering authors linked finance to insurance - and risk sharing to market exchange.[1]

Questions were framed within the Bernoulli paradigm of expected utility; see also Wilson (1968). That setting offers numerous insights, mostly qualitative [11], [22].[2] However, present several players, each worshipping his (ordinal) *non-transferable utility*, analysts face the intricacies of computing or quantifying solutions.

In response, during recent decades - partly spurred by needs for bank regulations [24] - finance theory has increasingly considered (cardinal) *transferable utilities,* reported via *monetary risk measures.*[3] This shift fits institutional or large investors, and it offers several advantages. To wit, analysis, computation and quantification all become more tractable. Also - on a substantial, albeit somewhat technical note - risk measures open doors to convex analysis, whence to duality, non-smooth objectives, bid-ask spreads, and boundary positions.

Accordingly, this paper presumes that utility (or dis-utility) be transferable across the members of a *syndicate*. Section 2 formulates the allocation problem as that of solving an *inf-convolution* [4]. That section also underscores precisely where and how risk aversion enters by way of convex preferences. Assuming such preferences, Pratt (1962) already considered *one* agent's "*risk aversion in the large and in the small.*" Transversal to his approach, present *several* agents here, their joint risk aversion "in the large" filters down to that of each single agent "in the small" [41].

Extremal convolution of agents' criteria facilitates the analysis. One-sided, global support of the resulting criterion helps to price risk and decentralize decisions. Optimal allocations may emerge as price-supported core solutions and market equilibria. For that, convex preferences are essential *only in the large* - for the convoluted syndicate.

These observations serve as backdrop to the main issues and novelties of the paper: *how might the agents, by and between themselves, eventually implement efficiency*? *In what way could they come to agree on common risk pricing?* Admittedly, to that end, convex criteria become crucial *also in the small* - for each member of the syndicate.

Section 3 prepares the ground by considering merely *one* bilateral, direct exchange. Section 4 shows, as main contribution, that repeated transactions, between just two parties at a time, may eventually generate efficiency. It's noteworthy that no objectives need be smooth.

The paper may interest diverse groups of readers. Included are economic, finance and insurance theorists who care about constructive attainment of stable equilibrium. Also addressed are optimizers and mathematicians who study stochastic or non-smooth data. Further, there are links to computer science concerned with block-coordinate methods and distributed or parallel programming [33], [34], [35], [36].

The paper can be read as a stylized story about risk-exposed agents' market behavior - *or* as displaying how equilibrium could emerge via agent-based computations. The measure-theoretic component, most prominent in many studies [19], [23] [40], is deliberately played down here.

[1] See also Malinvaud (1972-73), Bühlmann, Jewell (1979), Cass et al. (1996), and Dana, Scarsini (2007).

[2] Expected utility still dominates, often with strong assumptions as to differentiability of criteria or interiority of positions [17].

[3] Important references include [2], [10] and [19].

2. RISK AND SYNDICATED SHARING

This section formalizes two things. First, it explains what is meant here below by a *risk*? Second, it fixes the frames within which *syndicated sharing* might emerge.

Risk materializes from some underlying, elementary outcome (or state) ω in a specified set Ω. The latter comprises all relevant, but mutually exclusive, future "contingencies." Ω is endowed with a field $\mathcal{F}$ of subsets, each referred to as *measurable*. Members of $\mathcal{F}$ are just those events that can commonly be confirmed or refuted ex post - after ω has happened. The reader (and certainly the computer scientist) may prefer to regard Ω as finite, and then - with no loss of generality - let $\mathcal{F}$ be composed of all Ω-subsets.

Anyway, a *risk* is a $\mathcal{F}$-measurable mapping $\omega \mapsto x(\omega)$ from Ω into some Euclidean space $\mathcal{E}$ of consequences. For concreteness and interpretation, one may posit $\mathcal{E} = \mathbb{R}^C$, for a finite list C of "commodity" labels, and write $x(\omega) = [x(c, \omega)]_{c \in C}$. In standard finance and insurance, C is a singleton, $\mathcal{E} = \mathbb{R}$, and *money* is the only commodity in question [27], [28].

For a holder of risk x, outcome $\omega \in \Omega$ entails *loss* $x(\omega) \in \mathcal{E}$. A realized component $x(c, \omega) \geq 0$ of x activates his (contractual) commitment to *give up* the corresponding amount of commodity c; a *negative* component gives him the right to *receive* the same amount of c. With this convention, a holder of risk x has a "debit account" from which is "drawn" $x(\omega) \in \mathcal{E}$ in state ω.

In the leading interpretation, let $\mathbb{L}^0 = L^0(\Omega, \mathcal{F}; \mathcal{E})$ denote the space all $\mathcal{F}$-measurable mappings $x : \Omega \to \mathcal{E}$. Henceforth regard risks as vectors in some fixed real subspace $\mathbb{X} \subseteq \mathbb{L}^0$. That space inherits the natural order from $\mathcal{E}$.[4]

Syndicated sharing might come about when a fixed, finite family I of agents face risks (aggregate or individual). Here, attention goes beyond the most studied case in which I comprises just two agents: one demanding insurance, the other providing it.[5]

By standing assumption, member $i \in I$ valuates risk up front - prior to any realization $\omega \in \Omega$. For valuation, he uses an idiosyncratic, pecuniary *cost function* $\rho_i : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$.[6] Thus, $\rho_i(x_i)$ equals his personal liability, measured *ex ante* in money, of incurring loss $x_i(\omega) \in \mathcal{E}$, realized *ex post*. Put differently: $\rho_i(x_i)$ reports individual *dis-utility* or inconvenience, denominated in units of common account, of covering x_i.

