GENERATION OF $K-$CONVEX TEST PROBLEMS IN VARIABLE ORDERING SETTINGS

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ABSTRACT

Vector optimization problem with variable ordering structure is an extension of classical VOP with applications in Medicine and Economy. An approach that has been extended to this class is the projected gradient method. Although its theoretical convergence has been obtained, numerical experience is needed. In this work we propose two methods for generating test problems for this algorithm. Our proposals are based on Bishop-Phelps and simplicial cones respectively. In both cases we ensure that the hypothesis for the convergence of the projected gradient are satisfied.
KEYWORDS: K-convexity variable ordering structure vector optimization

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RESUMEN

El problema de optimización vectorial con orden variable es una extensión del VOP clásico con aplicaciones en la medicina y en la economía. El método del gradiente proyectado ha sido extendido para resolver este clase de problemas. Aunque se ha probado teóricamente su convergencia, es necesario realizar experimentos numéricos con el mismo. En este trabajo proponemos dos métodos para generar problemas prueba para este algoritmo. Nuestras propuestas se basan en los conos Bishop-Phelps y simpliciales respectivamente. En ambos casos aseguramos que se satisfacen las hipótesis para la convergencia del método del gradiente proyectado.
PALABRAS CLAVE: K-convexidad, estructura de orden variable, optimización vectorial.

1. INTRODUCTION

Let $F\colon \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function, $C \subset \mathbb{R}^n$ a convex set and $K\colon C \rightrightarrows \mathbb{R}^m$ a proper cone-valued map. The problem of finding a point $\bar{x} \in C$ such that

$$F(x) - F(\bar{x}) \notin -K(\bar{x}) \setminus \{0\},$$

[†]

for all $x \in C$ is called vector optimization problem with variable ordering structure. It is shortly denoted as

$$K - \min F(x), \tag{1.1}$$

Some applications of Problem (1.1) appear in medical diagnosis and portfolio optimization, see [7, 13]. For a summary of these applications see Section 1.3.1 in [6].

Recently, extensions of the steepest descent method and the Newton method for solving Problem (1.1) has been proposed in [1, 3]. Hence, test problems are needed to check the effectiveness of this algorithm. In this paper we propose two ways of generating these problems based on special families of cones. Roughly speaking, given a function F on the convex set $C \subset \mathbb{R}^n$, we obtain a cone-valued map $K\colon C \rightrightarrows \mathbb{R}^m$ such that F is $K-$convex on C. We call this procedure *K-convexification of F on C* and in this paper, it is carried out by using Bishop-Phelps and simplicial cones. In both cases we guarantee that the hypothesis for the convergence of the method appearing in [1] are fulfilled, namely, the compactness of C, the convexity of the function with respect to $K(x)$, the Lipschitz continuity of a generator of $K^*(x)$ and that there exists a proper cone $\mathcal{K}$ such that $K(x) \subset \mathcal{K}$ for all $x \in C$.

This paper is organized as follows. In Section 2. the notation and some preliminaries results are introduced. In Sections 3. and 4. we investigate the Bishop-Phelps and the simplicial $K-$convexification respectively. We conclude each of these sections showing a method for generating text functions for Problem (1.1). Finally in Section 5. conclusions and future research possibilities are given.

2. NOTATIONS AND PRELIMINARIES

This section contains the main notations and results that will be used in the paper. In the sequel, the inner product is denoted by $\langle \cdot, \cdot \rangle$, an arbitrary norm by $\| \cdot \|$ and the Euclidean norm by $\| \cdot \|_2$. The closed ball and the sphere centered at $x \in \mathbb{R}^n$ with radius $r > 0$ are $B(x, r) = \{y \in \mathbb{R}^n : \|y - x\| \leq r\}$ and $S(x, r) = \{y \in \mathbb{R}^n : \|y - x\| = r\}$ respectively. In addition, we denote by $\mathcal{C}^k(C, \mathbb{R}^m)$ the set of functions $F\colon C \subset \mathbb{R}^n \to \mathbb{R}^m$ that are k times continuously differentiable. Moreover, the Jacobian matrix of F at a point x is denoted by $JF(x)$.

We recall that a convex, closed, pointed cone with nonempty interior is called *proper cone*, see [4]. It is well known that a proper cone $\mathcal{K}$ defines a partial ordering on $\mathbb{R}^m$ as follows

$$z_1 \preceq_{\mathcal{K}} z_2 \Leftrightarrow z_2 - z_1 \in \mathcal{K}.$$

The next theorem provides a characterization for pointedness of a cone, see [5].

Theorem 2.1. *A convex and closed cone $\mathcal{K} \subset \mathbb{R}^m$ is pointed if and only if there exist $w \in S(0, 1)$ and $\delta > 0$ such that*

$$\langle w, z \rangle \geq \delta \, \|z\|_2 \, ,$$

for all $z \in \mathcal{K}$.

In the sequel, the *conic hull* of a set $\mathcal{G} \subset \mathbb{R}^m$ is denoted by $\operatorname{cone}\mathcal{G}$, this is $\operatorname{cone}\mathcal{G}$ is the smallest convex cone containing $\mathcal{G}$. If $\mathcal{K} = \operatorname{cone}\mathcal{G}$ for some convex cone $\mathcal{K}$, then we say that the set $\mathcal{G}$ is a *generator* of $\mathcal{K}$, see [4].

Now we introduce the two special classes of cones that are relevant for our work, see [10].

Definition 2.1. *The set $\mathcal{K}_{\mathrm{bp}} \subset \mathbb{R}^m$ is called a Bishop-Phelps cone if*

$$\mathcal{K}_{\mathrm{bp}} = \{z \in \mathbb{R}^m : \langle l, z\rangle \geq \|z\|\}, \tag{2.1}$$

where $l \in \mathbb{R}^m$ and $\|\cdot\|$ is a norm of $\mathbb{R}^m$.
A cone $\mathcal{K}_{\mathrm{sim}} \subset \mathbb{R}^m$ is called simplicial if $\mathcal{K}_{\mathrm{sim}} = \mathrm{cone}(\mathcal{B})$ for some basis $\mathcal{B}$ of $\mathbb{R}^m$.

