THE LAGRANGE MULTIPLIERS FOR CONVEX VECTOR FUNCTIONS IN BANACH SPACES

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ABSTRACT
This paper is devoted to vector-valued optimization problems in Banach spaces whose objective functions are cone-convex and the feasible sets are not assumed to be convex. By means of a well-known nonlinear scalarizing function and the oriented distance function, we derive optimality conditions for weak Pareto solutions and (ϵ, e)-Pareto solutions in terms of abstract subdifferentials and the Clarke subdifferential.

KEYWORDS: Lagrange multiplier, cone-convex function, Lipschitz function, (weak) Pareto minimal point, (ϵ, e)-Pareto minimal point, nonlinear scalarizing functional, oriented distance function, subdifferential.

MSC: 46A40, 49J53, 52A41, 90C30, 90C46.

RESUMEN
Sea el problema de optimización vectorial en espacios de Banach cuya función objetiva es convexa con respecto a un cono y el conjunto de soluciones factibles es no convexo. Usando funciones de escalarización conocidas y la función de distancia orientada, se derivan condiciones de optimalidad para soluciones débiles de Pareto y (ϵ, e)-soluciones de Pareto en términos de sub-diferenciales abstractos y del sub-diferencial de Clarke

PALABRAS CLAVE: Multiplicadoes de Lagrange, funciones convexas con respecto a un cono, funcion Lipschitziana, puntos mínimo (débiles) de Pareto

1. INTRODUCTION

In this paper, we investigate necessary optimality conditions for solutions of the optimization problem (VP) given by:

$$\text{minimize} \quad f(x) \quad \text{subject to} \quad x \in D, \tag{VP}$$

where f is a mapping between Banach spaces, $f : X \to Y$, D is a subset of X, D is not supposed to be convex and "minimization" is understood with respect to (w.r.t., for short) a partial order defined based on a proper cone C in Y. The problem (VP) is considered as a problem of ***vector-valued optimization***.

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In the specific case, D is a nonempty closed convex set in the normed vector space X and $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ is a proper convex function, it is well-known that the necessary and sufficient optimality condition for a minimum point $\overline{x}$ with $f(\overline{x}) \in \mathbb{R}$ of the problem (VP) is given by:

$$0 \in \partial_F f(\bar{x}) + N_F(\bar{x}; D),$$

where ∂_F and N_F are the Fenchel subdifferential and the normal cone in the sense of convex analysis, respectively. When D is not necessarily convex but $f : X \to \overline{\mathbb{R}}$ is Lipschitz and f attains a minimum over D at x with $f(x) \in \mathbb{R}$, then

$$0 \in \partial_C f(x) + N_C(x; D),$$

where ∂_C is the ***Clarke subdifferential*** and $N_C(x; D)$ is the ***Clarke normal cone*** to D at x; see [3].

It is of interest to extend the aforementioned results to the case of vector-valued functions. A natural way is using some scalarizing methods for the original problems in order to apply many results in the cases of real functionals. When f is a C-convex function and D is a closed convex set, Jahn [15] used a linear scalarizing function to characterize (weak) Pareto solutions of problem (VP); compare [15, Theorem 5.4 and Theorem 5.13]. In the case that C is non-solid ($\operatorname{int} C = \emptyset$), Durea et al. [4] investigated the problem (VP), where the objective function is continuous Frechet differentiable C-convex and D is convex. The case that $\operatorname{int} C = \emptyset$ and D is not supposed to be convex is also very interesting. However, to the best of our knowledge, there are not many papers concerning this problem. Bao et al. [1] constructed a new appropriate nonempty interior cone such that the Pareto minimal points w.r.t. the original cone C are also the Pareto minimal points w.r.t. the new cone. Durea et al. [5] introduced and investigated optimality conditions for the so-called (ϵ, e)-Pareto minimum. It is worth to mention that both [1] and [5] used the scalarizing functional introduced by Tammer et al. [7, 8] and the objective function f was supposed to be Lipschitz.

In this paper, we derive necessary optimality conditions for solutions of (VP) equipped with both solid and non-solid ordering structures under the assumptions that f is a cone-convex function and D is not supposed to be closed or convex. To study weak Pareto solutions in case that $\operatorname{int} C \neq \emptyset$, we scalazize our problem by the nonlinear scalarizing functional. In the case C is non-solid, we investigate (ϵ, e)-Pareto solutions by a different approach, that is using the oriented distance function introduced by Hiriart-Urruty [10, 11]; see for instance [9, 20]. By means of these functions, we derive optimality conditions for the solutions in terms of abstract subdifferentials and the Clarke subdifferential.

The paper is organized as follows: In Section 2, we present some preliminaries which will be used in the next parts. We recall the definitions and some important properties for two nonlinear scalarizing functions used to scalarize our problem. Section 3 introduces the abstract subdifferential satisfying some certain axioms which still hold for other special subdifferentials of Ioffe or Mordukhovich in infinite dimensional spaces. We recall the definition as well as some inportant properties of the Clarke subdifferential. This section also contains our main results: Necessary optimality conditions in terms of the abstract subdifferential and the Clarke subdifferential for weak Pareto optimal solutions and (ϵ, e)-Pareto solutions of the problem (VP).

