# A REVIEW OF TERMINATION RULES OF AN INEXACT PRIMAL-DUAL INTERIOR POINT METHOD FOR LINEAR PROGRAMMING PROBLEMS

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**ABSTRACT**
In this paper we apply the Inexact Newton theory on the perturbed KKT-conditions that are derived from the Karush-Kuhn-Tucker optimality conditions for the standard linear optimization problem. We discuss different formulations and accuracy requirements for the linear systems and show global convergence properties of the method.

**KEYWORDS:** global convergence, inexact search direction, infeasible interior point algorithm, linear optimization, primal-dual.

**MSC:** 49M15, 90C51, 15A23.

**RESUMEN**
En este trabajo, se aplica el método inexacto de Newton a la solución del sistema de KKT perturbado, que se deriva de las condiciones de optimalidad de KKT del problema de optimización lineal standard. Presentamos distintas formulaciones. Se prueban propiedades relativas a la convergencia global del método.

**PALABRAS CLAVE:** convergencia global, dirección de búsqueda inexacta, método interior infactible, optimización lineal, primal-dual.

## 1. INTRODUCTION

Consider the primal linear programming problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to:} & Ax = b, \;\; x \geq 0, \end{array} \tag{1.1}$$

where $A$ is an $m$-by-$n$ matrix of full rank $m$, $b$ an $m$-vector, and $c$ an $n$-vector and its dual problem

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to:} & A^T y + z = c, \;\; z \geq 0. \end{array} \tag{1.2}$$

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The optimality conditions for the linear program pair (1.1) and (1.2) are the Karush-Kuhn-Tucker (KKT) conditions:

$$F(x,y,z) \equiv \begin{pmatrix} Ax-b \\ A^T y + z - c \\ XZe \end{pmatrix} = 0, \quad x \geq 0,\ z \geq 0, \tag{1.3}$$

where $X = \mathrm{diag}(x)$, $Z = \mathrm{diag}(z)$ and $e$ is the vector of all ones in $\Re^n$.

An interior point algorithm solves a linear programming problem by generating a sequence of interior points from an initial interior point. An interior point is said to be feasible if it satisfies all the equality constraints, otherwise its infeasible. Further, an interior point algorithm that starts with an infeasible point is known as an *infeasible interior point algorithm.*

Bellavia [1] proved global convergence of an inexact interior point method. Mizuno and Jarre [2] proved global and polynomial-time convergence of an infeasible interior point algorithm using inexact computation. Portugal et al. [3] presented a truncated primal-infeasible dual-feasible interior point algorithm for linear programming. Portugal et al. [4] presented a truncated primal-infeasible dual-feasible interior point algorithm for solving monotone linear complementarity problems. The methods suggested in [3, 4, 5] have the major drawback of remaining primal-feasible once they become primal-feasible. Thus if they happen to become primal-feasible before the complementarity gap is significantly reduced satisfying the termination criteria in the iterative linear system solver is computationally expensive. Also the termination criteria suggested in these methods can not be used with the hybrid interior point algorithm where we alternate between a direct linear system solver and an iterative linear system solver in some manner. Kojima et al. [11] proved global convergence of an infeasible interior point algorithm and Zhang [7] proved polynomial complexity of a long-step path following infeasible interior point method.

The inexact Newton framework in the context of interior point algorithms can be extended to convex quadratic programming [9] and to nonlinear programming [10].

In this paper we discuss the convergence properties of an inexact interior point method which is a variant of the algorithm by Kojima, Megiddo, and Mizuno [11]. The algorithm in [11] has been studied by many researchers [13, 14, 15] and is known to be practically efficient among the numerous variations and extensions of the primal-dual interior point algorithm. In this paper we extend the discussions and results in [16, 17].

### 1.1. Overview and Notation

In Section 2. we formulate the linear programming problem in the augmented and normal equations forms. We state the inexact interior point algorithm. Section 3. discusses the global convergence results. We state the global convergence theorem and give its proof in this section. Lastly in Section 4. we give our concluding remarks.

Throughout this paper we use the following notation: For any vector $x$, $x^k$ denotes $x$ at the $k$-th computation step and $x_j^k$ denotes the $j$-th component of $x^k$. For any matrix $X$, $X^k$ denotes $X$ at the $k$-th computation step, $X_j^k$ denotes the $j$-th column of $X^k$, and $X_{ij}^k$ denotes the element in the $i$-th row and $j$-th column of $X^k$.

## 2. AN INEXACT INTERIOR POINT ALGORITHM

The perturbed KKT conditions for (1.3) with a positive $\mu$ is the nonlinear system

$$F_\mu(x,y,z) \equiv \begin{pmatrix} Ax-b \\ A^Ty+z-c \\ XZe-\mu e \end{pmatrix} = 0, \quad x \geq 0,\ z \geq 0. \tag{2.1}$$

The parameter $\mu > 0$ is referred to as the $\mu$-complementarity parameter and the set of triples $(x,y,z)$ that satisfy (2.1) for all $\mu > 0$ is called the (primal-dual) central path. Here we have adopted the notation $(u,v,w) = (u^T, v^T, w^T)^T$.