For every cone $\mathcal{K} \subset \mathbb{R}^m$ we define the dual cone

$$\mathcal{K}^* = \{l \in \mathbb{R}^m : \langle l, z\rangle \geq 0 \quad \forall z \in \mathcal{K}\}.$$

The next two results allow us to compute the dual cone for Bishop-Phelps and simplicial cones. They can be found in [10] and [5] respectively.

Proposition 2.1. *Let $\|\cdot\|_*$ be the dual norm of $\|\cdot\|$. If the cone $\mathcal{K}_{\mathrm{bp}}$ is given by (2.1) then*

$$\mathcal{K}^*_{\mathrm{bp}} = \mathrm{cl}[\mathrm{cone}\,(B_*(l,1))],$$

where $B_*(l,1) = \{z \in \mathbb{R}^m : \|z - l\|_* \leq 1\}$.

Proposition 2.2. *Let $A \in \mathbb{R}^{m\times m}$ and $\mathcal{K} = \left\{A\lambda : \lambda_i \geq 0,\ \ i = \overline{1,n}\right\}$. If $\det A \neq 0$ then*

$$\mathcal{K}^* = \left\{\left[A^{-1}\right]^T \nu : \nu_i \geq 0,\ \ i = \overline{1,n}\right\}.$$

We consider the variable ordering structure in the image space of the objective function F given by

$$F(\bar{x}) \preceq_{\bar{x}} F(x) \Leftrightarrow F(x) - F(\bar{x}) \in K(\bar{x}), \tag{2.2}$$

where $K\colon C \rightrightarrows \mathbb{R}^m$ is a cone-valued mapping such that $K(x)$ is a proper cone for all $x \in C$, see [1]. In this case, $K(x)$ is usually called *variable ordering cone mapping.* Other kinds of variable ordering structures can be found in [6]. Note that (2.2) allows us to formulate the vector optimization problem with variable ordering structure as follows: Problem (1.1) is to find $\bar{x} \in C$ such that $F(\bar{x})$ is minimal on the set $F(C)$ w. r. t. the partial ordering induced by the proper cone $\mathcal{K} = K(\bar{x})$.
As usual $\mathrm{gr}(K)$ denotes the graph of the application K. Now, we define a metric between sets and the Lipschitz continuity of set-valued mappings see [11, 12].

Definition 2.2. *The directed Hausdorff distance between two sets $A, B \subset \mathbb{R}^m$ is given by*

$$\Delta_H(A,B) = \sup_{a\in A}\inf_{b\in B} d(a,b),$$

where $d(a,b) = \|a - b\|$. Moreover, the number

$$d_H(A,B) = \max\{\Delta_H(A,B), \Delta_H(B,A)\},$$

is called Hausdorff distance between A and B.

Definition 2.3. *We say that a set-valued map $\Psi\colon C \rightrightarrows \mathbb{R}^m$ is Lipschitz continuous if there exists $\mu > 0$ such that*

$$d_H(\Psi(x), \Psi(y)) \leq \mu\,\|x - y\|,$$

for all $x, y \in C$.

Note that a cone-valued application with closed images is Lipschitz continuous if and only if it is constant. Indeed, if $z_1 \in K(x_1) \setminus K(x_2)$, as $K(x_1)$ is closed, there exists $r > 0$ such that

$$B(z_1, r) \cap K(x_2) = \emptyset. \tag{2.3}$$

Let $\alpha > 0$, if $z \in B(\alpha z_1, \alpha r) \cap K(x_2)$, as $K(x_2)$ is a cone then,

$$\frac{z}{\alpha} \in B(z_1, r) \cap K(x_2),$$

contradicting (2.3). So, $B(\alpha z_1, \alpha r) \cap K(x_2) = \emptyset$. But $\alpha z_1 \in K(x_1)$. As

$$\alpha r < d(\alpha z_1, K(x_2)) \leq d_H(K(x_1), K(x_2) \leq \mu \|x_1 - x_2\|,$$

we obtain a contradiction taking $\alpha \to \infty$.
So, the Lipschitz condition will be taken in the sense that a compact generator $G(x) \subset S(0,1)$ of the cone $K(x)$ is Lipschitz continuous.
As in the vector case, convexity plays an important role in variable ordering structures.

Definition 2.4. *Let $C \subset \mathbb{R}^n$ be a convex set. We say that $F \colon C \to \mathbb{R}^m$ is a $K-$convex function if*

$$\lambda F(x) + (1-\lambda)F(y) - F(\lambda x + (1-\lambda)y) \in K(\lambda x + (1-\lambda)y), \tag{2.4}$$

for all $\lambda \in [0,1]$ and any $x, y \in C$.
A problem is convex if C is a convex set a F is a K-convex function.

Next proposition provides a characterization of the differentiable case.

Proposition 2.3 (See [2])**.** *Let $C \subset \mathbb{R}^n$ be a convex set with $\operatorname{int} C \neq \emptyset$. If $F \in \mathcal{C}^1(C, \mathbb{R}^m)$ and $\operatorname{gr} K$ is closed. Then F is $K-$convex if and only if*

$$F(x) - F(y) - JF(y)(x-y) \in K(y),$$

for all $x, y \in C$.

The steepest descent algorithm proposed in [1] provides another motivation for studying K-convex functions. In the next two sections we characterize theses types of functions for variable ordering structures given by Bishop-Phelps and simplicial cones.

3. BISHOP-PHELPS $K-$CONVEXIFICATION

In this section we investigate a $K-$convexification related to Bishop-Phelps cones. We also deal with some of its properties in order to ensure the convergence of numerical methods. Throughout this section, we assume that $l \colon C \to \mathbb{R}^m$, $\|\cdot\|$ is a norm of $\mathbb{R}^m$ and that

$$K_{\mathrm{bp}}(y) = \{z \in \mathbb{R}^m : \langle l(y), z\rangle \geq \|z\|\}, \tag{3.1}$$

is the variable ordering mapping. Note that the variability of $K_{\mathrm{bp}}(y)$ is given by the mapping l and that the norm $\|\cdot\|$ is fixed.