2. VARIATIONAL ANALYSIS

Throughout the paper, unless otherwise specified, X and Y are two Banach spaces with their topological duals X^* and Y^*, respectively. We use the notations $\|.\|_X$, U_X for the norm and the closed unit ball in space X, respectively, we omit the subscript X if there is no risk of confusion. For a nonempty set A, $\operatorname{int} A$, $\operatorname{cl} A$ and $\operatorname{bd} A$ stand for the interior, closure and boundary of A, respectively. We denote that $d_A(.)$ is the distance function associated with A and $\delta_A(.)$ is the indicator function associated with A, i.e. $\delta_A(x) = 0$ if $x \in A$ and $\delta_A(x) = +\infty$ otherwise. Let C be a proper, convex, closed and pointed cone which specifies a partial order $\geq_C$ on Y as follows:

$$\text{for all } x, y \in Y, \qquad x \geq_C y \qquad \Longleftrightarrow \quad x - y \in C.$$

The continuous dual cone of C is given by

$$C^+ := \{y^* \in Y^* \mid \forall c \in C : \quad y^*(c) \geq 0\}.$$

We begin this section by recalling the notions of the (weak) Pareto minimality of a subset $A \subseteq Y$ and corresponding solution concepts for the vector optimization problem (VP).

Definition 2.1. *Let A be a nonempty subset of Y and $C \subset Y$ be a proper, convex and pointed cone.*

(i) We define the set of ***Pareto minimal points*** *of A w.r.t. C by*

$$\operatorname{Min}(A; C) := \big\{\bar{y} \in A |\ A \cap (\bar{y} - C) = \{\bar{y}\}\big\}.$$

If $f : X \to Y$ is a vector-valued function and $D \subset X$ is nonempty set, a point $\bar{x} \in D$ is said to be a Pareto minimizer of problem (VP) *if $f(\bar{x}) \in \operatorname{Min}(F(D); C)$.*

(ii) If $\operatorname{int} C \neq \emptyset$, then the set of ***weak Pareto minimal points*** *of A w.r.t. C is given by*

$$\operatorname{WMin}(A; C) := \big\{\bar{y} \in A |\ A \cap (\bar{y} - \operatorname{int} C) = \emptyset\big\}.$$

Similarly, a point $\bar{x} \in D$ is said to be a weak Pareto minimizer of problem (VP) *if $f(\bar{x}) \in \operatorname{WMin}(F(D); C)$.*

Obviously, if $\bar{x}$ is a Pareto solution of problem (VP) then it is also a weak Pareto solution. In this paper, we will investigate necessary optimality conditions for solutions of problem (VP), where the interior of C is empty or nonempty. For the case $\operatorname{int} C \neq \emptyset$, we will study optimality conditions for weak Pareto solutions by means of the scalarizing function introduced by Tammer and Weidner in [7, 8]. For the case $\operatorname{int} C = \emptyset$, we will derive optimality conditions for (ϵ, e)-Pareto minimal solutions, given by Durea, Dutta and Tammer [4].
The definition of (ϵ, e)-Pareto minimum and relationship between this concept and Pareto minimum are given as follows.

Definition 2.2. *Let A be a subset of Y and $C \subset Y$ be a proper, convex and pointed cone. Let $e \in C$ with $\|e\| = 1$ and $\epsilon > 0$. We say that*

- *$\bar{a} \in A$ is an (ϵ, e)-Pareto minimal point of A w.r.t. C if $(A - \bar{a}) \cap (-C - \epsilon e) = \emptyset$. The set of all these minima is denoted by $(\epsilon, e) - \operatorname{Min}(A, C)$.*

- *$\bar{x} \in D$ is an (ϵ, e)-Pareto minimal solution of problem* (VP) *if* $f(\bar{x}) \in (\epsilon, e) - \mathrm{Min}(f(D), C)$.

Note that [4] studied vector optimization problems, whose objective functions are locally Lipschitz. The reason for this choice is that they are very close to the efficient frontier and qualify in a better way as an approximate-minimum, cf. ([4, Page 199]). Furthermore, the concept of (ϵ, e)-Pareto minimizers is beneficial for us to get the nontrivial multipliers $y^* \neq 0$ using certain properties of the subdifferential of the distance function.

Proposition 2.1. *[4, Proposition 2.1] Let A be a subset of Y and $C \subset Y$ be a proper, convex and pointed cone. Then the following relation holds*

$$\mathrm{Min}(A, C) = \bigcap_{e \in C \cap S_Y} \bigcap_{\epsilon > 0} (\epsilon, e) - \mathrm{Min}(A, C).$$

The following definition presents some convexities of a vector-valued function where the ordering cone C is involved.

Definition 2.3. *Let $f : X \to Y$, $C \subset Y$ be a proper, convex and pointed cone and $t \in \mathbb{R}_+$, $e \in C$ be given. The function f is said to be*

(i) C-***convex*** *if for all $x, x' \in X, \lambda \in (0, 1)$, one has*

$$\lambda f(x) + (1 - \lambda) f(x') \in f(\lambda x + (1 - \lambda) x') + C.$$

(ii) ***strictly*** C-***convex*** *if* $\mathrm{int}\, C \neq \emptyset$ *and for all $x, x' \in X, \lambda \in (0, 1)$, one has*

$$\lambda f(x) + (1 - \lambda) f(x') \in f(\lambda x + (1 - \lambda) x') + \mathrm{int}\, C.$$

(iii) (C, e, t)-***strongly convex*** *if for all $x, x' \in X, \lambda \in (0, 1)$, one has*

$$\lambda f(x) + (1 - \lambda) f(x') \in f(\lambda x + (1 - \lambda) x') + (C + te).$$

Remark 2.1. *Obviously, we have that strictly C-convexity implies C-convexity. It is not necessary to suppose that C has nonempty interior in parts (i) and (iii) of the Defintion 2.3.. In additiion, (C, e, t)-strong convexity is considered as an extension of C-convexity since these two definitions are coincident if we take $e = 0 \in C$ and (C, e, t)-strongly convex is an extension of C-strictly convex if $e \in intC$. If the element e belongs to the boundary of C, then the (C, e, t)-strong convexity does not follow from the strict C-convexity.*

In this paper, we will derive some necessary optimality conditions for solutions of the optimization problem (VP) using the Fenchel subdifferential. Let X be a Banach space. Recall that the Fenchel subdifferential of a convex function $f : X \to \overline{\mathbb{R}}$ at $\overline{x}$ with $f(\overline{x}) \in \mathbb{R}$ is defined by

$$\partial_F f(\bar{x}) := \{x^* \in X^* \mid \forall x \in X : f(x) - f(\bar{x}) \geq x^*(x - \bar{x})\}. \tag{2.1}$$

For $\overline{x}$ with $f(\overline{x}) \notin \mathbb{R}$, one puts $\partial_F f(\bar{x}) = \emptyset$. If $\partial_F f(\bar{x})$ is nonempty, f is said to be subdifferentiable at $\bar{x}$.