Let $(x^k, y^k, z^k)$ with $(x^k, z^k) > 0$ be the iterate at interior point iteration $k$ and consider the perturbed KKT condition (2.1). Newton's method defines the equation of directional change (the Newton step equation)

$$F'_{\mu^k}(x^k,y^k,z^k) \begin{pmatrix} \Delta x^k \\ \Delta y^k \\ \Delta z^k \end{pmatrix} = -F_{\mu^k}(x^k,y^k,z^k). \tag{2.2}$$

If the linear system of equations is solved approximately, then equation (2.2) has a residual error $r^k$ that satisfies

$$F'_{\mu^k}(x^k,y^k,z^k) \begin{pmatrix} \Delta x^k \\ \Delta y^k \\ \Delta z^k \end{pmatrix} = -F_{\mu^k}(x^k,y^k,z^k) + r^k. \tag{2.3}$$

The residual $r^k$ will be partioned, $r^k = (\bar{r}^k, \hat{r}^k, \tilde{r}^k)$ into the primal infeasibility $\bar{r}^k$, dual infeasibility $\hat{r}^k$, and deviation in complementarity $\tilde{r}^k$. An inexact Newton step (2.3) is an approximate solution of the Newton step equation from (1.3)

$$F'(x^k,y^k,z^k) \begin{pmatrix} \Delta x^k \\ \Delta y^k \\ \Delta z^k \end{pmatrix} = -F(x^k,y^k,z^k) + r_F^k, \tag{2.4}$$

where

$$r_F^k = \mu^k \begin{pmatrix} 0 \\ 0 \\ e \end{pmatrix} + r^k.$$

This follows immediately since $F' = F'_\mu$ and $F_\mu = F - \mu\,(0,0,e)$.

A typical choice of the $\mu$-complementarity parameter at iteration $k$ is

$$\mu_k = \beta_1 \frac{(x^k)^T z^k}{n}, \tag{2.5}$$

where $\beta_1 \in [0,1)$ is the centering parameter. Bellavia [1] observed that

$$\|F(x^k,y^k,z^k)\|_2 \geq \frac{(x^k)^T z^k}{\sqrt{n}},$$

so if

$$\|r^k\|_2 \leq \eta^k (x^k)^T z^k, \tag{2.6}$$

then

$$\begin{aligned} \|r_F^k\|_2 &\leq \mu^k \sqrt{n} + \|r^k\|_2 \\ &\leq \frac{(x^k)^T z^k}{\sqrt{n}} \left(\beta_1 + \eta^k \sqrt{n}\right) \\ &\leq \left(\beta_1 + \eta^k \sqrt{n}\right) \|F(x^k, y^k, z^k)\|_2. \end{aligned} \tag{2.7}$$

Hence the sequence $\{\beta_1 + \eta^k \sqrt{n}\}$ can be regarded as a forcing sequence of inexact Newton methods [18] applied to the nonlinear system (1.3) ignoring the nonnegativity.
Numerical testing [20] indicates that $\eta^k < (1 - \beta_1)/\sqrt{n}$ is overly restrictive and that we can tolerate a much larger error in (2.3). To see this, consider the $\| \cdot \|_1$, then

$$\|F(x^k, y^k, z^k)\|_1 \geq (x^k)^T z^k.$$

If

$$\|r^k\|_1 \leq \eta^k (x^k)^T z^k,$$

then

$$\begin{aligned} \|r_F^k\|_1 &\leq \mu^k n + \|r^k\|_1 \\ &\leq (x^k)^T z^k \left(\beta_1 + \eta^k\right) \\ &\leq \left(\beta_1 + \eta^k\right) \|F(x^k, y^k, z^k)\|_1. \end{aligned}$$

This suggests solving the linear system (2.3) with an accuracy

$$\|r^k\|_1 \leq \eta^k \, (x^k)^T z^k \text{ for } \; 0 \leq \eta^k < 1 - \beta_1 \tag{2.8}$$

will be sufficient to achieve convergence. This forcing sequence is independent of the number of primal unknowns $n$.
Similar derivation can be done using $\| \cdot \|_\infty$ that leads to an even more restrictive forcing sequence than (2.7). To see this consider

$$\|r_F^k\|_\infty \leq \frac{(x^k)^T z^k}{n} \left(\beta_1 + \eta^k n\right) \leq \left(\beta_1 + \eta^k n\right) \|F(x^k, y^k, z^k)\|_\infty. \tag{2.9}$$

The number of unknowns of problem (2.3) is $m + 2n$. However, by noticing the structure of the Jacobian matrix

$$F'_{\mu^k}(x^k, y^k, z^k) = \begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ Z^k & 0 & X^k \end{bmatrix},$$

it will be evident that the linear system can be reduced to a system with either $m+n$ or $m$ unknowns. However, in both cases the error $\tilde{r}^k$ in the complementarity equation is zero.

Bellavia [1] establishes global convergence results for an inexact interior point method by interpreting it as an inexact Newton method. The convergence theory and implementation [1, 21] is based on solving the inexact Newton equation (2.4) with a forcing sequence $\eta^k < (1-\beta_1)/\sqrt{n}$.

In this paper, we prove global convergence results based on the reduced linear systems (the augmented systems or the normal equations), and we bound the residual $r^k$ by

$$\|r^k\| \le \eta^k (x^k)^T z^k \quad \text{for} \quad 0 \le \eta^k < 1. \tag{2.10}$$

The norm $\|\cdot\|$ can be any norm that satsfies

$$\max\{\|\bar{r}^k\|, \|\hat{r}^k\|\} \le \|(\bar{r}^k, \hat{r}^k)\|.$$

The idea behind inexact interior point algorithms is to derive a stopping criterion to the iterative linear system solvers that minimizes the computational effort involved in computing the search directions that guarantee global convergence. It is evident that termination criteria (2.8) may be more demanding to satisfy than termination criteria (2.10) especially for larger (close to 1) centering parameter $\beta_1$.

Considering the inaccuracy in the reduced systems that are actually solved in interior point methods seems to be a realistic approach. In our approach, we show global convergence of the complementarity gap, the norms of the primal residual and the dual residual to zero. The global convergence of these three imply the global convergence of the inexact Newton method (2.4). As shown in [1], the global convergence of the inexact Newton method further implies the global convergence of the inexact interior point method. Thus our approach deals directly with the inexact interior point algorithm other than the inexact Newton method as in [1]. In [12] an inexact interior point method is derived using error tolerances

$$\|\bar{r}^k\| \le \varepsilon_x^k \text{ and } \|\hat{r}^k\| \le \varepsilon_y^k,$$

where $\varepsilon_x^k, \varepsilon_y^k \to 0$. These tolerances depend on an estimate on the smallest singular value of $A$.