Now we define the function $\widehat{F}\colon C^2 \to \mathbb{R}^m$ by

$$\widehat{F}(x,y) = F(x) - F(y) - JF(y)(x-y).$$

In addition, let $\widehat{m}\colon C^2 \setminus \widehat{F}_1^{-1}(\{0\}) \to \mathbb{R}$ be given by

$$\widehat{m}(x,y) = \frac{\widehat{F}(x,y)}{\widehat{F}_1(x,y)}. \tag{3.2}$$

We will provide a sufficient condition for the convexity of F based on $\widehat{m}$. We start with the following lemma on the boundedness of $\widehat{m}$.

Lemma 3.1. *If $C \subset \mathbb{R}^m$ is a compact set and $\nabla^2 F_1(x) \succ 0$ for all $x \in C$. Then there exists $M > 0$ such that*

$$\|\widehat{m}(x,y)\| < M,$$

for all $x, y \in C$ with $x \neq y$.

Proof. It follows from Lema 3.2 in [9] and the compactness of C. □

The first part of the next proposition is a sufficient condition for the $K_{\mathrm{bp}}-$convexity of F on C. The second part contains the main idea for the $K_{\mathrm{bp}}-$convexification of F on C.

Proposition 3.1. *If $K_{\mathrm{bp}}\colon C \rightrightarrows \mathbb{R}^m$ is given by (3.1) then*

1. *If $\nabla^2 F_1(x) \succ 0$ for all $x \in C$ and*

$$l_1(y) \geq \|\widehat{m}(x,y)\| - \sum_{i=2}^{m} l_i(y)\widehat{m}_i(x,y), \tag{3.3}$$

for all $x, y \in C$ with $x \neq y$. Then F is $K_{\mathrm{bp}}-$convex.

2. *If $\nabla^2 F_1(x) \succ 0$ for all $x \in C$, the set C is compact and $l_2, \ldots, l_m$ are continuous functions, then there exists a continuous function l_1 such that (3.3) is satisfied.*

Proof. 1. Since $\nabla^2 F_1(x) \succ 0$ for all $x \in C$, it follows that $\widehat{F}_1(x,y) \geq 0$ for all $x, y \in C$. Hence, multiplying by $\widehat{F}_1(x,y)$ in both sides of (3.3) and then transforming the inequality we get

$$\langle l(y), \widehat{F}(x,y)\rangle \geq \left\|\widehat{F}(x,y)\right\|, \tag{3.4}$$

for all $x, y \in C$. From (3.4) and Proposition 2.3 it follows that F is $K_{\mathrm{bp}}-$convex.
2. If the set C is compact then the functions $l_2, \ldots, l_m$ are bounded on C. In addition, according to Proposition 3.1 the function $\widehat{m}$ is bounded on C. Thus there exists $U \in \mathbb{R}$ such that

$$\|\widehat{m}(x,y)\| - \sum_{i=2}^{m} l_i(y)\widehat{m}_i(x,y) \leq U, \tag{3.5}$$

for all $x, y \in C$ with $x \neq y$, therefore taking $l_1(y) = U$ yields the desired result. □

Note that our $K_{\mathrm{bp}}-$convexification follows from taking $l_2, \ldots, l_m$ continuous functions on the compact set C and then fixing $l_1(y) = U$ such that (3.5) holds.
It is worth to mention that for proving the convergence of numerical method it is often needed that there exists a proper cone $\mathcal{K}$ such that

$$K(y) \subset \mathcal{K},$$

for all $y \in C$, see [1, 3]. However, as we show in the next example, Proposition 3.1 does not guarantee this.

Example 3.1. *Let $F\colon [-1,1] \to \mathbb{R}^2$ be given by $F(y) = \big(y^2, y^3\big)$. Consider*

$$l(y) = (5, y),$$

in Proposition 3.1. It is easy to check that $(0,-1) \in K_{\mathrm{bp}}(-1)$ and $(0,1) \in K_{\mathrm{bp}}(1)$. Therefore, in this case such proper cone does not exist. ■

Despite the fact shown in Example 3.1, we can expect that for every $y_0 \in C$ there exists a proper cone $\mathcal{K}$ such that $K_{\mathrm{bp}}(y) \subset \mathcal{K}$ locally around y_0. This is formally stated in the next result.

Proposition 3.2. *If $l(y)$ is a continuous function then for all $y_0 \in C$ there exist $\delta > 0$ and a proper cone $\mathcal{K}$ such that*

$$K_{\mathrm{bp}}(y) \subset \mathcal{K}, \tag{3.6}$$

for all $y \in B(y_0, \delta) \cap C$.

Proof. Let $\mathcal{K}$ be the proper cone given by

$$\mathcal{K} = \{z \in \mathbb{R}^m : 2\langle l(y_0), z\rangle \geq \|z\|\}.$$

Suppose that (3.6) does not hold. Then there exists a sequence $\{y_p\}_{p\in\mathbb{N}}$ such that $y_p \to y_0$ and

$$K_{\mathrm{bp}}(y_p) \not\subset \mathcal{K},$$

for all $p \in \mathbb{N}$. This means that for some $z_p \in K_{\mathrm{bp}}(y_p)$ we have

$$2\langle l(y_0), z_p\rangle < \|z_p\|.$$

Assume without loss of generality that $\|z_p\| = 1$ for all $p \in \mathbb{N}$ and that $z_p \to z_0$ for some $z_0 \in S(0,1)$. Hence, it follows that

$$\langle l(y_p), z_p\rangle \geq 1, \tag{3.7}$$

and

$$2\langle l(y_0), z_p\rangle < 1. \tag{3.8}$$

Letting $p \to +\infty$ in (3.7) and (3.8) we get a contradiction and this completes the proof. □

Now we consider Bishop-Phelps cone-valued mappings with $\|\cdot\| = \|\cdot\|_2$, this is

$$K_{\text{rev}}(y) = \{z \in \mathbb{R}^m : \langle l(y), z\rangle \geq \|z\|_2\}. \tag{3.9}$$

Such cones are called revolution cones, see Section 2.2 in [8] . It turns out that in this case it is easy to compute δ in Proposition 3.2. Example 3.3.2 in [9] provides more details about the geometric idea behind its computation. Let $\gamma(z, y_0)$ be the angle between the vectors $z \in K_{\text{rev}}(y)$ and $l(y_0) \in \mathbb{R}^m$. Let

$$\bar{\gamma}(y, y_0) = \max_{z \in K_{\text{rev}}(y)} \gamma(z, y_0).$$

In the next lemma $\bar{\gamma}(y, y_0)$ is computed in $\mathbb{R}^2$.