Presuming that money and risk both be perfectly transferable - with no fees or frictions - any *aggregate risk* $x_I \in \mathbb{X}$ could be allocated by solving the *inf-convolution*

$$\rho_I(x_I) := \inf \left\{ \sum_{i \in I} \rho_i(x_i) : \sum_{i \in I} x_i = x_I \right\}. \tag{2.1}$$

$\rho_I(x_I)$ equals the money amount the syndicate would pay, in toto, to rid itself of aggregate liability x_I.

[4] Note that if Ω is finite, $\mathbb{L}^0$ becomes Euclidean.

[5] Typically, $\#I > 2$.

[6] The value $+\infty$ is a conceptual device. It signals infeasibility - that is, violation of underlying constraints, not spelled out here.

Remark 2.1. (On insurance versus finance) . *A twin version of problem (2.1) regards any* $\chi_i(\omega) := -x_i(\omega)$ *as realized revenue, put into* $i's$ *"credit account." In this optic, a "financial position"* $\chi_i \in \mathbb{X}$ *is worth transferable utility* $u_i(\chi_i) = -\rho_i(-x_i)$ *to him. Then,* $u_i(\chi_i)$ *equals the ex ante pecuniary value, to agent* i, *of owning contingent claim* $\omega \mapsto \chi_i(\omega) \in \mathcal{E}$. *Thus, while problem format (2.1) suits insurance, its mirror image, the sup-convolution*

$$u_I(\chi_I) := \sup \left\{ \sum_{i \in I} u_i(\chi_i) : \sum_{i \in I} \chi_i = \chi_I \right\} \tag{2.2}$$

might be better tailored to finance. Henceforth format (2.1) stays in focus.

Remark 2.2. (On links to risk measures) . *In important and tractable cases, each criterion* ρ_i *qualifies as a risk measure [15], [19], [23], [40]. As formalized or axiomatized, such measures satisfy one or more special properties - be it cash invariance, comonotone additivity, convexity, law invariance, monotonicity, positive homogeneity, or subadditivity. Just one of these properties is chief in the sequel, namely: convexity. What imports is first, that dis-utility be transferable, and second, that it comes convex.*

Any solution to (2.1) is called an efficient or *optimal* allocation of the prescribed aggregate x_I.[7] The next section considers how optimality could be captured and characterized via risk pricing. For this purpose, given any function $f : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ and linear functional $x^* : \mathbb{X} \to \mathbb{R}$, the *subgradient* expression $x^* \in \partial^- f(x)$ means that

$$x \in \arg\min \{f - x^*\} (\cdot). \tag{2.3}$$

By tacit assumption, (2.3) features a *finite* minimal value. Hereafter, to alleviate notations, it's convenient to write x^*x in place of $x^*(x)$.
If an extended indicator $\delta_X : \mathbb{X} \to \{0, +\infty\}$ defines a non-empty subset $X \subseteq \mathbb{X}$ by $x \in X \iff \delta_X(x) = 0$, then it subdifferential

$$\partial^- \delta_X(x) = \{x^* : x^*(\chi - x) \leq 0 \ \forall \chi \in X\} =: N_X(x). \tag{2.4}$$

is called the outward *normal cone* to X at $x \in X$.
When $f : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ is convex, $\partial^- f$ coincides with the customary subdifferential operator ∂f in convex analysis [39].

3. OPTIMAL RISK SHARING

This section aims at characterizing optimal solutions to problem (2.1). By assumption, at least one such solution exists.

[7] Not presuming transferable utility, many studies must contend with the weaker normative criterion of *Pareto optimality* [9], [21], [41]. Here that criterion has too little bite.

For simplicity, any subgradient $x^* \in \partial^- \rho_I(x_I)$ (2.3) of the convoluted criterion (2.1), at the aggregate risk x_I, is henceforth named a *shadow price.* Every such price system makes it easier to come to grips with efficient allocations:

Theorem 3.1. (Price characterization of optimal risk sharing)

- ***On shadow pricing and equal margins.*** *For any optimal allocation (x_i) of aggregate risk x_I,*
$$x^* \in \partial^- \rho_I(x_I) \Longleftrightarrow x^* \in \partial^- \rho_i(x_i) \text{ for each } i. \tag{3.1}$$
- ***On optimality and market equilibrium.*** *Any optimal allocation (x_i) of x_I, alongside any shadow price $x^* \in \partial^- \rho_I(x_I)$, constitutes a market equilibrium in that*
$$x_i \in \arg\min \{\rho_i - x^*\} \quad \forall i \in I \text{ and } \sum_{i \in I} x_i = x_I. \tag{3.2}$$
Conversely, every such equilibrium allocation (x_i) is optimal, and every associated price x^* must be a corresponding shadow entity.
- ***Cost-sharing core solutions.*** *Suppose, just here, that each agent i, at the outset, be "endowed" with a risk $e_i \in \mathbb{X}$ such that $\sum_{i \in I} e_i = x_I$. For any non-empty coalition $C \subseteq I$, let $x_C := \sum_{i \in C} e_i$, and, like (2.1), posit*
$$\rho_C(x_C) := \inf \left\{ \sum_{i \in C} \rho_i(x_i) : \sum_{i \in C} x_i = x_C \right\}.$$
Then, given any shadow price $x^* \in \partial^- \rho_I(x_I)$, the cost-sharing scheme (κ_i) defined by
$$\kappa_i := \inf \{\rho_i - x^*\} + x^* e_i \tag{3.3}$$
constitutes a core solution to the transferable-cost, cooperative game with characteristic (cost) function $C \mapsto \rho_C(x_C)$. That is,
$$\sum_{i \in C} \kappa_i \le \rho_C(x_C) \ \forall C \subset I, \text{ and } \sum_{i \in I} \kappa_i = \rho_I(x_I).$$
- ***Existence of a shadow price.*** *Let $\check{\rho}_I : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ denote the largest convex function pointwise $\le \rho_I$, the latter being defined in (2.1). If $\partial \check{\rho}_I(x_I)$ is non-empty and $\check{\rho}_I(x_I) = \rho_I(x_I)$, then each $x^* \in \partial \check{\rho}_I(x_I)$ is a shadow price. In particular,when $\partial \check{\rho}_I(x_I)$ reduces to a singleton, x^* is unique.*
- ***Existence of an optimal allocation.*** *In case $\mathbb{X}$ is Euclidean, assume that each ρ_i be lower semicontinuous. Suppose there is a subset $\mathcal{I} \subseteq I$such that $\rho_i^{-1}(\mathbb{R})$ is compact when $i \in \mathcal{I}$, and $\sum_{i \notin \mathcal{I}} \rho_i(x_i) \to +\infty$ as $\sum_{i \notin \mathcal{I}} \|x_i\| \to \infty$. Then there exists an optimal allocation.*