Let $\tau_1, \tau_2 \epsilon (0,1]$ and $\tau_3 \epsilon [0,1)$. For error tolerances

$$\begin{aligned} \|\bar{r}^k\| &\le (1-\tau_1)\|Ax^k - b\|, \\ \|\hat{r}^k\| &\le (1-\tau_2)\|A^T y^k + z^k - c\|, \\ \|\tilde{r}^k\| &\le \tau_3 \frac{(x^k)^T z^k}{n}, \end{aligned}$$

[19, Theorem 4.3] shows that the method terminates after a finite number of steps and thus has global convergence. For $(x^k, y^k, z^k)$ primal and dual feasible all further iterates will remain feasible.

To simplify the notation, we follow [5] and introduce the residuals

$$\begin{aligned} \xi^k &= b - Ax^k \\ \zeta^k &= c - A^T y^k - z^k. \end{aligned}$$

Let $G^k$ be defined by $G^k = (Z^k)^{-1} X^k$. Solving for $\Delta z^k$ in (2.2) leads to a $(n+m)\times(n+m)$ indefinite augmented system

$$\begin{bmatrix} O & A \\ A^T & -(G^k)^{-1} \end{bmatrix} \begin{pmatrix} \Delta y^k \\ \Delta x^k \end{pmatrix} = \begin{pmatrix} \xi^k \\ z^k + \zeta^k - (X^k)^{-1}\mu^k e \end{pmatrix} \tag{2.11}$$

and

$$\Delta z^k = (X^k)^{-1}(\mu^k e - Z^k \Delta x^k) - z^k. \tag{2.12}$$

An inexact solution of (2.11) satisfies

$$\begin{bmatrix} O & A \\ A^T & -(G^k)^{-1} \end{bmatrix} \begin{pmatrix} \Delta y^k \\ \Delta x^k \end{pmatrix} = \begin{pmatrix} \xi^k \\ z^k + \zeta^k - (X^k)^{-1}\mu^k e \end{pmatrix} + \begin{pmatrix} \bar{r}^k \\ \hat{r}^k \end{pmatrix}. \tag{2.13}$$

After computing $(\Delta y^k, \Delta x^k)$, $\Delta z^k$ can be found from (2.12). The approximate step $(\Delta y^k, \Delta x^k, \Delta z^k)$ is thus an inexact Newton step of (2.3) with $r^k = (\bar{r}^k, \hat{r}^k, 0)$. This is the form used in [5, 6].
Solving for $\Delta x^k$ in (2.11) gives the normal equations with $m$ unknowns

$$AG^k A^T \Delta y^k = AG^k(z^k - (X^k)^{-1}\mu^k e + \zeta^k) + \xi^k. \tag{2.14}$$

An inexact solution for (2.14) satisfies

$$AG^k A^T \Delta y^k \quad = \quad AG^k(z^k - (X^k)^{-1}\mu^k e + \zeta^k) + \xi^k + \bar{r}^k. \tag{2.15}$$

After computing $\Delta y^k$ then $\Delta x^k$ and $\Delta z^k$ are computed from

$$\Delta x^k \quad = \quad -G^k(c - A^T(y^k + \Delta y^k) - \mu^k (X^k)^{-1} e) \tag{2.16}$$
$$\Delta z^k \quad = \quad (X^k)^{-1}(\mu^k e - Z^k \Delta x^k) - z^k. \tag{2.17}$$

The linear system (2.13) is equivalent to (2.15) and (2.16) when $\hat{r}^k = 0$ and the approximate step $(\Delta y^k, \Delta x^k, \Delta z^k)$ is an inexact Newton step of (2.3) with $r^k = (\bar{r}^k, 0, 0)$.
Consider the dual and complementarity equations (2.3) and for simplicity eliminate the iteration index $k$.

$$\begin{bmatrix} O & A^T & I \\ Z & 0 & X \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} \zeta \\ -XZe + \mu e \end{pmatrix} + \begin{pmatrix} \tilde{r} \\ \hat{r} \end{pmatrix}. \tag{2.18}$$

Let $\widetilde{\Delta z} = \Delta z - \tilde{r}$. The linear shift $-\tilde{r}$ in $\Delta z$ will give

$$\begin{bmatrix} A & 0 & 0 \\ O & A^T & I \\ Z & 0 & X \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \widetilde{\Delta z} \end{pmatrix} = \begin{pmatrix} \xi \\ \zeta \\ -XZe + \mu e \end{pmatrix} + \begin{pmatrix} \bar{r} \\ 0 \\ \hat{r} - X\tilde{r} \end{pmatrix}.$$

Moving the residual from the dual equation to the complementarity equation is used to simplify the analysis in [8]. If $A = [B\ N]$ where $B$ is a $m \times m$ nonsingular matrix then a linear shift $-(B^{-1}\bar{r}, 0)$ in $\Delta x$ will give a zero residual in the primal equation also

$$\begin{bmatrix} A & 0 & 0 \\ O & A^T & I \\ Z & 0 & X \end{bmatrix} \begin{pmatrix} \widetilde{\Delta x} \\ \Delta y \\ \widetilde{\Delta z} \end{pmatrix} = \begin{pmatrix} \xi \\ \zeta \\ -XZe + \mu e \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \hat{r} - X\tilde{r} - Z\begin{pmatrix} B^{-1}\bar{r} \\ 0 \end{pmatrix} \end{pmatrix}.$$

The algorithm we discuss in this paper is a variant of the infeasible primal-dual interior point algorithm by Kojima et al. [11]. For any given accuracy $\epsilon > 0$ required for the total complementarity, any tolerance $\epsilon_p > 0$ for the primal feasibility, any tolerance $\epsilon_d > 0$ for the dual feasibility define

$$
\begin{aligned}
\mathcal{N} = \{(x,y,z) \in \mathcal{Q} :& \\
x_j z_j \quad & \geq \gamma x^T z/n \text{ for } j = 1,2,\ldots,n, && (2.19)\\
x^T z \quad & \geq \gamma_p \|Ax - b\| \text{ or } \|Ax - b\| \leq \epsilon_p, && (2.20)\\
x^T z \quad & \geq \gamma_d \|A^T y + z - c\| \text{ or } \|A^T y + z - c\| \leq \epsilon_d\}, && (2.21)
\end{aligned}
$$

where $\mathcal{Q} = \{(x,y,z) \in \Re^{n+m+n} : x > 0, z > 0\}$. The constants $0 < \gamma < 1$, $\gamma_p > 0$, $\gamma_d > 0$ will in a weak sense depend on the starting point, but will be chosen so that the neighborhood $\mathcal{N}$ is as large as possible.

