A GENERALIZATION OF THE OUTRANKING APPROACH BY INCORPORATING UNCERTAINTY AS INTERVAL NUMBERS

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ABSTRACT
The outranking approach is an important method in multi-criteria decision aid. Nevertheless, it requires information about the preferences of the decision maker that sometimes is difficult to determine. When it is ignored, often this difficulty causes vagueness or imprecision generating a poor performance in the overall model. The vagueness and imprecision generated this way can be considered in the form of interval numbers containing all possible values for the actual parameters. Evidently, the consideration of interval numbers instead of real numbers in the outranking approach demands restructuring the original method. In this study, we present a generalization of the outranking method that incorporates vagueness and imprecision in its parameters through interval numbers. Additionally, we propose a novel function to establish the possibility that an interval number is greater than or equal to another. The results of the experiments indicate that the method proposed is able to characterize uncertainty satisfactorily. Finally, we conclude that the classic outranking approach is a specification of the proposed method and that several intuitive requirements are accomplished.

KEYWORDS: decision aiding, outranking approach, handling uncertainty, preference modelling, interval numbers theory.

MSC: 90B50

RESUMEN
El enfoque de outranking es un método importante en la ayuda a la decisión multicriterio. Sin embargo, requiere información acerca de las preferencias del tomador de decisiones que algunas veces es difícil determinar. Cuando es ignorada, a menudo esta dificultad causa vaguedad o imprecisión que genera un desempeño pobre en el modelo general. La vaguedad e imprecisión generadas de esta forma pueden ser consideradas en forma de números intervalos conteniendo todos los valores posibles de los parámetros reales. Evidentemente, la consideración de números intervalos en lugar de números reales en el método de outranking demanda reestructurar el método original. En este trabajo, presentamos una generalización del método de outranking que incorpora vaguedad e imprecisión en sus parámetros a través de números intervalo. Además, proponemos una función novedosa para establecer la posibilidad de que un número intervalo sea mayor o igual a otro. Los resultados de los experimentos indican que el método propuesto es capaz de caracterizar la incertidumbre de manera satisfactoria. Finalmente, concluimos que el método clásico de outranking es una especificación del método propuesto y que varios requerimientos intuitivos son conseguidos.

PALABRAS CLAVE: ayuda a la decisión, enfoque de outranking, manejo de la incertidumbre, modelación preferencial, teoría de números intervalo.

1. INTRODUCTION

Multi-criteria decision aid (MCDA) is a relevant field within operational research. It encompasses several methods to support decision making. The outranking approach, and in particular the methodologies in the ELECTRE family methods [1, 12] are a popular research field within MCDA. The ELECTRE approach was first introduced in [3]. Later, this first version evolved into a series of variants. From which the most mentioned version is ELECTRE III [11]. A comprehensive survey of the ELECTRE family methods for multiple criteria decision analysis is provided by Figueira et al. in [4].
The ELECTRE family of methods is based on a common concept, the outranking relation. Through the so-called concordance and discordance sets, we can compare pairs of alternatives and exploit an outranking

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relation that leads to a particular choice or ranking of the alternatives. The construction of the outranking relation demands the elicitation of a set of parameters. Generally, this elicitation is not trivial. Probably, the most outstanding limitation of the ELECTRE methods is the need for precise measurements of their parameters. Elicitation of the performance of alternatives, thresholds, and weights and veto power of the criteria, frequently comprises some part of arbitrariness and imprecision. We conclude that the values of the parameters are assigned under conditions of uncertainty that, if ignored, can cause a poor performance in the overall decision model. The information about the value of the parameters is imprecise or uncertain because of limited information-processing capabilities, lack of data and/or ill-determined information. This type of uncertainty may be satisfactorily represented by interval analysis theory.
In this context, some authors have developed interesting extensions of the ELECTRE method for decision-making problems. In [1], Amiri et al. proposed the use of interval data in ELECTRE I and illustrated the approach for the assessment of 15 bank branches in Iran. Vahdani et al. [16] go beyond the interval data and generalize the ELECTRE methods incorporating also interval weights. Balali et al. [2] look for an integration of ELECTRE III and PROMETHEE II decision-making methods with an interval approach. Vahdani et al. [17] propose an extension of the ELECTRE method based on interval-valued fuzzy sets.
Here, we assume that the uncertainty related with the definition of all the outranking's parameters is embraced by interval numbers. Thus, generalizing the classic outranking approach in terms of the interval analysis theory. We propose a novel possibility function based on interval theory's arithmetic that allows us to obtain the credibility index of the outranking relation. We show that this possibility function has some attractive properties, such as a [0, 1] range and monotonicity.
The structure of the paper is structured as follows. In section 2 a brief background fundamental for this work is presented. Our proposal for generalizing the outranking approach through interval theory is presented in section 3, as well as an overview of some of the most outstanding properties of the generalization. Section 4 shows the experimental validation. Finally, section 5 concludes this work.