Proof. Let (x_i) be any optimal allocation. Then, $x^* \in \partial^- \rho_I(x_I)$
$$\Longleftrightarrow \sum_{i \in I} \rho_i(\chi_i) \ge \rho_I(\chi) \ge \rho_I(x_I) + x^*(\chi - x_I) \text{ with } \chi = \sum_{i \in I} \chi_i$$

$$\Longleftrightarrow \sum_{i\in I}\{\rho_i(\chi_i) - x^*\chi_i\} \geq \sum_{i\in I}\{\rho_i(x_i) - x^* x_i\} \ \forall (\chi_i) \in \mathbb{X}^I. \tag{3.4}$$

Fix any $i \in I$. In (3.4), posit $\chi_j = x_j$ for all $j \neq i$ to have

$$\rho_i(\chi_i) - x^*\chi_i \geq \rho_i(x_i) - x^* x_i \ \forall \chi_i \in \mathbb{X}, \forall i \in I. \tag{3.5}$$

By (2.3) it obtains $x^* \in \partial^- \rho_i(x_i), \forall i$. Conversely, summing inequalities (3.5) over $i \in I$ yields (3.4). This proves the first bullet on common margins (3.1).

As to market equilibrium, simply note that system (3.5) amounts to $x_i \in \arg\min \{\rho_i - x^*\} \ \forall i \in I$. Moreover, when (χ_i) allocates x_I, summation of (3.5) gives $\sum_{i\in I}\rho_i(\chi_i) \geq \sum_{i\in I}\rho_i(x_i)$. That is, any equilibrium profile is optimal. Further, by repeating the same summation, but now with $\sum_{i\in I}\chi_i = \chi$, one obtains

$$\rho_I(\chi) - x^*\chi \geq \rho_I(x_I) - x^* x_I \ \text{ for all } \chi \in \mathbb{X},$$

hence, again by (2.3), $x^* \in \partial^- \rho_I(x_I)$.

For core solutions, given any non-empty coalition $C \subseteq I$, note that

$$\begin{aligned}
\sum_{i\in C}\kappa_i &= \sum_{i\in C}\inf\{\rho_i(\chi_i) - x^*(\chi_i - e_i)\} \\
&\leq \inf\sum_{i\in C}\{\rho_i(\chi_i) - x^*(\chi_i - e_i)\} \\
&\leq \inf\sum_{i\in C}\left\{\rho_i(\chi_i) : \sum_{i\in C}(\chi_i - e_i) = 0\right\} = \rho_C(x_C).
\end{aligned}$$

In particular, $\sum_{i\in I}\kappa_i \leq \rho_I(x_I)$. That the converse of the last inequality also holds follows from (3.4).

For existence of a shadow price, simply observe that $\partial\check{\rho}_I(x_I) \subseteq \partial^- \rho_I(x_I)$.

Finally, *for existence of an optimal solution*, suppose a feasible allocation (χ_i) satisfies $\sum_{i\notin \mathcal{I}}\|\chi_i\| = r$. Then, provided r be large enough, $\sum_{i\in I}\rho_i(\chi_i) > \rho_I(x_I)$. Hence existence of an optimal allocation derives from the lower semicontinuity of each ρ_i, coupled with coercivity and compactness. □

Miscellaneous remarks conclude this section. None are essential.

(*On shadow pricing*). Any shadow price can be seen as a *common* Lagrange multiplier - a dual object, hence a linear price system $x^* : \mathbb{X} \to \mathbb{R}$ associated to relaxations of the coupling constraint $\sum_{i\in I} x_i = x_I$ [17]. A chief issue here below is how differences between agents' *individual* multipliers (alias their margins) could motivate exchanges. If such differences thereby dwindle, a common shadow price might emerge.

(*On market clearing and welfare*). By the *second welfare theorem* [29], [32], equilibrium (3.2) is a price-taking (alias *competitive*) *Walras equilibrium* for the pure exchange economy in which agent i, at the outset, already holds his part x_i of an optimal allocation (x_i). Note though that x^* is unaffected by the distribution (e_i) of initial risk *endowments*, satisfying $\sum_{i\in I} e_i = x_I$. Most likely, $x^* x_i \neq x^* e_i$ for at least two agents. Thus, commonplace budget constraints disappear here. They are obviated by endogenous side payments for demand/supply of insurance.

(*On core outcomes*). Core imputation (3.3) debits agent i for the risk e_i he sheds onto the syndicate, but credits him for the risk x_i he shoulders. *Individual rationality* prevails because $\kappa_i \leq \rho_i(e_i) \ \forall i$.