Lemma 3.2. *Let $l\colon C \to \mathbb{R}^2$ and $K\colon C \to \mathbb{R}^2$ be given by*

$$K(y) = \{z \in \mathbb{R}^2 : \langle l(y), z\rangle \geq \|z\|_2\}.$$

If $\|l(y_0)\|_2 \geq \|l(y) - l(y_0)\|_2$ and $\theta = \theta(y, y_0)$ is the angle between $l(y)$ and $l(y) - l(y_0)$, then

$$\begin{aligned}\bar{\gamma}(y, y_0) = \arcsin&\left(\frac{\|l(y) - l(y_0)\|\sin\theta}{\|l(y_0)\|_2}\right) + \\ &+ \arccos\left(\frac{1}{\|l(y) - l(y_0)\|_2\cos\theta + \sqrt{\|l(y_0)\|_2^2 - \|l(y) - l(y_0)\|_2^2 \sin^2\theta}}\right).\end{aligned}$$

Proof. Let α be the angle between $l(y_0)$ and $l(y)$ and β be the angle between $l(y)$ and $\bar{z}$. Applying the Sine and the Cosine Laws respectively, we get

$$\alpha = \arcsin\left(\frac{\|l(y) - l(y_0)\|\sin\theta}{\|l(y_0)\|_2}\right),$$

and

$$\beta = \arccos\left(\frac{1}{\|l(y) - l(y_0)\|_2\cos\theta + \sqrt{\|l(y_0)\|_2^2 - \|l(y) - l(y_0)\|_2^2 \sin^2\theta}}\right).$$

Obviously, we have $\bar{\gamma}(y, y_0) = \alpha + \beta$ so the desired result follows immediately. $\square$

The following result is very simple, however we did not find it in the literature and therefore we add a short proof.

Proposition 3.3. *If $u, v, w \in \mathbb{R}^m$ are vectors such that*

$$\max\left(\sphericalangle(u, v), \sphericalangle(v, w)\right) \leq \frac{\pi}{2},$$

then

$$\sphericalangle(u, w) \leq \sphericalangle(u, v) + \sphericalangle(v, w),$$

where $\sphericalangle(u, w)$ denotes the angle between u and v.

Proof. Let $H = \{x \in \mathbb{R}^m : \langle a, x\rangle = 0\}$, with $\|a\|_2 = 1$, be an hyperplane such that $u, w \in H$. Define

$$P_H(x) = x - \langle a, x\rangle a.$$

It is easy to verify that

$$\sphericalangle(u, P_H(v)) \leq \sphericalangle(u, v), \tag{3.10}$$

and

$$\sphericalangle(P_H(v), w) \leq \sphericalangle(v, w). \tag{3.11}$$

Using (3.10) and (3.11) we get

$$\begin{aligned}\sphericalangle(u, w) &= \sphericalangle(u, P_H(v)) + \sphericalangle(P_H(v), w)\\ &\leq \sphericalangle(u, v) + \sphericalangle(v, w),\end{aligned}$$

and this finishes the proof. □

Now we propose a method for computing δ in Proposition 3.2 for revolution cone-valued mappings. But first, fix $y_0 \in C$ and let $\gamma_A \colon [0, \pi] \times [0, \|l(y_0)\|_2) \to \mathbb{R}$ and $\gamma_M \colon [0, \|l(y_0)\|_2) \to \mathbb{R}$ be given by

$$\begin{aligned}\gamma_A(x_1, x_2) = \arcsin&\left(\frac{x_2 \sin x_1}{\|l(y_0)\|_2}\right) +\\ &+ \arccos\left(\frac{1}{x_2 \cos x_1 + \sqrt{\|l(y_0)\|_2^2 - x_2^2 \sin^2 x_1}}\right),\\ \gamma_M(\varepsilon) = \max_{\substack{x_1 \in [0,\pi]\\ x_2 \in [0,\varepsilon]}}& \gamma_A(x_1, x_2).\end{aligned} \tag{3.12}$$

Next we give our method for computing δ for $y_0 \in C$ in Proposition 3.2 when $\|\cdot\| = \|\cdot\|_2$ and then prove its effectiveness.

Algorithm 3.1 Computing δ for $y_0 \in C$ in Proposition 3.2

> **set** $\varepsilon \in (0, \|l(y_0)\|_2)$
> **while** $\gamma_M(\varepsilon) \geq \frac{\pi}{2}$ **do**
> $\varepsilon = \frac{\varepsilon}{2}$
> **end while**
> **set** $\delta > 0$
> **while** $\max_{y \in B(y_0,\delta) \cap C} \|l(y) - l(y_0)\|_2^2 \geq \varepsilon^2$ **do**
> $\delta = \frac{\delta}{2}$
> **end while**
> **return** δ

Proposition 3.4. *If $l \colon C \to \mathbb{R}^m$ is a continuous function and $K_{\text{rev}}(y)$ is given by* (3.9) *then Algorithm 3.1 returns $\delta > 0$ such that*

$$K_{\text{rev}}(y) \subset \mathcal{K}.$$

for all $y \in B(y_0, \delta) \cap C$, where $\mathcal{K}$ is a proper cone.