Definition 2.4. *Let Y be a Banach space, $g : Y \to \mathbb{R} \cup \{+\infty\}$ and $C \subset Y$ be a proper, convex and pointed cone. We say that g is a C-nondecreasing function at $y_0 \in Y$ if*

$$y \in Y \bigcap \big(y_0 - C\big) \Longrightarrow g(y) \leq g(y_0).$$

In the following, we recall a well known result presenting the formula of the Fenchel subdifferential of a composition. The proof is omitted since it was presented in [17].

Lemma 2.1. ***([17, Lemma 2.2])*** *Let X, Y be Banach spaces, $C \subset Y$ be a proper, convex and pointed cone. Suppose that $f : X \to Y$ is a C-convex vector-valued function and $g : Y \to \mathbb{R} \cup \{+\infty\}$ is a convex, C-nondecreasing function on Y. If there exists $(\bar{x}; \bar{y}) \in \operatorname{epi} f$ such that g is continuous at $\bar{y}$, then for $\bar{y} = f(\bar{x}) \in \operatorname{dom} g := \{y \in Y \mid g(y) < +\infty\}$ one has*

$$\partial_F(g \circ f)(\bar{x}) = \bigcup_{y^* \in \partial_F g(\bar{y})} \partial_F(y^* \circ f)(\bar{x}).$$

The following lemma shows a sufficient condition for the Lipschitz property of a scalar convex function.

Lemma 2.2. *([21, Corollary 2.2.12])*
Let X be a Banach space, $f : X \to \overline{\mathbb{R}}$ be a proper convex function on X and consider $x_0 \in X$ with $f(x_0) \in \mathbb{R}$ and $D(x_0, \theta) := \{x \in X| \; ||x - x_0||_X \leq \theta\}$ with $\theta \geq 0$. Suppose that for some $\theta > 0$, $m \geq 0$,

$$\forall x \in D(x_0, \theta) : f(x) \leq f(x_0) + m.$$

Then f is Lipschitz around x_0 with a Lipschitz constant given by

$$\forall \theta' \in (0, \theta), \forall x, x' \in D(x_0, \theta') : |f(x) - f(x')| \leq \frac{m}{\theta} \cdot \frac{\theta + \theta'}{\theta - \theta'} \cdot \|x - x'\|_X.$$

Definition 2.5. *Consider $f : X \to Y$ and $C \subset Y$ is a proper, convex and pointed cone. We say that*

*(i) f is **C-bounded from above** on a subset $A \subseteq X$ if there exists a constant $\mu > 0$ such that*

$$f(A) \subset \mu U_Y - C.$$

*(ii) f is **C-bounded from above** around a point $x \in X$ if it is C-bounded from above on a neighborhood of x.*

In the following, we recall "the nonlinear scalarizing functional" or "Gerstewitz scalarizing functional", which was widely used in vector optimization and set optimization. It was intensively studied by Tammer and Weidner in [7, 8]. This function will be used in this work to derive necessary optimality conditions for weak Pareto solutions of the problem (VP). Let A be a given proper and closed subset of Y and $e \in Y \setminus \{0\}$ such that

$$A + [0, +\infty) \cdot e \subseteq A. \tag{2.2}$$

We consider the scalarizing functional $\varphi_{A,e} : Y \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\varphi_{A,e}(y) := \inf\{\lambda \in \mathbb{R} \mid \lambda \cdot e \in y + A\}, \tag{2.3}$$

where we use the conventions $\inf \emptyset := +\infty$, $\sup \emptyset := -\infty$ and $(+\infty) + (-\infty) = +\infty$. To simplify the notation, we use the symbol $\varphi := \varphi_{A,e}$ if no confusion arises.
In the following, we present some important properties of $\varphi_{A,e}$ that will be used in the sequel. For the proof, we refer the reader to [5, 8, 18].

Proposition 2.2. *([8, Theorem 2.3.1]) Let Y be a topological vector space and $A \subset Y$ be a proper, closed set. Let e be given in $Y \setminus \{0\}$ such that* (2.2) *holds, then the following properties hold for $\varphi := \varphi_{A,e}$:*

(a) *φ is lower semi-continuous and* $\operatorname{dom}\varphi = \mathbb{R}e - A$.

(b) $\forall y \in Y,\ \forall t \in \mathbb{R} : \varphi(y) \le t$ *if and only if* $y \in te - A$.

(c) $\forall y \in Y,\ \forall t \in \mathbb{R} : \varphi(y + te) = \varphi(y) + t$.

(d) *φ is convex if and only if A is convex.*

(e) *$\varphi(\lambda y) = \lambda\varphi(y)$ for all $\lambda > 0$ and $y \in Y$ if and only if A is a cone.*

(f) *φ is proper if and only if A does not contain lines parallel to e, i.e., $\forall y \in Y, \exists t \in \mathbb{R} : y + te \notin A$.*

(g) *φ takes finite values if and only if A does not contain lines parallel to e and $\mathbb{R}e - A = Y$.*

(h) *If we suppose that*

$$A + [0, +\infty) \cdot e \subseteq \operatorname{int} A, \tag{2.4}$$

then φ is continuous.