Further, let $\omega^*$ be any large number and let $\epsilon^* = \min\{\epsilon, \gamma_p \epsilon_p, \gamma_d \epsilon_d\}$. Then the neighborhood [11]

$$\mathcal{N}^* = \{(x,y,z) \in \mathcal{N} : \epsilon^* \leq x^T z^k \leq \omega^*\}, \tag{2.22}$$

is a compact set. We will show that either $(x^k, y^k, z^k) \in \mathcal{N}^*$ or satisfies the *termination criteria*

$$(x^k)^T z^k \leq \epsilon, \|Ax^k - b\| \leq \epsilon_p \text{ and } \|A^T y^k + z^k - c\| \leq \epsilon_d, \tag{2.23}$$

or

$$(x^k)^T z^k > \omega^*, \tag{2.24}$$

after a finite number of steps.

The inequalities $x_j^k z_j^k \geq \gamma (x^k)^T z^k/n$ for $j = 1,2,\ldots,n$, in (2.19) prevent the generated sequence $\{(x^k, y^k, z^k)\}$ from reaching the boundary of $\mathcal{Q}$ before the total complementarity $(x^k)^T z^k$ becomes 0. On the other hand, the inequalities (2.20) and (2.21) prevent the possibility of the generated sequence $\{(x^k, y^k, z^k)\}$ converging to an infeasible complementary solution [11].

We choose any initial point $(x^1, y^1, z^1) \in \mathcal{Q}$ and parameters $\gamma, \gamma_p, \gamma_d$ and $\omega^*$ so that $(x^1, y^1, z^1) \in \mathcal{N}$ and $(x^1)^T z^1 \leq \omega^*$. Let $0 < \beta_1 < \beta_2 < \beta_3 < 1$. We state the algorithm below:

**Algorithm 2.1.** *Inexact Infeasible Primal-Dual Algorithm*

Step 1. *Set* $k = 1$. *Assume* $(x^1, y^1, z^1) \in \mathcal{N}$ *such that* $(x^1)^T z^1 \leq \omega^*$.

Step 2. *If (2.23) or (2.24) is satisfied then terminate.*

Step 3. *Let* $\mu^k = \beta_1 (x^k)^T z^k/n$.

Step 4. *Compute the inexact solution* $(\Delta x^k, \Delta y^k, \Delta z^k)$ *of (2.2).*

Step 5. *Let* $0 < \bar{\alpha}^k < 1$ *such that*

$$(x^k, y^k, z^k) + \alpha(\Delta x^k, \Delta y^k, \Delta z^k) \in \mathcal{N}, \tag{2.25}$$

$$(x^k + \alpha \Delta x^k)^T (z^k + \alpha \Delta z^k) \leq (1 - \alpha(1 - \beta_2))(x^k)^T z^k, \tag{2.26}$$

*hold for every* $\alpha \in (0, \bar{\alpha}^k]$.

Step 6. *Choose a primal step length* $\alpha_p^k \in [\bar{\alpha}^k, 1]$, *a dual step length* $\alpha_d^k \in [\bar{\alpha}^k, 1]$ *and a new iterate*

$$(x^{k+1}, y^{k+1}, z^{k+1}) = (x^k + \alpha_p^k \Delta x^k, y^k + \alpha_d^k \Delta y^k, z^k + \alpha_d^k \Delta z^k), \tag{2.27}$$

*such that*

$$\begin{aligned}(x^{k+1}, y^{k+1}, z^{k+1}) &\in \mathcal{N}, & (2.28)\\ {(x^{k+1})}^T z^{k+1} &\leq (1 - \bar{\alpha}^k(1 - \beta_3)){(x^k)}^T z^k. & (2.29)\end{aligned}$$

Step 7. *Increase k by 1. Go to Step 2.*

In this algorithm we take relatively short steps when the search directions are computed to a relatively low accuracy. In Section 4. we will show the existance of an $\bar{\alpha}^k > 0$ for all $k$ as long as (2.23) is not satisfied for feasible problems.

## 3. CONVERGENCE

For $(x^k, y^k, z^k) \in \mathcal{N}$ the following inequalities will be used in the discussion of the algorithm.

$${(x^k)}^T z^k \geq \gamma_p \|Ax^k - b\| \text{ or } \|Ax^k - b\| \leq \epsilon_p, \quad (3.1)$$

$${(x^k)}^T z^k \geq \gamma_d \|A^T y^k + z^k - c\| \text{ or } \|A^T y^k + z^k - c\| \leq \epsilon_d, \quad (3.2)$$

$${(x^k)}^T \Delta z^k + {(\Delta x^k)}^T z^k = -(1 - \beta_1){(x^k)}^T z^k, \quad (3.3)$$

$$x_i^k \Delta z_i^k + \Delta x_i^k z_i^k = \beta_1 {(x^k)}^T z^k / n - x_i^k z_i^k, \quad (3.4)$$

Inequalities (3.1) and (3.2) follow from $(x^k, y^k, z^k) \in \mathcal{N}$ and (2.19) and (2.20). Equation (3.3) and (3.4) follow from simple manipulations of (2.12) with $\mu^k = \beta_1 {(x^k)}^T z^k / n$.