2. BACKGROUND

2.1. Outranking approach

The outranking approach proceeds by a pair-wise comparison of alternatives in order to rank those alternatives in terms of their priority. Within the framework of the ELECTRE family of methods, the outranking approach supposes that the following is given (see [12]).

- A set A of potential actions (or alternatives).
- A consistent family F of n criteria g_j that allow the actors involved in the decision process (decision maker, DM) to reflect their points of view on the performance of the alternatives. Furthermore, $g_j(\boldsymbol{a}) \in \mathbb{R}$ is called the performance of $\boldsymbol{a} \in A$ in the j^{th} criterion. And, $\forall \boldsymbol{a}' \in A$ and $\forall \boldsymbol{a} \in A$, $g_j(\boldsymbol{a}') \geq g_j(\boldsymbol{a}) \Rightarrow \boldsymbol{a}'$ is at least as good as $\boldsymbol{a}$ in the j^{th} criterion.

The outranking approach assumes that by considering the whole family of criteria it is possible to define a binary relation between $\boldsymbol{a}'$ and $\boldsymbol{a}$ called outranking relation, S. $\boldsymbol{a}'\boldsymbol{S}\boldsymbol{a}$ is true if the values of the performances of $\boldsymbol{a}'$ and $\boldsymbol{a}$ give a sufficiently strong argument for considering the following statement as true (see [12]): "$\boldsymbol{a}'$, with respect to the n criteria, is at least as good as $\boldsymbol{a}$". The sentence "at least as good as" is synonymous with "not worse than".
In order to evaluate the outranking relation S, it is necessary to consider that no all criteria in F are equally important from the DM's perspective. When building S, this situation is considered by means of two parameters assigned to each criterion: the weight and the veto of the criterion. The weight (or importance coefficient) of the j^{th} criterion, w_j, sets the importance of this criterion in relation with the rest of the criteria and is only considered in the definition of the concordance degree (defined below). The veto threshold of the j^{th} criterion, v_j, reflects the capacity given to the j^{th} criterion for rejecting the assertion $\boldsymbol{a}'\boldsymbol{S}\boldsymbol{a}$ without any help of other criteria.
It is also possible to build a binary *marginal* relation between $\boldsymbol{a}'$ and $\boldsymbol{a}$ called S_j. $\boldsymbol{a}'\boldsymbol{S_j}\boldsymbol{a}$ holds if the values $g_j(\boldsymbol{a}')$ and $g_j(\boldsymbol{a})$ give a sufficiently strong argument for considering the following statement as true: "$\boldsymbol{a}'$, with respect to the j^{th} criterion only, is at least as good as $\boldsymbol{a}$".