Definition 2.6. *Let Y be a topological vector space and D be a nonempty subset of Y. A functional $\varphi : Y \to \bar{\mathbb{R}}$ is called D-monotone, if*

$$\forall y_1, y_2 \in Y : y_1 \in y_2 - D \Rightarrow \varphi(y_1) \le \varphi(y_2).$$

Moreover, φ is said to be strictly D-monotone, if

$$\forall y_1, y_2 \in Y : y_1 \in y_2 - D \setminus \{0\} \Rightarrow \varphi(y_1) < \varphi(y_2).$$

The following result provides some further properties of the scalarizing functional $\varphi_{A,e}$.

Proposition 2.3. *([8, Theorem 2.3.1]) Under the assumptions of Proposition 2.2., and we take $\emptyset \neq D \subseteq Y$. Then the following properties hold:*

(a) *$\varphi_{A,e}$ is D-monotone if and only if $A + D \subseteq A$.*

(b) *$\varphi_{A,e}$ is subadditive if and only if $A + A \subseteq A$.*

Note that if A is a closed, convex proper set and does not contain lines parallel to e then $\varphi_{A,e}$ is a proper convex function. Therefore, we can provide in the following some calculus for its sub**d**ifferential in the sense of convex analysis.

Proposition 2.4. *([5, Theorem 2.2]) Let Y be a topological vector space and $A \subset Y$ be a closed, convex, proper set. Let e be given in $Y \setminus \{0\}$ such that* (2.2) *holds and for every $y \in Y$ there exists $t \in \mathbb{R}$ such that $y + te \notin A$. Consider the scalarizing function $\varphi := \varphi_{A,e}$ determined by* (2.3) *and $\bar{y} \in \operatorname{dom}\varphi$. Then*

$$\partial_F \varphi(\bar{y}) = \{y^* \in Y^* \mid y^*(e) = 1,\ y^*(d) + y^*(\bar{y}) - \varphi(\bar{y}) \ge 0 \quad \forall d \in A\}. \tag{2.5}$$

Remark 2.2. *Observe that if C is a proper, convex, closed and pointed cone, $A := C$ then $\varphi_{C,e}(\bar{y}) = 0$. Taking into account Proposition 2.4., we obtain that*

$$\partial_F \varphi_{C,e}(0) = \{y^* \in C^+ \mid y^*(e) = 1\}. \tag{2.6}$$

The following proposition presents the Lipschitzianity of the function $\varphi_{C,e}$ provided that C is a proper, convex, closed cone with $\operatorname{int} C \neq \emptyset$. We also illustrate how a (weak) Pareto minimizer of a set A w.r.t. the cone C can be characterized by the scalarizing function $\varphi_{C,e}$. This result will be used in the proofs of necessary optimality conditions for solutions of the problem (VP) in next sections.

Proposition 2.5. *([5, Lemma 2.4]) Let Y be a Banach space, $C \subset Y$ be a proper, convex, closed cone with nonempty interior. Let e be a given point in $\operatorname{int} C$. Then the function $\varphi_{C,e} : Y \to \mathbb{R}$ defined by* (2.3) *is continuous, sublinear and strictly-*$\operatorname{int} C$*-monotone. Moreover, $\varphi_{C,e}$ is $d(e, \operatorname{bd}(C))^{-1}$-Lipschitz and for every $y \in Y$ and $y^* \in \partial\varphi_{C,e}(y)$ one has $||e||^{-1} \leq ||y^*|| \leq d(e, \operatorname{bd}(C))^{-1}$.*
If $A \subset Y$ is a nonempty set such that $\bar{y} \in \operatorname{WMin}(A; C)$, then one has

$$\varphi_{C,e}(y - \bar{y}) \geq 0 \quad \text{for all} \quad y \in A. \tag{2.7}$$

We introduce in the following the second scalarizing function, namely **"oriented distance function"**, which was introduced in [10, 11] to analyse the geometry of nonsmooth optimization problems.

Definition 2.7. *The oriented distance function $\Delta_A : Y \to \overline{\mathbb{R}}$ defined for a set $A \subset Y, A \neq Y$, by*

$$\Delta_A(y) := d_A(y) - d_{Y \setminus A}(y), \tag{2.8}$$

with convention that $d_\emptyset(y) = +\infty$.

Some important properties of the oriented distance function are presented as follows. For the proof, we refer the reader to [20, Proposition 3.2] and [20, Theorem 4.3].

Proposition 2.6. *Consider the oriented distance function Δ_A given by* (2.8)*, where A is a subset of Y. We have that*

(i) Δ_A *is Lipschitzian of rank 1.*

(ii) $\Delta_A(y) < 0$ *for every* $y \in \operatorname{int} A$, $\Delta_A(y) = 0$ *for every y in the boundary of A and* $\Delta_A(y) > 0$ *for every* $y \in \operatorname{int}(Y \setminus A)$.

(iii) If A is convex, then Δ_A is convex and if A is cone, then Δ_A is positively homogeneous.

(iv) If A is closed and convex cone and $y_1, y_2 \in Y$ with $y_1 - y_2 \in A$, then $\Delta_A(y_1) \leq \Delta_A(y_2)$.

(v) If $\operatorname{int} A = \emptyset$, *then* $\operatorname{cl}(Y \setminus A) = Y$, *it follows that $d_{Y \setminus A}(y) = 0$ for every $y \in Y$, hence $\Delta_A = d_A$.*

(vi) Let A be a proper, convex and closed cone in Y. A point $\bar{y} \in M \subset Y$ is a Pareto minimal point of M w.r.t. A if and only if $\bar{y}$ is a solution of the problem $\min_{y \in M} \Delta_{-A}(y - \bar{y})$, *i.e.,* $\Delta_{-A}(y - \bar{y}) > 0$, *for all* $y \in M, y \neq \bar{y}$.