Let

$$0 \leq \eta_{\max} < \min\left\{\frac{\beta_1}{\max\{\gamma_p, \gamma_d\}}, 1\right\}. \quad (3.5)$$

The neighborhood $\mathcal{N}$ is made large by making $\gamma_p$ and $\gamma_d$ small and $\eta_{\max} < 1$.

The inexact Newton direction $(\Delta x^k, \Delta y^k, \Delta z^k)$ computed in (2.13) and (2.12) satisfies

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ Z^k & 0 & X^k \end{bmatrix} \begin{pmatrix} \Delta x^k \\ \Delta y^k \\ \Delta z^k \end{pmatrix} = \begin{pmatrix} b - Ax^k \\ c - A^T y^k - z^k \\ \mu^k e - X^k Z^k e \end{pmatrix} + \begin{pmatrix} \bar{r}^k \\ \hat{r}^k \\ 0 \end{pmatrix}, \quad (3.6)$$

where the residual satsifies

$$\|(\bar{r}^k, \hat{r}^k)\| \leq \eta^k \, {(x^k)}^T z^k, \quad (3.7)$$

for

$$0 \leq \eta^k \leq \eta_{\max}, \quad (3.8)$$

Note that $\eta^k {(x^k)}^T z^k = \frac{\eta^k n}{\beta_1} \mu^k$.

The coefficient matrix on the left hand side of (3.6) is $F'_\mu$ which is nonsingular and continuous for $(x, y, z) \in \mathcal{N}^*$ defined in (2.22). Since $\|(\bar{r}^k, \hat{r}^k)\|$ in (3.7) is bounded for $(x^k, y^k, z^k) \in \mathcal{N}^*$ the inexact Newton direction $(\Delta x^k, \Delta y^k, \Delta z^k)$ determined in (3.6) of equations is well defined (in the sense that for a residual that satisfies (3.7) and (3.15) is unique) and is bounded over the compact set $\mathcal{N}^*$. Hence there exists a positive constant $\tau$ such that the inexact Newton direction $(\Delta x^k, \Delta y^k, \Delta z^k)$ from (3.6) computed at Step 4 of every interior point iteration satisfies the inequalities

$$|\Delta x_i^k \Delta z_i^k - \gamma {(\Delta x^k)}^T \Delta z^k / n| \leq \tau \text{ and } |{(\Delta x^k)}^T \Delta z^k| \leq \tau. \quad (3.9)$$

From equation (3.5) we see that the choice of $\eta$ depends on the values of $\gamma_p$ and $\gamma_d$. Thus we can relax the termination criteria (3.7) for the iterative method if we carefully choose $\gamma_p$ and $\gamma_d$. Further, to avoid primal or dual infeasibility $\gamma_p$ and $\gamma_d$ should have approximately the same values. This prevents the norm of the primal or dual residual from converging to zero faster than the other which might lead to an infeasible solution. Since by the definition of the neighborhood $\mathcal{N}$ the complementarity gap, the norms of the primal and dual residuals are supposed to converge to zero at approximately the same rate then termination criteria (3.7) avoids unnecessarily many iterations as would have been the case if we had used

$$\|(\bar{r}^k, \hat{r}^k)\| \leq \eta^k \ \max \left\{ \|Ax^k - b\|, \|A^T y^k + z^k - c\| \right\},$$

or $\|\bar{r}^k\| \leq \eta^k \ \|Ax^k - b\|\|$, see [5].

We will now show that for an inexact Newton direction from Step 4 we can find $0 < \bar{\alpha}^k < 1$ so that the conditions (2.25) and (2.26) in Step 5 are satisfied. It will be shown that for a given $\mathcal{N}^*$ there exists $\alpha^* > 0$ such that $\alpha^* \leq \bar{\alpha}^k$. Let $\xi(\alpha)$ and $\zeta(\alpha)$ be defined by

$$\begin{aligned} \xi(\alpha) &\equiv b - A(x^k + \alpha \Delta x^k) = \xi^k - \alpha A \Delta x^k, && (3.10) \\ \zeta(\alpha) &\equiv c - A^T(y^k + \alpha \Delta y^k) - z^k = \zeta^k - \alpha(A^T \Delta y^k + \Delta z^k). && (3.11) \end{aligned}$$

For $\Delta y^k$ and $\Delta x^k$ given by (2.13), and $\Delta z^k$ by (2.12) it follows from (3.6) that the expressions for $\xi(\alpha)$ and $\zeta(\alpha)$ simplify to

$$\begin{aligned} \xi(\alpha) &= (1-\alpha)\xi^k - \alpha \bar{r}^k, && (3.12) \\ \zeta(\alpha) &= (1-\alpha)\zeta^k - \alpha \hat{r}^k. && (3.13) \end{aligned}$$

Define the real-valued functions $f_i, i = 1, 2, \ldots, n$, $g_p, g_d$, and $h$ as follows:

$$\begin{aligned} f_i(\alpha) &= (x_i^k + \alpha \Delta x_i^k)(z_i^k + \alpha \Delta z_i^k) - \gamma (x^k + \alpha \Delta x^k)^T (z^k + \alpha \Delta z^k)/n, \\ g_p(\alpha) &= (x^k + \alpha \Delta x^k)^T (z^k + \alpha \Delta z^k) - \gamma_p \|\xi(\alpha)\|, \\ g_d(\alpha) &= (x^k + \alpha \Delta x^k)^T (z^k + \alpha \Delta z^k) - \gamma_d \|\zeta(\alpha)\|, \text{ and} \\ h(\alpha) &= (1 - \alpha(1-\beta_2))(x^k)^T z^k - (x^k + \alpha \Delta x^k)^T (z^k + \alpha \Delta z^k). \end{aligned}$$

Consider Step 5 in the Inexact Infeasible Primal-Dual Algorithm. From the definition of the neighborhood $\mathcal{N}$ condition (2.25) is equivalent to

$$\begin{aligned} &f_i(\alpha) \geq 0 \ (i = 1, 2, \ldots, n), \\ &g_p(\alpha) \geq 0 \quad \text{or} \quad \|\xi(\alpha)\| \leq \epsilon_p, \\ &g_d(\alpha) \geq 0 \quad \text{or} \quad \|\zeta(\alpha)\| \leq \epsilon_d, \end{aligned}$$

for $0 < \alpha \leq \bar{\alpha}^k$. Similarly, (2.26) is equivalent to

$$h(\alpha) \geq 0,$$

for $0 < \alpha \leq \bar{\alpha}^k$.