(*On attainment of equilibrium*). Syndicate members can hardly leap straight to market equilibrium (3.2) - or to core imputations (3.3). They need time to learn. Further, where do prices come from? These two queries motivate the subsequent sections.

(*On composition of risks*). Many studies elaborate on important effects of adding background or non-insurable risk [9], [13], [20]. Here the aggregate risk is rather taken as primitive - not as decomposed across different sources. While risk sharing remains the chief issue, nothing was, or will be, said about stochastic dependence among various components or terms in x_I. Thus, the paper is silent on crucial policy consequences - say, on comonotonicity, co-insurance or deductibles [9]. Note though, that when $\dim \mathcal{E} > 1$, risk becomes multivariate [13].

(*On functional properties*). It's remarkable that, so far, no criterion ρ_i needs be continuous, convex, monotone, normalized, or positively homogenous [38]. Broadly, what imports is only that the convoluted criterion ρ_I be convex with respect to *one* point, namely x_I. In other words: convexity - hence monotone margins - enters chiefly at the aggregate level. As is easily verified, inf-convolution (2.1) tends to lift useful properties - for example, convexity - from individual terms ρ_i to ρ_I. What is no less important, it also contributes towards "creating" such properties. For instance, on a qualitative note, adding many and minor members to the syndicate I, renders $\rho_I(\cdot)$ "more convex" [3], [12], [30].

(*On vector spaces*). Except for the last bullet, Theorem 3.1 holds for any real linear space; no topological arguments were invoked.

(*On outside insurance*). Background components, if any, in x_I, whether difficult to diversity or insure, ought nonetheless be evaluated - at least at their margins. Using shadow prices to do so, the syndicate could transact with exogenous parties. To illustrate, suppose some outsider $o \notin I$ offers to cover liability $\Delta \in \mathbb{X}$ for premium π. Since

$$x^* \in \partial\rho_I(x_I) \Longrightarrow \rho_I(x_I - \Delta) \geq \rho_I(x_I) - x^*\Delta,$$

the syndicate should care that $\pi \leq \inf \{x^*\Delta : x^* \in \partial\rho_I(x_I)\}$.[8] Also, because

$$\rho_{I\cup o}(x_I) \leq \rho_I(x_I - \Delta) + \rho_o(\Delta),$$

and reasonably, $\pi \geq \rho_o(\Delta)$, the outside offer might be attractive if $\pi \leq \rho_{I\cup o}(x_I) - \rho_I(x_I - \Delta)$. Thus, this paper opens towards - but does not develop - extended syndicates. Including more members could fit say, partial cover for syndicate-wide catastrophes.

(*On decision making*). Naming the ensemble I a *syndicate* could lure one into thinking that the members must make a common decision in face of uncertainty. Here, that image isn't quite fitting. In what follows, decisions on sharing will be fully decentralized.

4. BILATERAL RISK REALLOCATION

Theorem 3.1 invites a top-down perspective - from coordinated shadow pricing (of risk) at the upper, aggregate level down to equal margins across agents.

[8] Likewise, if the syndicate were to carry additional risk Δ for outsider o, it ought charge him a premium $\pi \geq \sup \{x^*\Delta : x^* \in \partial\rho_I(x_I)\}$.

This section embarks on an alternative, more constructive, totally opposite approach. It starts from unequal margins deep down and moves upward, by way of trade, to make for possible emergence of common risk pricing. Upon travelling this turned-around path, convexity becomes handy - in fact, almost indispensable.

Some motivating remarks are in order. It appears natural to regard each syndicate member as autonomous and self-interested - more or less. Then, quite likely, data or decisions are distributed. Consequently, a coordinator (or central processing unit) could hardly direct all choices, know every detail - or solicit necessary information. This observation indicates the expediency of letting market-like platforms mobilize the members, but leave their choices to themselves. For that, some useful instruments are most classical. In fact, *direct exchange* will serve here as chief vehicle. Moreover, just two parties transact at any time.

Recall that a monetary measure may take the value $+\infty$, thereby signalling non-admissible risk exposure. Thus, agent $i \in I$ is subject to the viability restriction that x_i always belong to his *effective domain*

$$X_i := \{x_i \in \mathbb{X} : \rho_i(x_i) < +\infty\} = \rho_i^{-1}(\mathbb{R}) =: dom\rho_i. \tag{4.1}$$

From here onwards, the paper's orientation becomes more computational - and practical. So, *assume now that $\mathbb{X}$ be finite-dimensional Euclidean with norm $\|\cdot\|$, and that each criterion ρ_i be convex.*

To begin with, this section fixes two agents $i, j \in I$, and it considers the following episode. Prior to realization of any contingency $\omega \in \Omega$, the two parties meet with respective holdings $x_i \in X_i$ (4.1) and $x_j \in X_j$. Suppose they detect that

$$\partial\rho_i(x_i) \cap \partial\rho_j(x_j) = \varnothing. \tag{4.2}$$

Put differently: suppose they disagree on ex ante valuations of marginal risks. Then, why not shift some "small risk" $\Delta \in \mathbb{X}$ away from first agent to the second? With no loss of generality, the said shift takes the form $\Delta = sd$ for some *step* $s > 0$ along some *direction* $d \in \mathbb{X}$. Thus, they could arrive at updated, still feasible positions

$$x_i^{+1} := x_i - sd \in X_i \quad \text{and} \quad x_j^{+1} := x_j + sd \in X_j. \tag{4.3}$$

Such updating just reallocates risks; it preserves aggregates in that $x_i^{+1} + x_j^{+1} = x_i + x_j$. The last part of (4.3) implies that d belongs to the convex cone

$$D_j(x_j) := \{d : x_j + sd \in X_j \text{ for sufficiently small step } s \geq 0\}. \tag{4.4}$$

This cone comprises all *feasible directions* for agent j at x_j. By assumption, $D_j(x_j)$ *is closed with non-empty interior.*[9] Similarly, the first statement in (4.3) tells that $-d \in D_i(x_i)$. Hence the interlocutors must, out of concerns for feasibility, agree on some direction

$$d \in D_{ij}(x_i, x_j) := [-D_i(x_i)] \cap D_j(x_j). \tag{4.5}$$

Besides feasibility, they have, of course, additional concerns. Regarding these, it appears natural that

$$d = x_i^* - x_j^* \text{ with } x_i^* \in \partial\rho_i(x_i) \text{ and } x_j^* \in \partial\rho_j(x_j). \tag{4.6}$$

[9] $intD_j(x_j)$ is non-empty if $intX_j$ is likewise.