By the above proposition, the functional Δ_A is convex, positively homogeneous and 1-Lipschitz for every closed and convex cone A. Note that both Δ_A and d_A are convex functions provided that A is convex, so we can take their subdifferentials in the sense of Fenchel. For the convenience of the reader, we recall the calculus of subdifferential of the distance function d_A in the following proposition. Since the proof is presented in [2, Theorem 1], we omit it in this paper.

Proposition 2.7. *([2, Theorem 1]) Let A be a nonempty, closed and convex subset of Y. Then d_A is a convex function on Y with the Fenchel subdifferential*

$$\partial_F d_A(y) = \begin{cases} S_{Y^*} \cap N_F(y; A_y), & \text{if } y \notin A \\ U_{Y^*} \cap N_F(y; A), & \text{if } y \in A, \end{cases}$$

where U_{Y^*}, S_{Y^*} are the closed unit ball and the unit sphere in Y^*, $A_y := A + d_A(y)U_Y$ and $N_F(\bar{a}; A)$ is the normal cone at a point $\bar{a} \in A$ and given as

$$N_F(\bar{a}; A) = \left\{ y^* \in Y^* \middle| y^*(a - \bar{a}) \leq 0, \forall a \in A \right\}.$$

Remark 2.3. *Suppose that C is a proper, convex, closed and pointed cone in Y with $\operatorname{int} C = \emptyset$. It follows from Proposition 2.6.(v) and Proposition 2.7. that*

$$\partial \Delta_{-C}(0) = \partial d_{-C}(0) = U_{Y^*} \cap N_F(0; -C) = U_{Y^*} \cap C^+.$$

In addition, for every $\epsilon > 0$ and $e \in C$, the interior of $-C - \epsilon e$ is empty. Durea et al. [4, Remark 2.2] proved that the following relation holds true for every $y \notin (-C - \epsilon e)$

$$\partial \Delta_{-C-\epsilon e}(y) = \partial d_{-C-\epsilon e}(y) \subseteq S_{Y^*} \cap C^+. \tag{2.9}$$

3. THE LAGRANGE MULTIPLIER RULES

3.1. Abstract subdifferential

We begin this section by presenting in the following the definition of abstract subdifferentials ∂, which will be used in the sequel. Let $\mathcal{X}$ be a class of Banach spaces, $X \in \mathcal{X}$. For every lower-semicontinuous function $f : X \to \overline{\mathbb{R}}$ and $x \in X$, we denote by $\partial f(x)$ the abstract subdifferential of f at x with $f(x) \in \mathbb{R}$. The abstract subdifferential is a (possible empty) subset in X^* and satisfies the following axioms:

(A1) If f is convex, then $\partial f(x)$ coincides with the Fenchel subdifferential.

(A2) If $\bar{x}$ is a local minimum point for f, then $0 \in \partial f(\bar{x})$.

(A3) If f is Lipschitz around $\bar{x}$ and g is proper lower-semicontinuous around this point. Then one has the inclusion

$$\partial(f + g)(\bar{x}) \subseteq \partial f(\bar{x}) + \partial g(\bar{x}).$$

We can mention here some subdifferentials with the above properties:

- the approximate (or Ioffe) subdifferential when $\mathcal{X}$ is a class of Banach spaces [12, 13, 14].
- the limiting (or Mordukhovich) subdifferential when $\mathcal{X}$ is a class of Asplund spaces [16].

We denote by $N_\partial(x; A)$ the normal cone to a set A at a point $x \in A$ w.r.t. the abstract subdifferential ∂ and this normal cone is given by $N_\partial(x; D) := \partial\delta_A(x)$. Of course, if D is convex, then $N_\partial(x; D) = N_F(x; D)$.

Remark 3.4.

- *In [5], Durea et al. derived optimality conditions for solutions of the problem* (VP) *where the objective function f is Lipschitz. They also defined the abstract subdifferentials with chain rule axiom. Observe that, the class of our abstract subdifferentials is larger than one given in [5] since the conditions related to chain rules in [5] are omitted in our work.*
- *We can see that our abstract subdifferential can be considered as a special case of the so-called generic subdifferential given in [6]. The reason is that the axioms (A2) and (A3) imply the second axiom of the generic subdifferential, see [6, page 638]. In [6], Theorems 2.1 and 2.2 presented optimality conditions for minimal elements of geometric vector optimization problems* $\mathrm{Min}(A; C)$ *using the generic subdifferential. Therefore, it is possible to use this generic subdifferential for dealing with the problem* (VP), *since* $f(x_0) \in \mathrm{Min}(f(D), C)$ *provided that x_0 is a Pareto minimizer of* (VP). *However, these optimality conditions only concerned the image set $f(D)$ but not the properties of the objective function f. Therefore, the chain rule axiom of generic subdifferential is not necessary to be assumed in [6].*

In the following, we present our main results: the Lagrange multipliers for the vector-valued optimization problem (VP) in Banach spaces, in which f is cone-convex and the feasible set D is not supposed to be convex.

Theorem 3.1. *Let $X, Y \in \mathcal{X}$, D be a closed subset of X, C be a proper, convex, closed and pointed cone in Y with nonempty interior. Consider the problem* (VP) *where the objective function $f : X \to Y$ is C-convex. Suppose that ∂ satisfies the axioms (A1), (A2), (A3) and ∂_F is the Fenchel subdifferential given by* (2.1). *If $f(\bar{x}) \in \mathrm{WMin}(f(D); C)$ and f is C-bounded from above around $\bar{x} \in D$ then for every $e \in \operatorname{int} C$ there exists $y^* \in C^+$ with $y^*(e) = 1$ such that*

$$0 \in \partial_F(y^* \circ f)(\bar{x}) + N_\partial(\bar{x}; D). \tag{3.10}$$

Proof: Let $e \in \operatorname{int} C$. We consider the corresponding function $\varphi := \varphi_{C,e}$ given by (2.3).
By using the assumption $f(\bar{x}) \in \mathrm{WMin}(f(D); C)$ and applying Proposition 2.5., we get that $\bar{x}$ is a minimum of the problem