Proof. Due to the continuity of l and γ_M, it follows that both *while* loops are finite. Note that the first loop ensures that

$$\gamma_A(x_1, x_2) \leq \gamma_M(\varepsilon) < \frac{\pi}{2}, \tag{3.13}$$

for all $(x_1, x_2) \in [0, \pi] \times [0, \varepsilon]$ and that the second one guarantees that

$$\|l(y) - l(y_0)\|_2 < \varepsilon, \tag{3.14}$$

for all $y \in B(y_0, \delta) \cap C$. Using (3.13) and (3.14) we get

$$\gamma_A(x_1, \|l(y) - l(y_0)\|_2) \leq \gamma_M(\varepsilon), \tag{3.15}$$

for all $x_1 \in [0, \pi]$ and all $y \in B(y_0, \delta) \cap C$. Let α and β be the angles between $l(y_0)$ and $l(y)$, and between $l(y)$ and $\bar{z}$ respectively. Note that Lemma 3.2 remains valid for an arbitrary hyperplane of $\mathbb{R}^m$. Therefore, it follows that

$$\gamma_A(\theta, \|l(y) - l(y_0)\|_2) = \alpha + \beta. \tag{3.16}$$

In addition, applying Proposition 3.3 we obtain

$$\bar{\gamma}(y, y_0) \leq \alpha + \beta. \tag{3.17}$$

Using (3.15), (3.16) and (3.17) we get

$$\bar{\gamma}(y, y_0) \leq \gamma_M(\varepsilon), \tag{3.18}$$

for all $y \in B(y_0, \delta) \cap C$. Now, define

$$\mathcal{K} = \left\{ z \in \mathbb{R}^m : \left\langle \frac{l(y_0)}{\|l(y_0)\|_2 \cos[\gamma_M(\varepsilon)]}, z \right\rangle \geq \|z\|_2 \right\}.$$

From (3.18) it follows that

$$K_{\text{rev}}(y) \subset \mathcal{K}.$$

for all $y \in B(y_0, \delta) \cap C$. □

Remark 3.1. *It is also possible to compute δ in Proposition 3.2 for a Bishop-Phelps $K_{\text{bp}}-$convexification with an arbitrary norm. Since all the norms in $\mathbb{R}^m$ are equivalent, for every norm $\|\cdot\|$ there exists $a > 0$ such that*

$$\|z\| \geq a\|z\|_2,$$

for all $z \in \mathbb{R}^m$. Consider the revolution cone-valued mapping

$$K_a(y) = \left\{ z \in \mathbb{R}^m : \left\langle \frac{l(y_0)}{a}, z \right\rangle \geq \|z\|_2 \right\}.$$

Obviously, we have $K_{\text{bp}}(y) \subset K_a(y)$ for all $y \in C$. Thus, it is enough to apply Algorithm 3.1 to $K_a(y)$ in order to find $\delta > 0$ such that (3.6) *holds for some proper cone $\mathcal{K}$.*

Now we turn our attention on the Lipschitz continuity of a generator of $K^*_{\text{bp}}(y)$. As a preliminary step, we give the following lemma.

Lemma 3.3. *Let $\|\cdot\|_*$ be the dual norm of $\|\cdot\|$. If $l\colon C \to \mathbb{R}^m$ is a continuous function such that $\|l(y)\|_* > 1$ for all $y \in C$ and the set C is compact, then there exists $\eta > 0$ such that*

$$\|l(y) + u\|_2 \geq \eta,$$

for all $y \in C$ and all $u \in B_*(0,1)$, where $B_*(0,1) = \{z \in \mathbb{R}^m : \|z\|_* \leq 1\}$.

Proof. Since all the norms in $\mathbb{R}^m$ are equivalent, there exists $a > 0$ such that

$$\|l(y) + u\|_2 \geq a\|l(y) + u\|_*, \tag{3.19}$$

for all $y \in C$ and all $u \in B_*(0,1)$. Besides, since $\|l(y)\|_* > 1 \geq \|u\|_*$ we have $l(y) \neq -u$. Thus, it follows that

$$\|l(y) + u\|_* > 0, \tag{3.20}$$

for all $y \in C$ and all $u \in B_*(0,1)$. Since C and $B_*(0,1)$ are compact sets and the function $\|l(y)+u\|_*$ is continuous, from Weierstrass theorem and (3.20), we deduce that there exist $y_0 \in C$ and $u_0 \in B_*(0,1)$ such that

$$\|l(y) + u\|_* \geq \|l(y_0) + u_0\|_* > 0. \tag{3.21}$$

for all $y \in C$ and all $u \in B_*(0,1)$. Taking $\eta = a\|l(y_0) + u_0\|_*$ and using (3.19) and (3.21) the result follows. □

We also need two lemmas that can be found in [9].

Lemma 3.4. *If $A, B \subset \mathbb{R}^n$ are compact sets then there exist $\bar{a} \in A$ and $\bar{b} \in B$ such that*

$$\Delta_H(A,B) = \|\bar{a} - \bar{b}\|_2,$$

and

$$\Delta_H(A,B) \leq \|\bar{a} - b\|_2,$$

for all $b \in B$.

Lemma 3.5. *If $a, b \in \mathbb{R}^m \setminus \{0\}$, then*

$$\left\| \frac{a}{\|a\|_2} - \frac{b}{\|b\|_2} \right\|_2 \leq \frac{\|a-b\|_2}{\sqrt{\|a\|_2\,\|b\|_2}},$$

and the equality holds if and only if $\|a\|_2 = \|b\|_2$.

The next theorem provides a sufficient condition for the Lipschitz continuity of the set-valued map given by $G^*_{\mathrm{bp}}(y) = K^*_{\mathrm{bp}}(y) \cap S_2(0,1)$.