Broadly, the chosen direction should equal a subgradient difference. For a partial and *economic rationale* behind (4.6), suppose $\mathbb{X} = \mathbb{R}^{C\times\Omega}$, C and Ω being finite sets of commodities and contingencies respectively. Here, for argument, let $D_{ij}(x_i, x_j) = \mathbb{X}$. Then, by (4.6), component (c, ω) of d should satisfy $d(c, \omega) > 0 \Longleftrightarrow x_i^*(c, \omega) > x_j^*(c, \omega)$. Thus, the party who has lowest marginal cost becomes a net receiver; he should shoulder a larger part of loss in the setting (c, ω).

For another partial, but *mathematical rationale* behind (4.6), still with $D_{ij}(x_i, x_j) = \mathbb{X}$, suppose, here again just for argument, that ρ_i, ρ_j, are differentiable at x_i, x_j respectively. Then, $d = \rho_i'(x_i) - \rho_j'(x_j) = x_i^* - x_j^*$, and, using directional derivatives,

$$\rho_i'(x_i; -d) + \rho_j'(x_j; d) = -x_i^* d + x_j^* d = -\left[x_i^* - x_j^*\right] d = -\left\|x_i^* - x_j^*\right\|^2.$$

Consequently, provided $x_i^* \neq x_j^*$ and $s > 0$ be sufficiently small, updates (4.3) imply

$$\rho_i(x_i^{+1}) + \rho_j(x_j^{+1}) < \rho_i(x_i) + \rho_j(x_j).$$

In this case, modulo *zero-sum monetary side payments* μ_i, $\mu_j \in \mathbb{R}$, both parties can see some satisfaction in so far as

$$\rho_i(x_i^{+1}) + \mu_i < \rho_i(x_i), \quad \rho_j(x_j^{+1}) + \mu_j < \rho_j(x_j) \quad \& \quad \mu_i + \mu_j = 0.$$

To argue more fully for (4.6), it remains to discuss instances where $D_{ij}(x_i, x_j) \neq \mathbb{X}$ - or where some function ρ_i, ρ_j isn't differentiable at the respective agent's actual holding. For either case, suppose that each subgradient $x_i^* \in \partial\rho_i(x_i)$ has a decomposition

$$x_i^* = \underline{x}_i^* + n_i \text{ with } \underline{x}_i^* \in \underline{X}_i^*(x_i) \text{ and } n_i \in N_{X_i}(x_i). \tag{4.7}$$

The correspondence $x_i \in X_i \rightrightarrows \underline{X}_i^*(x_i) \subset \mathbb{X}$ is supposed outer semicontinuous [39] with non-empty, uniformly bounded, closed, convex values. $N_{X_i}(x_i)$ is the normal cone (2.4) at x_i of the effective domain X_i (4.1). By assumption, each subgradient $x_j^* \in \partial\rho_j(x_j)$ of agent j decomposes likewise.

Invoking decompositions (4.7), can (4.5) comply with (4.6)? To address this question, write simply $\mathcal{P}_{ij}[\cdot]$ for the *orthogonal projection* onto the closed convex cone $D_{ij}(x_i, x_j)$ (4.5).

Proposition 4.1. (Projected subgradient difference) *Any two subgradients $x_i^* \in \partial\rho_i(x_i)$, $x_j^* \in \partial\rho_j(x_j)$ admit decompositions (4.7) such that*

$$d = \underline{x}_i^* - \underline{x}_j^* \in D_{ij}(x_i, x_j).$$

Proof. Recall the assumption that both cones $D_i(x_i)$, $D_j(x_j)$ are closed convex. Hence, by a decomposition theorem of Moreau, the particular choice

$$d := \mathcal{P}_{ij}\left[x_i^* - x_j^*\right] = (x_i^* - x_j^*) - n \in D_{ij}(x_i, x_j) \tag{4.8}$$

is well defined for some unique normal vector n, belonging to the dual cone

$$N_{ij}(x_i, x_j) := \{n : nd \leq 0 \text{ for all } d \in D_{ij}(x_i, x_j)\}.$$

By assumption, both cones $-D_i(x_i)$, $D_j(x_j)$ have non-empty interior. Hence

$$N_{ij}(x_i, x_j) = -N_{X_i}(x_i) + N_{X_j}(x_j).$$

So, the above outward pointing vector n equals $-n_i + n_j$ for some normals $n_i \in N_{X_i}(x_i)$, $n_j \in N_{X_j}(x_j)$. Now, (4.6) follows from (4.7) which implies $x_i^* - n_i = \underline{x}_i^*$, $x_j^* - n_j = \underline{x}_j^*$, and thereby

$$d = (x_i^* - x_j^*) - (-n_i + n_j) = \underline{x}_i^* - \underline{x}_j^* \in \partial\rho_i(x_i) - \partial\rho_j(x_j). \quad \square$$

5. REPEATED RISK EXCHANGE

The preceding section dealt with just one encounter - and the attending direct exchange of risk - between two syndicate members.
Each such transaction is henceforth seen as a recurrent episode of repeated exchange. For convergence of the resulting process, *step-sizes* should provide sufficient progress. Further, some *protocol* should regulate *who meets next whom.* These issues are taken up next.