$$\min\ z_e(x) + \delta_D(x), \tag{3.11}$$

where $z_e : X \to \mathbb{R}$ and $z_e(x) = \varphi(f(x) - f(\bar{x}))$.
Since f is C-bounded from above around $\bar{x}$, there is a neighborhood $U_{\bar{x}}$ of $\bar{x}$ and $\mu > 0$ such that

$$f(U_{\bar{x}}) \subset \mu U_Y - C.$$

It follows that

$$f(U_{\bar{x}}) - f(\bar{x}) \subset \mu U_Y - f(\bar{x}) - C.$$

As $e \in \operatorname{int} C$, we have that $Y = \mathbb{R}e - C$. Hence, there exists a real number t such that $\mu U_Y - f(\bar{x}) \subseteq te - C$. Taking into account Proposition 2.3. (a), φ is C-monotone. Therefore, it holds that

$$\varphi(f(U_{\bar{x}}) - f(\bar{x})) \leq \varphi(\mu U_Y - f(\bar{x})) \leq t.$$

Hence, $z_e(x) = \varphi(f(x) - f(\bar{x}))$ is bounded from above around $\bar{x}$.
Since f is C-convex, so is $g(x) := f(x) - f(\bar{x})$. Taking into account Propositions 2.2. and 2.3., φ is convex and C-nondecreasing on Y and thus $z_e = \varphi \circ g$ is convex. Because z_e satisfies all the assumptions given in Lemma 2.2., we have that z_e is Lipschitz around $\bar{x}$. Moreover, since D is a closed subset in X, δ_D is a proper lower-semicontinuous function. By the axioms (A2) and (A3), we have that

$$0 \in \partial z_e(\bar{x}) + N_\partial(\bar{x}; D). \tag{3.12}$$

As z_e is convex, its abstract subdifferential coincides with the Fenchel subdifferential, i.e., $\partial z_e(\bar{x}) = \partial_F z_e(\bar{x})$. Applying Lemma 2.1., we obtain that

$$\partial_F z_e(\bar{x}) = \bigcup_{y^* \in \partial_F \varphi(0)} \partial_F(y^* \circ g)(\bar{x}) = \bigcup_{y^* \in \partial_F \varphi(0)} \partial_F(y^* \circ f)(\bar{x}).$$

Substituting the above formula into (3.12) we can conclude that (3.10) holds for some $y^* \in \partial_F \varphi(0)$. From Remark 2.2. we get that $y^* \in C^+$ and $y^*(e) = 1$, which completes the theorem. □

Remark 3.5.

- *In order to obtain necessary optimality conditions for solutions of the problem* (VP) *for the case X is Asplund and ∂ is the Mordukhovich subdifferential, Tuan et al. [19, Theorem 5] assumed that C is a closed normal cone, f is C-convex and C-bounded whereas Bao et al. [1] needed the strict Lipschitzianity of f. Theorem 3.1. considered that f is C-convex and C-bounded, however the assumption that C is a closed normal cone is omitted since we only used the Lipschitzianity of the function $z_e(\cdot)$ instead of Lipschitzianity of the objective function f.*
- *Observe that if the abstract subdifferential in [5, Theorem 3.1] is the Mordukhovich subdifferential (the Ioffe subdifferential) one needs to suppose that Y is finite dimensional (f is strongly compactly Lipschitz, respectively), whereas, Theorem 3.1. holds true for general dimentional spaces provided that the objective function is C-convex.*

The following result presents optimality conditions in terms of the abstract subdifferential for (ϵ, e)-Pareto solutions of the problem (VP) in the sense of Definition 2.2. where $\operatorname{int} C = \emptyset$.

Theorem 3.2. *Let $X, Y \in \mathcal{X}$, D be a closed subset of X, C be a closed, convex and pointed cone in Y with empty interior. Assume that $\epsilon > 0$, $e \in C$, $\|e\| = 1$ and x_0 is an (ϵ, e)-Pareto minimal solution of the problem* (VP). *Suppose that ∂ satisfies the axioms (A1), (A2) and (A3) and ∂_F is the Fenchel subdifferential given by* (2.1). *If f is a (C, e, ϵ)- strongly convex function and f is C-bounded from above around $x_0 \in D$ then there exist $\bar{x} \in B(x_0, \sqrt{\epsilon}) \cap D$, $y^* \in C^+, \|y^*\| = 1$ such that*

$$0 \in \partial_F(y^* \circ f)(\bar{x}) + \sqrt{\epsilon} U_{X^*} + N_\partial(\bar{x}; D), \tag{3.13}$$

Proof: Since $e \in C$, $\|e\| = 1$, it is obvious that $d_{-C-\epsilon e}(0) \leq \epsilon$. Because $f(x_0)$ is an (ϵ, e)-Pareto point of f over D, for every $x \in D$, we have that

$$d_{-C-\epsilon e}(0) \leq \epsilon \leq d_{-C-\epsilon e}(f(x) - f(x_0)) + \epsilon. \tag{3.14}$$

Consider a function $z_{\epsilon,e} : X \to \mathbb{R}$ given by $z_{\epsilon,e}(x) = d_{-C-\epsilon e}(f(x) - f(x_0))$. From (3.14), we obtain

$$z_{\epsilon,e}(x_0) \leq \inf_{x \in X} z_{\epsilon,e}(x) + \epsilon.$$

As f is C-bounded from above around x_0, there is a neighborhood U_0 of x_0 and $\mu > 0$ such that

$$f(U_0) \subset \mu U_Y - C.$$

It follows that

$$f(U_0) - f(x_0) \subset \mu U_Y - f(x_0) - C \subseteq \big(\mu U_Y - f(x_0) + \epsilon e\big) - C - \epsilon e.$$