Theorem 3.1. *If $l\colon C \to \mathbb{R}^m$ is a Lipschitz continuous function such that $\|l(y)\|_* > 1$ for all $y \in C$ and the set C is compact, then $G^*_{\mathrm{bp}}(y)$ is a Lipschitz continuous set-valued mapping.*

Proof. It is enough to show that there exist $\mu > 0$ and $\eta > 0$ such that

$$\Delta_H(G^*_{\mathrm{bp}}(y_1), G^*_{\mathrm{bp}}(y_2)) \leq \frac{\mu}{\eta}\|y_1 - y_2\|_2,$$

for all $y_1, y_2 \in C$. Clearly the set $G^*_{\mathrm{bp}}(y)$ is compact for all $y \in C$. Hence, applying Lemma 3.4 it follows that there exists $\bar{z}_1 \in G^*_{\mathrm{bp}}(y_1)$ such that

$$\Delta_H(G^*_{\mathrm{bp}}(y_1), G^*_{\mathrm{bp}}(y_2)) \leq \|\bar{z}_1 - z_2\|_2, \tag{3.22}$$

for all $z_2 \in G^*_{\mathrm{bp}}(y_2)$. Since $\bar{z}_1 \in G^*_{\mathrm{bp}}(y_1)$, according Proposition 2.1 there exists $u \in B_*(0,1)$ such that

$$\bar{z}_1 = \frac{l(y_1) + u}{\|l(y_1) + u\|_2}. \tag{3.23}$$

Taking

$$z_2 = \frac{l(y_2) + u}{\|l(y_2) + u\|_2}, \tag{3.24}$$

and then substituting (3.23) and (3.24) into (3.22) we get

$$\Delta_H(G^*(y_1), G^*(y_2)) \leq \left\| \frac{l(y_1) + u}{\|l(y_1) + u\|_2} - \frac{l(y_2) + u}{\|l(y_2) + u\|_2} \right\|_2. \tag{3.25}$$

Applying Lemma 3.5 in (3.25) it follows that

$$\Delta_H(G^*_{\mathrm{bp}}(y_1), G^*_{\mathrm{bp}}(y_2)) \leq \frac{\|l(y_1) - l(y_2)\|_2}{\sqrt{\|l(y_1) + u\|_2 \|l(y_2) + u\|_2}}.$$

Now, using the Lipschitz continuity of $l(y)$ and Lemma 3.3 we deduce that there exist $\mu > 0$ y $\eta > 0$ such that

$$\begin{aligned}
\|l(y_1) - l(y_2)\|_2 &\leq \mu\|y_1 - y_2\|_2, \\
\|l(y_1) + u\|_2 &\geq \eta, \\
\|l(y_2) + u\|_2 &\geq \eta.
\end{aligned}$$

The rest of the proof is straightforward. □

As a result of this section we can formulate a method for generating test problems based on Bishop-Phelps cones.

Method 3.1. *Given*

- *A compact and convex set C such that $\operatorname{int} C \neq \emptyset$.*
- *A vector function $F \in \mathcal{C}^2(C, \mathbb{R}^n)$ such that $\nabla^2 F_1(x) \succ 0$ for all $x \in C$.*
- *Lipschitz continuous functions $l_i \colon C \to \mathbb{R}$, $i = \overline{2, m}$.*
- *A norm $\|\cdot\|$ in $\mathbb{R}^m$.*

Step 1 *Find $l_1(y)$ according to the second part of Proposition 3.1. By the first part of this proposition, it follows that F is $K_{\mathrm{bp}}-$convex on C.*

Step 2 *Fix $y_0 \in C$. If $\|\cdot\| = \|\cdot\|_2$ then apply Algorithm 3.1 for computing δ. Otherwise, proceed according to Remark 3.1. Set $\overline{C} = B(y_0, \delta) \cap C$. By Proposition 3.4, there exists a proper cone $\mathcal{K} \subset \mathbb{R}^m$ such that $K_{\mathrm{bp}}(x) \subset \mathcal{K}$ for all $x \in \overline{C}$.*

Step 3 *Define $G^*_{\mathrm{bp}}(y) = K^*_{\mathrm{bp}}(y) \cap S_2(0,1)$, where $K^*_{\mathrm{bp}}(y)$ is given as in Proposition 2.1 for all $y \in \overline{C}$. By Theorem 3.1, $G^*_{\mathrm{bp}}(y)$ is Lipschitz continuous on $\overline{C}$.*

The following corollary summarizes the result of the previous method by referring to the assumptions for the convergence of the Projected Gradient method for solving Problem (1.1).

Corollary 3.1. *Let C, F, l_i be given as in Method 3.1. As a result of Method 3.1, the following three assumptions for the convergence of the Projected Gradient method for solving Problem* (1.1) *are fulfilled*

- *F is $K_{\mathrm{bp}}-$convex on $\overline{C}$.*
- *There exists a proper cone $\mathcal{K} \subset \mathbb{R}^m$ such that $K_{\mathrm{bp}}(x) \subset \mathcal{K}$ for all $x \in \overline{C}$.*
- *$G^*_{\mathrm{bp}}(y)$ is Lipschitz continuous on $\overline{C}$.*

4. SIMPLICIAL $K-$CONVEXIFICATION

Now we apply the same ideas of the previous section to a $K-$convexification related to simplicial cones. Let $g_1, \ldots, g_m \colon C \to \mathbb{R}^m$ be vector functions such that $\det[g_1(y), \ldots, g_m(y)] \neq 0$ for all $y \in C$. We define the simplicial cone-valued mapping by

$$K_{\mathrm{sim}}(y) = \mathrm{cone}\{g_1(y), \ldots, g_m(y)\}. \tag{4.1}$$

Furthermore, we define $g(y) = (g_1(y), \ldots, g_m(y))$ for all $y \in C$. The next result has several common points with Proposition 3.1.

Proposition 4.1. *Let $h\colon \mathbb{R}^n \to \mathbb{R}^{m\times m}$ be a mapping such that $\det h(y) \neq 0$ for all $y \in C$. If $g(y) = [h(y)]^{-1}$ and $K_{\mathrm{sim}}(y)$ is given by* (4.1) *then*

1. *If $\nabla^2 F_1(x) \succ 0$ for all $x \in C$ and*

$$h_{i1}(y) \geq -\sum_{j=2}^{m} h_{ij}(y)\widehat{m}_j(x,y), \quad i = \overline{1,m}, \tag{4.2}$$

for all $x, y \in C$ with $x \neq y$. Then F is $K_{\mathrm{sim}}-$convex.