The protocol: Henceforth suppose agents meet pair-wise, just *one* pair at each stage, selected independently and randomly with uniform probability $1/\binom{n}{2}$, $n := \#I$. This stochastic mechanism complements the deterministic ones considered in [16]. Random matching is also used in [14], [33], [34], [35]. This paper differs by allowing extended-valued, non-Lipschitz objectives.

Step-sizes $s_k > 0$ *should dwindle, but not too fast* along stages $k = 0, 1, ...$ These requirements are cared for provided

$$\sum_{k=0}^{\infty} s_k^2 < +\infty \text{ and } \left\{ \sum_k s_k : i \text{ meets } j \right\} = +\infty \text{ almost surely } \forall i, j \in I. \tag{5.1}$$

Thus, each encounter is a single step of a discrete-time stochastic process. For argument, that process is cast next in the form of an **algorithm**:

Repeated exchange proceeds at stages $k = 0, 1, ...$
Start *with some profile* $\mathbf{x} = (x_i) \in \mathbb{X}^I$ *which satisfies* $\sum_{i \in I} x_i = x_I$ *and* $x_i \in X_i := dom\rho_i$ (4.1) *for each* $i \in I$.
Select two members $i, j \in I$ *independently, with uniform probability.*
Choose any of their marginal valuation schemes $x_i^* \in \partial_i(x_i)$, $x_j^* \in \partial\rho_j(x_j)$ *such that* (4.7) *and* (4.8) *hold.*
Update their actual holdings *by* (4.3) *for some suitable step-size* $s > 0$.
Continue *to* **Select two members** *until convergence.*

Theorem 5.1. (Convergence [18].) *Under (5.1) the profile* $k \mapsto \mathbf{x}^k = (x_i^k)$ *generated by repeated risk exchange converges almost surely (a.s.) to the set* $\bar{\mathbf{X}}$ *of optimal allocations of the aggregate risk* x_I*. That is,* $dist(\mathbf{x}^k, \bar{\mathbf{X}}) \to 0$ *a.s.*

Proof. [18] provides proof for the mirror-image instance (2.2). For completeness, and to save translation from concave to convex functions, direct arguments are included here.
Consider a stage k at the moment when $\mathbf{x}^k = (x_i^k)$ is about to be updated by members i, j. For simpler notation, temporarily suppress mention of k. In compliance with (4.3) and (4.6), define a direction $\mathbf{d} \in \mathbb{X}^I$ with components

$$d_j := x_i^* - x_j^* =: -d_i, \text{ and all other } \mathbf{d}\text{-components are nil.}$$

Using step-size $s > 0$, direct exchange, as modelled by (4.3), takes the form $\mathbf{x}^{+1} := \mathbf{x} + s\mathbf{d}$. Now, for any optimal allocation $\bar{\mathbf{x}} \in \bar{\mathbf{X}}$,

$$\begin{aligned} \left\|\mathbf{x}^{+1} - \bar{\mathbf{x}}\right\|^2 &= \|\mathbf{x} + s\mathbf{d} - \bar{\mathbf{x}}]\|^2 = \|\mathbf{x} - \bar{\mathbf{x}}\|^2 + 2s(\mathbf{x} - \bar{\mathbf{x}})\mathbf{d} + s^2 \|\mathbf{d}\|^2 \\ &= \|\mathbf{x} - \bar{\mathbf{x}}\|^2 + 2s\left\{(x_i - \bar{x}_i)d_i + (x_j - \bar{x}_j)d_j\right\} + 2s^2 \|d_i\|^2 . \end{aligned}$$

Recall that the correspondences $x_i \rightrightarrows \underline{X}_i^*(x_i)$ and $x_j \rightrightarrows \underline{X}_j^*(x_j)$, mentioned in (4.7), have uniformly bounded values. On that premise, Proposition 4.1 implies that $\|d_i\|$ is bounded. Hence $2\|d_i\|^2 \leq \gamma$ for some constant $\gamma \geq 0$.
The pair (i, j) of distinct agents is randomly drawn with uniform probability. Repeated draws are independent of one another. In the above equalities take expectation E with respect to drawing the agent pair (i, j). This gives

$$E\left\|\mathbf{x}^{+1} - \bar{\mathbf{x}}\right\|^2 \leq \|\mathbf{x} - \bar{\mathbf{x}}\|^2 - \frac{8s}{n(n-1)} \sum_i \sum_j [x_i^* - x_j^*](x_i - \bar{x}_i) + s^2\gamma. \tag{5.2}$$

Next, consider the orthogonal projection $\mathcal{P}_{\mathbf{\Delta}}$ from the product space $\mathbf{X} := \mathbb{X}^I$ onto its linear subspace

$$\mathbf{\Delta} := \left\{(\Delta_i) \in \mathbf{X} : \sum_{i \in I} \Delta_i = 0\right\}.$$

Given *any* vector $\mathbf{x}^* = (x_i^*) \in \mathbf{X}$, it holds for each block component $i \in I$ of its projection $\mathcal{P}_{\mathbf{\Delta}}[\mathbf{x}^*]$ onto $\mathbf{\Delta}$ that

$$(\mathcal{P}_{\mathbf{\Delta}}[\mathbf{x}^*])_i = \frac{1}{n} \sum_{j \in I} \left\{x_i^* - x_j^*\right\}.$$