Consider the function $g_p$ and the simplified expression for $\xi$ in (3.12). Then

$$\begin{aligned} g_p(\alpha) &\geq (x^k + \alpha\Delta x^k)^T(z^k + \alpha\Delta z^k) - \gamma_p(1-\alpha)\|\xi^k\| - \gamma_p\alpha\|\bar{r}^k\| \\ &\geq (x^k)^T z^k + \alpha[(\Delta x^k)^T z^k + (x^k)^T\Delta z^k] + \alpha^2(\Delta x^k)^T\Delta z^k \\ &\quad -(1-\alpha)(x^k)^T z^k - \gamma_p\alpha\|\bar{r}^k\| \\ &\geq \alpha\beta_1(x^k)^T z^k + \alpha^2(\Delta x^k)^T\Delta z^k - \gamma_p\alpha\|\bar{r}^k\|. \end{aligned} \tag{3.14}$$

using (2.20), and (3.3). By (3.7)

$$\|\bar{r}^k\| \quad \leq \quad \|(\bar{r}^k, \hat{r}^k)\| \leq \eta^k\,(x^k)^T z^k, \tag{3.15}$$

the condition on the norms $\|\cdot\|$, and $\eta^k \leq \eta_{\max}$ lead to

$$g_p(\alpha) \geq \alpha\,(\beta_1 - \eta_{\max}\,\gamma_p)\,(x^k)^T z^k + \alpha^2(\Delta x^k)^T\Delta z^k.$$

For $(x^k, y^k, z^k) \in \mathcal{N}^*$ and (3.9) $(x^k)^T z^k \geq \epsilon^*$ and $(\Delta x^k)^T\Delta z^k \geq -\tau$ we have

$$g_p(\alpha) \geq \alpha\,[(\beta_1 - \eta_{\max}\,\gamma_p)\,\epsilon^* - \alpha\tau]\,. \tag{3.16}$$

Hence for $\alpha \geq \frac{(\beta_1 - \eta_{\max}\,\gamma_p)\epsilon^*}{\tau}$ we see that $g_p(\alpha) \geq 0$.
Similarly, for the function $g_d$ and

$$\|\hat{r}^k\| \quad \leq \quad \|(\bar{r}^k, \hat{r}^k)\| \leq \eta^k\,(x^k)^T z^k,$$

then

$$\begin{aligned} g_d(\alpha) &\geq (x^k + \alpha\Delta x^k)^T(z^k + \alpha\Delta z^k) - \gamma_d(1-\alpha)\|\zeta^k\| - \alpha\gamma_d\|\hat{r}^k\| \\ &\geq \alpha\beta_1(x^k)^T z^k + \alpha^2(\Delta x^k)^T\Delta z^k - \alpha\gamma_d\|\hat{r}^k\| \\ &\geq \alpha\,(\beta_1 - \eta_{\max}\,\gamma_d)\,(x^k)^T z^k + \alpha^2(\Delta x^k)^T\Delta z^k \\ &\geq \alpha\,[(\beta_1 - \eta_{\max}\,\gamma_d)\,\epsilon^* - \alpha\tau] \geq 0. \end{aligned} \tag{3.17}$$

Hence for $\alpha \geq \frac{(\beta_1 - \eta_{\max}\,\gamma_d)\epsilon^*}{\tau}$, we see that $g_d(\alpha) \geq 0$.
Next consider

$$\begin{aligned} f_i(\alpha) &= (x_i^k + \alpha\Delta x_i^k)(z_i^k + \alpha\Delta z_i^k) - \gamma(x^k + \alpha\Delta x^k)^T(z^k + \alpha\Delta z^k)/n \\ &= (x_i^k z_i^k - \gamma{(x^k)}^T z^k/n)(1-\alpha) + \alpha\beta_1(1-\gamma){(x^k)}^T z^k/n + \\ &\quad (\Delta x_i^k\Delta z_i^k - \gamma{(\Delta x^k)}^T\Delta z^k/n)\alpha^2 \\ &\geq \beta_1(1-\gamma)(\epsilon^*/n)\alpha - \tau\alpha^2 = \alpha[\beta_1(1-\gamma)(\epsilon^*/n) - \tau\alpha]. \end{aligned} \tag{3.18}$$

Inequality (3.18) follows from application of (3.3), (3.4) and (3.9) on the expression for $f_i(\alpha)$ above.
Next consider

$$\begin{aligned} h(\alpha) &= (1 - \alpha(1-\beta_2)){(x^k)}^T z^k - (x^k + \alpha\Delta x^k)^T(z^k + \alpha\Delta z^k) \\ &= \alpha(\beta_2 - \beta_1){(x^k)}^T z^k + \alpha^2{(\Delta x^k)}^T\Delta z^k \\ &\geq \alpha[(\beta_2 - \beta_1)\epsilon^* - \alpha\tau]. \end{aligned} \tag{3.19}$$

The inequality (3.19) follows from application of (3.3) and (3.9).
Let $\eta_{\max}$ be given by (3.5), $\bar{\gamma} = \max\{\gamma_p, \gamma_d\}$ and define

$$\alpha^* = \min\left\{1, \frac{(\beta_1 - \eta_{\max}\,\bar{\gamma})\epsilon^*}{\tau}, \frac{\beta_1(1-\gamma)\epsilon^*}{n\tau}, \frac{(\beta_2 - \beta_1)\epsilon^*}{\tau}\right\}. \tag{3.20}$$