2. *If $\nabla^2 F_1(x) \succ 0$ for all $x \in C$, the set C is compact, the matrix $(h_{ij}(y))_{i=\overline{2,m}}^{j=\overline{2,m}}$ is non-singular and continuous on C, then there exist continuous functions $h_{11}, \ldots, h_{1m}$ and $h_{21}, \ldots, h_{m1}$ such that* (4.2) *holds.*

Proof. 1. Since $\nabla^2 F_1(x) \succ 0$ for all $x \in C$, it follows that $\widehat{F}_1(x,y) \geq 0$ for all $x, y \in C$. Hence, multiplying by $\widehat{F}_1(x,y)$ in both sides of (4.2) and then transforming the inequality we get

$$h_{i1}(y)\widehat{F}_1(x,y) \geq -\sum_{j=2}^{m} h_{ij}(y)\widehat{F}_j(x,y), \quad i = \overline{1,m}. \tag{4.3}$$

Transforming (4.3) and then rewriting it in matrix form we get

$$h(y)\widehat{F}(x,y) \succeq 0. \tag{4.4}$$

Moreover, it is clear that

$$\widehat{F}(x,y) = g(y)\left[h(y)\widehat{F}(x,y)\right]. \tag{4.5}$$

Thus, from (4.4), (4.5) and Proposition 2.3 it follows that F is $K_{\text{sim}}-$convex.

2. Take

$$\begin{aligned} h_{11}(y) &= 1, \\ h_{1j}(y) &= 0, \quad j = \overline{2,m}. \end{aligned}$$

This ensures that $\det h(y) \neq 0$ for all $y \in C$. Besides, if the set C is compact then $(h_{ij}(y))_{i=\overline{2,m}}^{j=\overline{2,m}}$ is bounded on C. From Proposition 3.1 we get that $\widehat{m}$ is bounded on C too. Hence, there exist $U_i \in \mathbb{R}$ such that

$$U_i \geq -\sum_{j=2}^{m} h_{ij}(y)\widehat{m}_j(x,y), \quad i = \overline{2,m},$$

for all $x, y \in C$, thus taking

$$h_{i1}(y) = U_i, \quad i = \overline{2,m},$$

the result follows. □

As in the previous section, here it is important to know whether there exists a proper cone $\mathcal{K}$ such that

$$K_{\text{sim}}(y) \subset \mathcal{K},$$

for all $y \in C$. Now we give an example for showing that Proposition 4.1 does not guarantee this.

Example 4.1. *Let* $F\colon [0,\pi] \to \mathbb{R}^3$ *and* $h\colon [0,\pi] \to \mathbb{R}^{3\times 3}$ *be given by*

$$F(y) = (y^2, y, -y),$$

and

$$h(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos y & -\sin y \\ 0 & \sin y & \cos y \end{pmatrix}.$$

Obviously we have $F_1''(y) = 2 > 0$ *and* $\det h(y) = 1$*. However, it follows that*

$$h(0)e_2 = h(\pi)(-e_2) = e_2,$$

which means that $e_2 \in K_{\text{sim}}(0)$ *and* $-e_2 \in K_{\text{sim}}(\pi)$*. Therefore, such proper cone does not exist.* ■

The next result is analogous to Proposition 3.2.

Proposition 4.2. *If the functions* $g_1(y), \ldots, g_m(y)$ *are continuous, then for all* $y_0 \in C$ *there exist* $\delta > 0$ *and a proper cone* $\mathcal{K}$ *such that*

$$K_{\text{sim}}(y) \subset \mathcal{K}. \tag{4.6}$$

for all $y \in B(y_0, \delta) \cap C$.

Proof. Since $K_{\text{sim}}(y_0)$ is pointed, from Theorem 2.1 it follows that there exist $v \in \mathbb{R}^m$ and $M \in \mathbb{R}$ such that

$$\langle v, g_i(y_0)\rangle \geq M\|g_i(y_0)\|_2, \quad i = \overline{1,m}. \tag{4.7}$$

Let $\mathcal{K} \subset \mathbb{R}^m$ be the revolution cone given by

$$\mathcal{K} = \{z \in \mathbb{R}^m : 2\langle v, z\rangle \geq M\|z\|_2\}.$$

Obviously the functions

$$\xi_i(y) = 2\langle v, g_i(y)\rangle - M\|g_i(y)\|_2, \quad i = \overline{1,m},$$

are continuous on C. In addition, from (4.7) we get

$$\xi_i(y_0) > 0, \quad i = \overline{1,m},$$

hence there exists $\delta > 0$ such that

$$\xi_i(y) > 0, \quad i = \overline{1,m},$$

for all $y \in B(y_0, \delta) \cap C$. This means that

$$g_i(y) \in \mathcal{K}, \quad i = \overline{1,m},$$

for all $y \in B(y_0, \delta) \cap C$, which finishes the proof. $\square$

Following the ideas of Proposition 4.2 we can find $\delta > 0$ such that (4.6) holds. Let $\zeta > 1$, $\bar{v} \in \mathbb{R}^m$ such that

$$\begin{aligned} &\bar{v} = \arg\min \|v\|_2^2 \\ \text{s. a.} \quad &\langle v, g_i(y_0)\rangle \geq \|g_i(y_0)\|_2, \quad i = \overline{1,m}. \end{aligned}$$

Define $\bar{\xi}\colon \mathbb{R}_+ \to \mathbb{R}$ by

$$\bar{\xi}(\delta) = \min_i \min_{y \in B(y_0,\delta)\cap C} \{\zeta\langle \bar{v}, g_i(y)\rangle - \|g_i(y)\|_2\}.$$

Next we give our method and then prove its effectiveness.