Hence,

$$d_{-C-\epsilon e}(f(U_0) - f(x_0)) \leq d_{-C-\epsilon e}(\mu U_Y - f(x_0) + \epsilon e\big),$$

Therefore, $z_{\epsilon,e}(x) = d_{-C-\epsilon e}(f(x) - f(x_0))$ is bounded around x_0.
From the assumption f is (C, e, ϵ)-strongly convex, so is $g(x) := f(x) - f(x_0)$. Taking into account $d_{-C-\epsilon e}$ is $(C+\epsilon e)$-nondecreasing on Y, $z_{\epsilon,e} = d_{-C-\epsilon e} \circ g$ is convex. By Theorem 2.2., $z_{\epsilon,e}$ is Lipschitz around x_0.
Applying the Ekeland variational principle (see [8, Proposition 3.10.1]) for $z_{\epsilon,e}$ on D, we get an element $\bar{x} \in B(x_0, \sqrt{\epsilon}) \cap D$ and $\bar{x}$ is a minimum of the problem

$$\min\, z_{\epsilon,e}(x) + \sqrt{\epsilon}\|x - x_0\| + \delta_D(x).$$

Since D is a closed subset in X, δ_D is a proper lower-semicontinuous function. Applying the calculus rules of the abstract subdifferential ∂, we obtain the following assertion

$$0 \in \partial z_{\epsilon,e}(\bar{x}) + \sqrt{\epsilon}\partial\| \cdot -x_0\|(\bar{x}) + N_\partial(\bar{x}; D). \tag{3.15}$$

As $z_{\epsilon,e}(\cdot)$ and $\| \cdot -x_0\|$ are convex, their abstract subdifferentials coincide with the Fenchel subdifferential, i.e., $\partial z_{\epsilon,e}(\bar{x}) = \partial_F z_{\epsilon,e}(\bar{x})$, $\partial\| \cdot -x_0\|(\bar{x}) = \partial_F\| \cdot -x_0\|(\bar{x})$. Then (3.15) becomes

$$\begin{aligned} & 0 \in \partial_F z_{\epsilon,e}(\bar{x}) + \sqrt{\epsilon}\partial_F\| \cdot -x_0\|(\bar{x}) + N_\partial(\bar{x}; D) \\ \iff & 0 \in \partial_F z_{\epsilon,e}(\bar{x}) + \sqrt{\epsilon}U_{X^*} + N_\partial(\bar{x}; D). \end{aligned} \tag{3.16}$$

By Lemma 2.1.,

$$\partial_F z_{\epsilon,e}(\bar{x}) = \bigcup_{y^* \in \partial_F d_{-C-\epsilon e}(0)} \partial_F(y^* \circ g)(\bar{x}) = \bigcup_{y^* \in \partial_F d_{-C-\epsilon e}(0)} \partial_F(y^* \circ f)(\bar{x}).$$

Substituting the above formular into (3.16) we can conclude that (3.13) holds for some $y^* \in \partial_F d_{-C-\epsilon e}(0)$. From (2.9), we get $y^* \in S_{Y^*} \cap C^+$ which completes the theorem. $\square$

Remark 3.6. *Observe that Theorem 3.2. considers the problem* (VP) *in Banach spaces while Durea et al. [4] dealt with vector-valued problems in Asplund spaces. In addition, instead of using the strictly Lipschitz property as in [4], we need the (C,e,t)-strong convexity of the objective funtion f.*
Note that if D is closed then the indicator function of D is lower-semicontinuous. Therefore, the closedness assumption of D is essential in all theorems of this section in order to apply the sum rule of the abstract subdifferential ∂. In the next section, we can omit the closedness of the feasible set D to get the necessary optimality conditions for solutions of the problem (VP) *in terms of the Clarke subdifferential.*

3.2. Clarke subdifferential

In this section, we show that it is possible to derive necessary optimality conditions for the problem (VP) in terms of the Clarke subdifferetial in Banach spaces; in particular, the feasible set is not supposed to be closed.
Let $\mathcal{X}$ be a class of Banach spaces, $X \in \mathcal{X}$. Recall that the Clarke generalized gradient or the Clarke subdifferential of a locally Lipschitz function $f : X \to \overline{\mathbb{R}}$ at $\bar{x}$ with $f(\overline{x}) \in \mathbb{R}$, is denoted by ∂_C and determined by

$$\partial_C f(\bar{x}) := \{x^* \in X^* | \ \forall v \in X : \ f^\circ(\bar{x}, v) \geq x^*(v)\},$$

where $f^\circ(\bar{x}, v) := \limsup\limits_{y \to \bar{x},\ t \downarrow 0^+} \frac{1}{t}\big(f(y+tv) - f(y)\big)$ is the generalized directional derivative of f at $\bar{x}$ in the direction v. The normal cone to a set $D \subseteq X$ at a point x in the sense of Clarke is denoted by $N_C(x; D)$ and defined as:

$$N_C(x; D) := \mathrm{cl}^*\big(\bigcup_{t \geq 0} t\ \partial_C d_D(x)\big), \tag{3.17}$$

where cl^* denotes weak* closure.
Observe that the Clarke subdifferential ∂_C satisfies the axioms $(A1)$ and $(A2)$; see Subsection 3.1.. However, instead of (A3) the sum rule of the Clarke subdifferential requires that all the involved functions are Lipschitz, i.e.,

(A3') If f, g is Lipschitz around $\bar{x}$ then one has the inclusion

$$\partial_C(f+g)(\bar{x}) \subseteq \partial_C f(\bar{x}) + \partial_C g(\bar{x}).$$

To derive necessary optimality conditions for solutions of the problem (VP), we present in the following an important result which will be used in the sequel. For the proof, we refer the reader to [3, page 52].