For $\alpha^*$ defined in (3.20) we observe that for every $\alpha \in [0, \alpha^*]$

$$f_i(\alpha) \geq 0 \ (i = 1, 2, \ldots, n),$$

$$\begin{aligned} &g_p(\alpha) \geq 0 \quad \text{if} \quad g_p(0) = (x^k)^T z^k - \gamma_p \|Ax^k - b\| \geq 0 \text{ or} \\ &\|A(x^k + \alpha\Delta x^k) - b\| \leq \epsilon_p \text{ if } g_p(0) < 0, \end{aligned} \tag{3.21}$$

$$\begin{aligned} &g_d(\alpha) \geq 0 \quad \text{if} \quad g_d(0) = (x^k)^T z^k - \gamma_d \|A^T y^k + z^k - c\| \geq 0 \text{ or} \\ &\|A^T(y^k + \alpha\Delta y^k) + (z^k + \alpha\Delta z^k) - c\| \leq \epsilon_d \quad \text{if} \quad g_d(0) < 0, \end{aligned} \tag{3.22}$$

$$h(\alpha) \geq 0.$$

Consider (3.21). The condition $g_p(0) < 0$ is equivalent to $\gamma_p\|Ax^k - b\| > (x^k)^T z^k$. Further

$$\begin{aligned} \|A(x^k + \alpha\Delta x^k) - b\| &\leq (1-\alpha)\|Ax^k - b\| + \alpha\,\eta^k (x^k)^T z^k \\ &\leq (1-\alpha)\|Ax^k - b\| + \alpha\,\eta^k\,\gamma_p\|Ax^k - b\| \\ &\leq (1 - \alpha(1-\beta_1))\|Ax^k - b\| \\ &\leq \|Ax^k - b\| \end{aligned} \tag{3.23}$$

from the choice of $\eta^k$ in (3.5). From the definition of $\mathcal{N}$ and Step 5 of Algorithm 2.1 we know the iterate $(x^k, y^k, z^k)$ generated satisfies $(x^k, y^k, z^k) \in \mathcal{N}$ for all $k$. Thus $g_p(0) < 0$ implies that $\|Ax^k - b\| \leq \epsilon_p$. From (3.23) it follows that $\|A(x^k + \alpha\Delta x^k) - b\| \leq \epsilon_p$ if $g_p(0) < 0$. This verifies observation (3.21).
Next we note also that $\eta^k\,\gamma_d < \beta_1$ for $\eta^k \leq \eta_{\max}$ and $\eta_{\max}$ defined by (3.5). By a similar argument as for (3.21), observation (3.22) follows.
Thus, we have shown that there exists an $\alpha^* > 0$ such that $\bar{\alpha}^k \geq \alpha^*$ in Algorithm 2.1 for all $k$. By the construction of the real-valued functions $f_i (i = 1, \ldots, n), g_p, g_d$, and $h$, this is equivalent to saying that (2.25) and (2.26) hold for every $\alpha \in [0,\ \alpha^*]$.
The following theorem was stated in [17] without proof. For the completeness we include the proof in this paper.

**Theorem 3.1.** *Choose $\eta_{\max}$ that satisfies (3.5), and let $\alpha^*$ be given by (3.20).*
*Let $\psi^k = \max\left\{\|\xi^k\|, \|\zeta^k\|, (x^k)^T z^k\right\}$. If the norm of $(\bar{r}^k, \hat{r}^k)$ in (2.13) is bounded as in (3.7) then*

$$\psi^{k+1} \leq (1 - \alpha^*(1 - \max\{\beta_3, \eta_{\max}\}))\psi^k < \psi^k.$$

**Proof:** From (3.12) we have

$$\begin{aligned} \|\xi(\alpha_p^k)\| &\leq (1-\alpha_p^k)\|\xi^k\| + \alpha_p^k\|\bar{r}^k\| \\ &\leq (1-\alpha_p^k)\|\xi^k\| + \alpha_p^k\eta_{\max}(x^k)^T z^k \\ &\leq (1 - \alpha_p^k(1-\eta_{\max}))\psi^k, \end{aligned} \tag{3.24}$$

since $\max\left\{\|\xi^k\|, (x^k)^T z^k\right\} \le \psi^k$. Similarly from (3.13) we get

$$\|\zeta(\alpha_d^k)\| \le (1 - \alpha_d^k(1 - \eta_{\max}))\psi^k. \tag{3.25}$$

We also observe from (2.29) that

$$(x^{k+1})^T z^{k+1} \le (1 - \bar{\alpha}^k(1 - \beta_3))\psi^k. \tag{3.26}$$

Using (3.24), (3.25) and (3.26) with $\psi^{k+1}$ we get

$$\begin{aligned}
\psi^{k+1} &\le \max\left\{1 - \alpha_p^k(1 - \eta_{\max}),\, 1 - \alpha_d^k(1 - \eta_{\max}),\, 1 - \bar{\alpha}^k(1 - \beta_3)\right\}\psi^k \\
&\le \max\left\{1 - \bar{\alpha}^k(1 - \eta_{\max}),\, 1 - \bar{\alpha}^k(1 - \beta_3)\right\}\psi^k \\
&\le \max\left\{1 - \bar{\alpha}^k\left(1 - \max\{\eta_{\max}, \beta_3\}\right)\right\}\psi^k \\
&\le \max\left\{1 - \alpha^*\left(1 - \max\{\eta_{\max}, \beta_3\}\right)\right\}\psi^k. \blacksquare
\end{aligned}$$

Theorem 3.1 implies that $\|\xi^k\| \to 0$, $\|\zeta^k\| \to 0$ and $(x^{k+1})^T z^{k+1} \to 0$ as $k \to \infty$. Thus as $k \to \infty$ there exists a $k$ after which the algorithm generates a point $(x^{k+1}, y^{k+1}, z^{k+1})$ which is either an approximate optimal solution $(x^{k+1}, y^{k+1}, z^{k+1})$ satisfying (2.23) or satisfies (2.24). Therefore Algorithm 2.1 is globally convergent.