So, returning to the sum in the last inequality,

$$\sum_i \frac{1}{n} \sum_j [x_i^* - x_j^*](x_i - \bar{x}_i) = \mathcal{P}_\Delta[\mathbf{x}^*](\mathbf{x} - \bar{\mathbf{x}}) = \mathbf{x}^*(\mathbf{x} - \bar{\mathbf{x}}) \leq \underline{\mathbf{x}}^*(\mathbf{x} - \bar{\mathbf{x}}).$$

The last inequality obtains from (4.7) because $x_i^* - n_i = \underline{x}_i^*$ and $-n_i^*(x_i - \bar{x}_i) \leq 0$. The upshot so far is that

$$E\left\|\mathbf{x}^{+1} - \bar{\mathbf{x}}\right\|^2 \leq \|\mathbf{x} - \bar{\mathbf{x}}\|^2 - \frac{8}{n-1} s\underline{\mathbf{x}}^*(\bar{\mathbf{x}} - \mathbf{x}) + s^2\gamma.$$

Reintroduce the state $\mathbf{x}^k$ which prevails at stage k, and let $\bar{\mathbf{x}}^k \in \bar{\mathbf{X}}$ realize $dist(\mathbf{x}^k, \bar{\mathbf{X}})$. It obtains now for $k = 0, 1, ...$ that

$$E\left\|\mathbf{x}^{k+1} - \bar{\mathbf{x}}^{k+1}\right\|^2 \leq \left\|\mathbf{x}^k - \bar{\mathbf{x}}^k\right\|^2 - \frac{8}{n-1} s_k \underline{\mathbf{x}}^{*k}(\bar{\mathbf{x}}^k - \mathbf{x}^k) + s_k^2\gamma.$$

With identifications

$$A_k := dist^2(\mathbf{x}^k, \bar{\mathbf{X}}), \quad B_k := 0, \quad C_k := \frac{8s_k}{(n-1)}\underline{\mathbf{x}}^{*k}(\bar{\mathbf{x}}^k - \mathbf{x}^k), \quad D_k := s_k^2\gamma,$$

all these items are non-negative random variables. Further, the last inequality takes the form

$$EA_{k+1} \leq A_k(1 + B_k) - C_k + D_k. \tag{5.3}$$

Next invoke the Robbins-Siegmund *Lemma* [5], [37]: *Suppose* $k \mapsto A_k, B_k, C_k, D_k$ *be sequences of non-negative random variables, each adapted to a filtration* $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$ *of some probability space. Further suppose* $\sum B_k$, $\sum D_k < +\infty$ *almost surely, and that* (5.3) *holds with conditional expectation* $E[\cdot\,|\mathcal{F}_k]$. *Then,* A_k *converges almost surely, and* $\sum C_k < +\infty$ *likewise.*
In the actual context, if $A := \lim A_k > 0$ on some set of trajectories which has strictly positive probability, the contradiction $\sum C_k = +\infty$ holds on that set. Hence, $A = 0$ a.s. □

The above proof of convergence was coached in primal terms; dual items - alias prices - never came to the fore.[10] Finally, in any limit $(\bar{x}_i)$, the forces that drive bilateral exchanges disappear. That is, each two agents i, j see a common price in $\partial\rho_i(\bar{x}_i) \cap \partial\rho_j(\bar{x}_j)$. To round up, if some agent has smooth objective (whence interior position), eventually everybody sees common prices:

Proposition 5.2. (On emergence of common prices) *Ultimately, at any limit point* $(\bar{x}_i)$, *provided* $\partial\rho_i(\bar{x}_i)$ *reduces to a singleton* x^* *for at least one agent* i, *that singleton becomes a common (shadow) price* $x^* \in \cap_{i\in I}\partial\rho_i(\bar{x}_i)$.

Remark 5.1. (On second-hand trade and linear pricing) *Exchange, as modelled above, does not preclude bundling or unbundling of risks. Further, holdings or transfers need not be first hand. Consequently, to block arbitrage, pricing had finally better become linear and common [27], [28].*

Remark 5.2. (On risk-free papers and money) *Following finance theory, it's common to single out a particular claim* $\mathbf{1} \in \mathbb{X}$, *referred to as risk-free.*[11] *Many risk measures [19], [40] are then presumed translation invariant along that claim, meaning*

$$\rho_i(x_i + r\mathbf{1}) = \rho_i(x_i) + r \quad \forall i \in I \quad \forall r \in \mathbb{R}. \tag{5.4}$$

No such properties have been invoked here. Note though, that given any shadow price x^*, it obtains from (5.4) and $x_i \in \arg\min\{\rho_i - x^*\}$ (3.2), that $x^*\mathbf{1} = 1$. Thus, a risk-free paper "monetizes" transactions; it serves as unit of account.

ACKNOWLEDGMENTS: Thanks for good comments and support are due to the referees and Røwdes Fond respectively.

RECEIVED: JULY 2017.
REVISED: JANUARY, 2018.

[10] Theorem 4.1 extends the convergence result in [14] by allowing viability constraints (4.1).
[11] In most of the literature, outcomes are univariate, meaning $\mathcal{E} = \mathbb{R}$. Then, naturally, $\mathbf{1}(\omega) \equiv 1$.

REFERENCES

[1] K. J. ARROW(1864): The role of securities in the optimal allocation of risk bearing, **Colloques Internationaux du CNRS** XL 41-48 (1953); translated in **Review of Economic Studies**, 31 (2), 86, 91-6.

[2] P. ARTZNER, F. DELBAEN, J.-M. EBER AND D. HEATH (1999): Coherent measures of risk, **Mathematical Finance** 9, 203-28.

[3] R. J. AUMANN (1964): Markets with a continuum of traders, **Econometrica** 32,39-50.