Algorithm 4.1 Computing δ for $y_0 \in C$ in Proposition 4.2

```
set δ > 0
while ξ̄(δ) < 0 do
    δ = δ/2
end while
return δ
```

Proposition 4.3. *If the functions $g_1(y), \ldots, g_m(y)$ are continuous and $K_{\text{sim}}(y)$ is given by (4.1) then Algorithm 4.1 returns $\delta > 0$ such that*

$$K_{\text{sim}}(y) \subset \mathcal{K}.$$

for all $y \in B(y_0, \delta) \cap C$, where $\mathcal{K}$ is a proper cone.

Proof. It is easy to verify that the *while* loop is finite and that it ensures that

$$\bar{\xi}(\delta) \geq 0,$$

or equivalently that

$$\zeta\langle \bar{v}, g_i(y)\rangle \geq \|g_i(y)\|_2, \quad i = \overline{1,m}, \tag{4.8}$$

for all $y \in B(y_0, \delta) \cap C$. Let $\mathcal{K}$ be given by

$$\mathcal{K} = \{z \in \mathbb{R}^m : \zeta\langle \bar{v}, z\rangle \geq \|z\|_2\}.$$

From (4.8) we deduce

$$g_i(y) \in \mathcal{K}, \quad i = \overline{1,m},$$

for all $y \in B(y_0, \delta) \cap C$ and this completes the proof. □

The next result is analogous to Theorem 3.1. Here we give a sufficient condition for the Lipschitz continuity of a generator of $K^*_{\text{sim}}(y)$ for the simplicial $K_{\text{sim}}-$convexification.

Theorem 4.1. *If* $\det g(y) \neq 0$, $g(y)$ *is Lipschitz continuous and* $g^*(y) = \left([g(y)]^{-1}\right)^T$ *for all* $y \in C$, *then the set-valued mapping*

$$G^*_{\text{sim}}(y) = \left\{ \frac{g^*_1(y)}{\|g^*_1(y)\|_2}, \ldots, \frac{g^*_m(y)}{\|g^*_m(y)\|_2} \right\},$$

is Lipschitz continuous on C *and* $K^*_{\text{sim}}(y) = \operatorname{cone} G^*_{\text{sim}}(y)$ *for all* $y \in C$.

Proof. Note that

$$g^*(y) = \frac{1}{\det g(y)} \operatorname{cof} g(y),$$

where $\operatorname{cof} g(y)$ is the cofactor matrix of $g(y)$. Obviously $\operatorname{cof} g(y)$ is Lipschitz continuous. Moreover, the function $\det g(y) \neq 0$ is Lipschitz continuous, then $g^*(y)$ is Lipschitz continuous on C, see [12]. In addition, since $\det g(y) \neq 0$, we get

$$\|g^*_i(y)\|_2 \neq 0, \quad i = \overline{1,m}.$$

Therefore each function in the set

$$G^*_{\text{sim}}(y) = \left\{ \frac{g^*_1(y)}{\|g^*_1(y)\|_2}, \ldots, \frac{g^*_m(y)}{\|g^*_m(y)\|_2} \right\},$$

is Lipschitz continuous. This implies that the set-valued mapping $G^*(y)$ is Lipschitz continuous on C. Finally, from Proposition 2.2 we obtain $K^*_{\text{sim}}(y) = \operatorname{cone} G^*_{\text{sim}}(y)$ for all $y \in C$. □

Analogously to the previous section, here we formulate a method for generating test problems based on simplicial cones.

Method 4.1. *Given*

- *A compact and convex set* C *such that* $\operatorname{int} C \neq \emptyset$.

- *A vector function $F \in \mathcal{C}^2(C, \mathbb{R}^n)$ such that $\nabla^2 F_1(x) \succ 0$ for all $x \in C$.*
- *Lipschitz continuous functions $h_{ij}\colon C \to \mathbb{R}$, $i = \overline{2,m}$, $j = \overline{2,m}$, such that the matrix $(h_{ij}(y))_{i=\overline{2,m}}^{j=\overline{2,m}}$ is nonsigular for all $y \in C$.*

Step 1 *Find $h_{11}(y)$, $h_{1j}(y)$, $j = \overline{2,m}$ and $h_{i1}(y)$, $i = \overline{2,m}$ according to the second part of Proposition 4.1. By the first part of this proposition, it follows that F is $K_{\text{sim}}-$convex on C.*

Step 2 *Fix $y_0 \in C$. Apply Algorithm 4.1 for computing δ. Set $\overline{C} = B(y_0, \delta) \cap C$. By Proposition 4.3, there exists a proper cone $\mathcal{K} \subset \mathbb{R}^m$ such that $K_{\text{sim}}(x) \subset \mathcal{K}$ for all $x \in \overline{C}$.*

Step 3 *Define*

$$G^*_{\text{sim}}(y) = \left\{ \frac{g_1^*(y)}{\|g_1^*(y)\|_2}, \ldots, \frac{g_m^*(y)}{\|g_m^*(y)\|_2} \right\}.$$

By Theorem 4.1, $G^*_{\text{sim}}(y)$ is Lipschitz continuous on $\overline{C}$.

With the aim of summarizing, we give a corollary which is analogous to Corollary 3.1.

Corollary 4.1. *Let C, F, h_{ij} be given as in Method 4.1. As a result of Method 4.1, the following three assumptions for the convergence of the Projected Gradient method for solving Problem* (1.1) *are fulfilled*

- *F is $K_{\text{sim}}-$convex on $\overline{C}$.*
- *There exists a proper cone $\mathcal{K} \subset \mathbb{R}^m$ such that $K_{\text{sim}}(x) \subset \mathcal{K}$ for all $x \in \overline{C}$.*
- *$G^*_{\text{sim}}(y)$ is Lipschitz continuous on $\overline{C}$.*

5. CONCLUSIONS

In this paper we proposed two methods for generating test problems for the projected gradient method for solving vector optimization problems with variable ordering structure. In both cases we used two well known classes of proper cones, namely Bishop-Phelps and simplicial cones. Note that, although we focus on testing a particular method, the hypothesis exposed in Corollaries 3.1 and 4.1 are also natural for other numerical methods in variable ordering structure. The issues on the practical implementation of Methods 3.1 and 4.1 are topics of further research.

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