Theorem 3.1 is based on the termination criteria $\|(\bar{r}^k, \hat{r}^k)\| \le \eta^k (x^k)^T z^k$. A criteria based on a measure that includes the sizes of the dual and primal infeasibility is $\|(\bar{r}^k, \hat{r}^k)\| \le \eta^k (\|\xi^k\| + \|\zeta^k\| + (x^k)^T z^k)$ and the next result shows that for this termination $\eta_k$ must be uniformly bounded away from a constant less than $\frac{1}{2}$ if the step size is close to 1.

Let $\lambda = \left(\frac{1}{\gamma_p} + \frac{1}{\gamma_d} + 1\right)$ and

$$0 < \eta^k \le \eta_{\max} < \min\left\{\frac{\beta_1}{\max\{\gamma_p, \gamma_d\}\,\lambda}, \frac{1}{2}\bar{\alpha}^k(1 - \beta_3)\right\}. \tag{3.27}$$

Let the norm of $(\bar{r}^k, \hat{r}^k)$ in (2.13) be bounded such that

$$\|(\bar{r}^k, \hat{r}^k)\| \le \eta^k \left(\|\xi^k\| + \|\zeta^k\| + (x^k)^T z^k\right),$$

where $\eta^k$ is given by (3.27). It follows that

$$\|(\bar{r}^k, \hat{r}^k)\| \le \eta^k \lambda (x^k)^T z^k. \tag{3.28}$$

Next we consider (3.14) and (3.28) to get

$$\begin{aligned}
g_p(\alpha) &\ge \alpha\beta_1 (x^k)^T z^k + \alpha^2 (\Delta x^k)^T \Delta z^k - \gamma_p \alpha\, \eta^k\, \lambda (x^k)^T z^k \\
&\ge \alpha(\beta_1 - \gamma_p \alpha\, \lambda\, \eta^k)(x^k)^T z^k + \alpha^2 (\Delta x^k)^T \Delta z^k \\
&\ge \alpha((\beta_1 - \gamma_p \alpha\, \lambda\, \eta_{\max})\epsilon^* - \alpha\tau) \ge 0.
\end{aligned} \tag{3.29}$$

Similarly, from (3.17) we can show that

$$g_d(\alpha) \ge \alpha((\beta_1 - \gamma_d \alpha\, \lambda\, \eta_{\max})\epsilon^* - \alpha\tau) \ge 0. \tag{3.30}$$

From (3.18), (3.19),(3.29) and (3.30) there exists a positive $\alpha^*$ that satisfies

$$\alpha^* = \min\left\{1, \frac{(\beta_1 - \max\{\gamma_p, \gamma_d\}\,\lambda\,\eta_{\max})\epsilon^*}{\tau}, \frac{\beta_1(1 - \gamma)\epsilon^*}{n\tau}, \frac{(\beta_2 - \beta_1)\epsilon^*}{\tau}\right\}. \tag{3.31}$$

**Theorem 3.2.** *Let $\psi^k = \|\xi^k\| + \|\zeta^k\| + (x^k)^T z^k$, and the norm of $(\bar{r}^k, \hat{r}^k)$ in (2.13) be bounded such that*

$$\|(\bar{r}^k, \hat{r}^k)\| \leq \eta^k \psi^k,$$

*where $\eta^k$ is given by (3.27). Then*

$$\psi^{k+1} \leq (1 - \alpha^*(1-\beta_3) + 2\eta_{\max})\psi^k.$$

**Proof:** Considering (3.12) and (3.13) leads to

$$\|\xi(\alpha_p)\| \leq (1-\alpha_p)\|\xi^k\| + \alpha_p\, \eta_{\max}\, \psi^k \tag{3.32}$$

$$\|\zeta(\alpha_d)\| \leq (1-\alpha_d)\|\zeta^k\| + \alpha_d\, \eta_{\max}\, \psi^k. \tag{3.33}$$

Hence from (2.29), (3.32) and (3.33) we get

$$\begin{aligned} \psi^{k+1} &\leq (1-\bar{\alpha}^k(1-\beta_3))\psi^k + (\alpha_p + \alpha_d)\eta_{\max}\, \psi^k \\ &\leq (1-\bar{\alpha}^k(1-\beta_3) + 2\, \eta_{\max})\psi^k. \blacksquare \end{aligned}$$

When we compute an exact search direction $(\Delta x^k, \Delta y^k, \Delta z^k)$ at every Step 4 of Algorithm 2.1, Algorithm 2.1 reduces to the globally convergent infeasible interior point algorithm in [11]. In other words the algorithm in this paper is an inexact variant of the algorithm in [11]. It follows that a mixed interior point algorithm, where we alternate between a direct method and an iterative method at distinct interior point iterations, in some manner, to solve the linear system (2.13) is globally convergent if the norm of $(\bar{r}^k, \hat{r}^k)$ is bounded as in (3.7). Hence the class of preconditioners in [20, 22, 23, 24] can be applied in the implementation of mixed interior point algorithm, and still maintain global convergence.

## 4. CONCLUDING REMARKS

In this paper we have reviewed different termination criteria for inexact infeasible interior-point method for linear optimization. Termination criteria for the iterative method that guarantees global convergence of the interior point algorithm have been suggested. For the algorithm discussed in this paper, we have established a relationship between the accuracy of the solution of the linear system and the step length parameter. In particular, for low accuracy we need to carry out short step length in order to have global convergence. We have shown that we can make a hybrid method that alternates between inexact search directions and exact search directions and still have global convergence.

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