[4] P. BARRIEU AND N. EL KAROUIR (2005): Inf-convolution of risk measures and optimal risk transfe **Finance and Stochastics** 9, 2, 269-98.

[5] A. BENVENISTE, M. MÉTIVIER AND P. PRIOURET (1990): **Adaptive Algorithms and Stochastic Approximations**, Springer, Berlin.

[6] K. BORCH (1962): Equilibrium in a reinsurance market, **Econometrica** 30, 3, 424-44.

[7] H. BÜHLMANN AND S. JEWELL (1979): Optimal risk exchanges, **Astin Bull.** 10, 243-62.

[8] D. CASS, G. CHILCHILNISKY AND H.-M. WU (1996): Individual risk and mutual insurance, **Econometrica** 64, 2, 333-41.

[9] R.-A. DANA AND M. SCARSINI (2007): Optimal risk sharing with background risk, **J. Econ. Theory** 133, 152-76.

[10] F. DELBAEN (2002): Coherent risk measures on general probability spaces, in K. Sandmann, P. Schönbucher (eds.) **Advances in Finance and Stochastics, Essays in Honour of Dieter Sondermann**, Springer, Berlin 1-38.

[11] L. EECKHOUDT, C. GOLLIER AND H. SCHLESINGER (2005): **Economic and Financial Decisions under Risk**, Princeton University Pres.

[12] I. V. EVSTIGNEEV AND S. D. FLÅM (2001): Sharing nonconvex cost, **J. Global Optimization** 20, 3-4, 257-71.

[13] I. FINKELSHTAIN, O. KELLA AND M. SCARSINI (1999): On risk aversion with two risks, **J. Mathematical Economics** 31, 239-50.

[14] S. D. FLÅM (2011): Exchanges and measures of risks, **Mathematics and Financial Economics** 5, 249–67.

[15] S. D. FLÅM (2015): Risk measures, convexity, and max-min shortfalls, **Journal of Convex Analysis** 23, 3.

[16] S. D. FLÅM (2015): Bilateral exchange and competitive equilibrium, **Set-Valued and Variational Analysis**

[17] S. D. FLÅM (2016): Borch's theorem, equal margins, and efficient allocation, **Insurance: Mathematics and Economics** 70, 162-8.

[18] S. D. FLÅM (2018): Blocks of coordinates, stochastic programming, and markets (typescript).

[19] H. FÖLLMER AND A. SCHIED (2004): **Stochastic Finance**, (sec. ed.) Walter de Gruyter, Berlin.

[20] C. GOLLIER AND J. W. PRATT (1999): Risk vulnerability and the tempering effect of background risk, **Econometrica** 64, 5, 1109-23.

[21] D. HEATH AND H. KU (2004): Pareto equilibria with coherent measures of risk, **Mathematical Finance** 14, 2, 163-72.

[22] D. HENRIET AND J-C ROCHET (1991): **Microecomie de l'assurance**, Economica, Paris.

[23] E. JOUINI, W. SCHACHERMAYER AND N. TOUZI (2008): Optimal risk sharing for law invariant utility functions, **Mathematical Finance** 18, 2, 269-92.

[24] S. KOU, X. PENG AND C. C. HEYDE (2013): External risk measures and Basel Accords, **Mathematics of Operations Research** 38, 3, 393-417.

[25] G. PFLUG (2006): Subdifferential representations of risk measures, **Mathematical Programming**, Ser. B 108, 339-54.

[26] J. W. PRATT (1962): Risk aversion in the small and in the large, **Econometrica** 32, 122-36.

[27] Y. LENGWILER (2004): **Microfoundations of Financial Economics**, Princeton University Press.

[28] S. F. LEROY AND J. WERNER (2001): **Principles of Financial Economics**, Cambridge University Press.

[29] D. G. LUENBERGER (1995): **Microeconomic Theory**, McGraw-Hill, New York.

[30] E. MALINVAUD (1972): The allocation of individual risks in large markets, **Journal of Economic Theory** 4, 312-28.

[31] E. MALINVAUD (1973): Markets for an exchange economy with individual risks, **Econometrica** 41, 383-410.

[32] A. MAS-COLELL, M. D. WHINSTON AND J. R. GREEN (1995):, **Microeconomic Theory**, Oxford University Press.

[33] I. NECOARA (2013): Random coordinate descent algorithms for multi-agent convex optimization over networks, **IEEE Transactions on Automatic Control** 58, 8, 2001-13.

[34] I. NECOARA AND A. PATRASCU (2014): A random coordinate descent algorithm for optimization problems with composite objective function and linear coupled constraints, **Computational Optimization and Applications** 57, 2, 307-37.

[35] I. NECOARA, YU. NESTEROV, F. GLINEUR (2017): Random block coordinate descent methods for linearly constrained optimization over networks, **J. Optimization Theory and Applic**. 173, 227-54.

[36] YU. NESTEROV (2012): Efficiency of coordinate descent methods on huge-scale optimization problems, **SIAM J. Optimization** 22, 2, 341-362.

[37] H. ROBBINS AND D. SIEGMUND (1971): A convergence theorem for non-negative almost supermartingales and some applications, in J. Rustagi (ed.) **Optimization Methods in Statistics**, Academic Press, New York 235-257.

[38] A. PICHLER (2017): A quantitative comparison of risk measures, **Ann. Oper. Res.** 254-275 DOI 10.1007/s10479-017-2397-3.

[39] R. T. ROCKAFELLAR AND J.-B. WETS (1998): **Variational Analysis**, Springer Verlag, Berlin.

[40] L RÜSCHENDORF (2013): **Mathematical Risk Analysis**, Springer-Verlag, Berlin.

[41] R. WILSON (1968): The theory of syndicates, **Econometrica** 36, 1, 119-32.