Proposition 3.8. *[3, Corollary 2.4.3] Let $X \in \mathcal{X}$ and $f : X \to \overline{\mathbb{R}}$ be locally Lipschitz at $\bar{x}$ with $f(\overline{x}) \in \mathbb{R}$. If f attains a minimum at $\bar{x}$ over $D \subseteq X$, then it holds that $0 \in \partial_C f(\bar{x}) + N_C(\bar{x}; D)$.*

The following result derives necessary optimality condition for weak Pareto miminizers of problem (VP) in terms of the Fenchel subdifferential ∂_F determined by (2.1) and the Clarke normal cone given by (3.17). Proposition 3.8. shows that the closedness of the feasible set D is not necessary for the proofs of next theorems.

Theorem 3.3. *Let $X, Y \in \mathcal{X}$, D be a subset of X, C be a proper, convex, closed and pointed cone in Y with nonempty interior. Consider the problem* (VP) *where the objective function $f : X \to Y$ is C-convex. If $f(\bar{x}) \in \mathrm{WMin}(f(D); C)$ and f is C-bounded from above around $\bar{x} \in D$ then for every $e \in \operatorname{int} C$ there exists $y^* \in C^+$ with $y^*(e) = 1$ such that*

$$0 \in \partial_F (y^* \circ f)(\bar{x}) + N_C(\bar{x}; D). \tag{3.18}$$

Proof: Let $e \in \operatorname{int} C$. We consider the corresponding function $\varphi := \varphi_{C,e}$ given by (2.3).
By using the assumption $f(\bar{x}) \in \mathrm{WMin}(f(D); C)$ and applying Proposition 2.5., we get that $\bar{x}$ is a minimum of the problem

$$\min_{x \in D} z_e(x), \tag{3.19}$$

where $z_e : X \to \mathbb{R}$ and $z_e(x) = \varphi(f(x) - f(\bar{x}))$.
By using the same arguments presented in Theorem 3.1., we have that z_e is locally Lipschitz at $\bar{x}$. Taking into account Proposition 3.8., we obtain that

$$0 \in \partial_C z_e(\bar{x}) + N_C(\bar{x}; D). \tag{3.20}$$

Because of the convexity of z_e, applying Remark 2.2. and Proposition 2.5., there exist $y^* \in C^+$ with $y^*(e) = 1$ such that (3.18) holds true. $\square$
To the best of our knowledge, Ha [9] used the Clarke coderivative to obtain optimality conditions for several types of solutions of set-valued optimization problems. In addition, optimality conditions in terms of the Clarke subdifferential were only derived for (weakly) Pareto minimal solution for geometric vector optimization problems; see e.g. [5, 22]. In this section, we obtain optimality conditions in terms of the Clarke subdifferential for solutions of the problem (VP) without the closedness assumption of the feasible set D.
We end this paper by the following theorem presenting a necessary optimality condition for (ϵ, e)-Pareto solutions of the problem (VP) in the sense of Definition 2.2. where $\operatorname{int} C = \emptyset$.

Theorem 3.4. *Let $X, Y \in \mathcal{X}$, D be a subset of X, C be a proper, closed, convex and pointed cone in Y with empty interior and $f : X \to Y$ be a vector-valued function. Assume that $\epsilon > 0$, $e \in C$, $\|e\| = 1$ and x_0 is an (ϵ, e)-Pareto minimal solution of the problem* (VP). *If f is a (C, e, ϵ)- strongly convex function and f is C-bounded from above around $x_0 \in D$ then there exist $\bar{x} \in B(x_0, \sqrt{\epsilon}) \cap D$, $y^* \in C^+, \|y^*\| = 1$ such that*

$$0 \in \partial_F (y^* \circ f)(\bar{x}) + \sqrt{\epsilon} U_{X^*} + N_C(\bar{x}; D). \tag{3.21}$$

Proof: Consider the functions $d_{-C-\epsilon e}(\cdot)$ and $z_{\epsilon,e}(\cdot) = d_{-C-\epsilon e}(f(\cdot) - f(x_0))$. As shown in Theorem 3.2., the function $z_{\epsilon,e}(\cdot)$ is Lipschitz around x_0. Applying the Ekeland variational principle for $z_{\epsilon,e}(\cdot)$ on D, we get an element $\bar{x} \in B(x_0, \sqrt{\epsilon}) \cap D$ such that $\bar{x}$ is a minimum of the problem

$$\min_{x \in D} z_{\epsilon,e}(x) + \sqrt{\epsilon}\|x - x_0\|.$$

Taking into account Proposition 3.8., we obtain that

$$0 \in \partial_C z_{\epsilon,e}(\bar{x}) + \sqrt{\epsilon}\partial_C \| \cdot - x_0\|(\bar{x}) + N_C(\bar{x}; D). \tag{3.22}$$

Following the same lines given in the proof of Theorem 3.2., we get the conclusion. $\square$

4. CONCLUSIONS

We have dealt with the problem (VP) for both cases $\operatorname{int} C \neq \emptyset$ and $\operatorname{int} C = \emptyset$ by using appropriate scalarizing functionals. The optimality conditions are derived in terms of the abstract subdifferential and the Clarke subdifferential. Although our abstract subdifferential is a special case of the generic subdifferential given in [6] and it is possible to use this generic subdifferential for solving the problem (VP), the optimality conditions in this paper concern the properties of the function f, not only the image set $f(D)$ as in [6]. In addition, in order to obtain the optimality conditions in terms of the Clark subdifferential we can omit the closedness of the feasible set.

ACKOWLEDGMENTS: The authors would like to thank the reviewer for the helpful report and important suggestions which improve the manuscript significantly. The second author also would like to thank the Ministry of Education and Training of Vietnam for financial support.

RECEIVED: NOVEMBER, 2017.
REVISED: FEBRUARY, 2018.

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