2.1.1. Concordance and Discordance Concepts

In order to define if the j^{th} criterion is in concordance with the assertion $\boldsymbol{a}'\boldsymbol{S}\boldsymbol{a}$ it must be true that $\boldsymbol{a}'\boldsymbol{S_j}\boldsymbol{a}$. $\boldsymbol{a}'\boldsymbol{S_j}\boldsymbol{a}$ is true iff $g_j(\boldsymbol{a}') \geq g_j(\boldsymbol{a}) - q_j$. Where q_j is a real positive indifference threshold, associated with g_j. q_j represents the hesitation in the internal preference model of the DM about when to declare that $g_j(\boldsymbol{a}')$ and $g_j(\boldsymbol{a})$ are equivalent. The subset of all criteria in F that are in concordance with the assertion $\boldsymbol{a}'\boldsymbol{S}\boldsymbol{a}$ is called the concordant coalition. It is denoted by $C(\boldsymbol{a}'S\boldsymbol{a})$.
The j^{th} criterion is in discordance with the assertion $\boldsymbol{a}'\boldsymbol{S}\boldsymbol{a}$ if the difference $g_j(\boldsymbol{a}) - g_j(\boldsymbol{a}')$ is sufficiently large. This situation is denoted as $\boldsymbol{a}P\boldsymbol{a}'$. $\boldsymbol{a}P\boldsymbol{a}'$ iff $g_j(\boldsymbol{a}) \geq g_j(\boldsymbol{a}') + p_j$, where p_j is the threshold that indicates when the difference is sufficiently large. The subset of all criteria in F that are in discordance with the assertion $\boldsymbol{a}'\boldsymbol{S}\boldsymbol{a}$ is called the discordant coalition. It is denoted by $C(\boldsymbol{a}P\boldsymbol{a}')$ since it can also be viewed as the concordant coalition with the assertion $\boldsymbol{a}P\boldsymbol{a}'$.
Evidently, there might be some criteria that are neither concordant nor discordant with the assertion $\boldsymbol{a}'\boldsymbol{S}\boldsymbol{a}$. These are the criteria j for which it is true $g_j(\boldsymbol{a}) - p_j \leq g_j(\boldsymbol{a}') < g_j(\boldsymbol{a}) - q_j$ with $p_j > q_j$. The subset of F defined by the criteria satisfying this condition is denoted by $C(\boldsymbol{a}Q\boldsymbol{a}')$.

Concordance index

The concordance index, $c(\boldsymbol{a}', \boldsymbol{a})$, reflects the strength of the assertion $\boldsymbol{a}'\boldsymbol{S}\boldsymbol{a}$. $c(\boldsymbol{a}', \boldsymbol{a})$ is defined in ELECTRE III as

$$c(\boldsymbol{a}', \boldsymbol{a}) = \sum_{j=1}^{n} w_j c_j(\boldsymbol{a}', \boldsymbol{a}) \qquad (1)$$

Where

$$c_j(\boldsymbol{a}', \boldsymbol{a}) = \begin{cases} 0 & g_j(\boldsymbol{a}') \leq g_j(\boldsymbol{a}) - p_{j,} \\ 1 & g_j(\boldsymbol{a}') \geq g_j(\boldsymbol{a}) - q_j, \\ \dfrac{g_j(\boldsymbol{a}') - g_j(\boldsymbol{a}) + p_j}{p_j - q_j} & \text{otherwise.} \end{cases}$$

Discordance index

We can be sure that when $g_j(\boldsymbol{a}) - g_j(\boldsymbol{a}')$ is too large (i.e., at least v_j) then the j^{th} criterion is incompatible with the assertion $\boldsymbol{a}'\boldsymbol{S}\boldsymbol{a}$ whatever the other performances are. Nevertheless, the veto effect can occur in different degree for a difference $g_j(\boldsymbol{a}) - g_j(\boldsymbol{a}')$ smaller than v_j as long as $g_j(\boldsymbol{a}) - g_j(\boldsymbol{a}') > p_j$. Based on this, the outranking method builds the following discordance index.

$$d_j(\boldsymbol{a}', \boldsymbol{a}) = \begin{cases} 1 & g_j(\boldsymbol{a}') \leq g_j(\boldsymbol{a}) - v_{j,} \\ 0 & g_j(\boldsymbol{a}') \geq g_j(\boldsymbol{a}) - p_j, \\ \dfrac{g_j(\boldsymbol{a}) - g_j(\boldsymbol{a}') - p_j}{v_j - p_j} & \text{otherwise.} \end{cases}$$

Which leads to a non-discordance predicate based on Eq. (2) (see [5]).

$$Nd(\boldsymbol{a}', \boldsymbol{a}) = \min_{j \in C(\boldsymbol{a}P\boldsymbol{a}')} \{1 - d_j(\boldsymbol{a}', \boldsymbol{a})\}. \qquad (2)$$

Credibility of $\boldsymbol{S}$ and some preferential relations

From $c(\boldsymbol{a}', \boldsymbol{a})$ and $Nd(\boldsymbol{a}', \boldsymbol{a})$ we can build a fuzzy relation $\sigma: A \times A \longrightarrow \mathbb{R}$, where $\sigma \in [0,1]$, such that $\sigma(\boldsymbol{a}', \boldsymbol{a})$ represents the degree of credibility of the statement "$\boldsymbol{a}'$ is at least as good as $\boldsymbol{a}$" as given in Eq. (3).

$$\sigma(\boldsymbol{a}', \boldsymbol{a}) = c(\boldsymbol{a}', \boldsymbol{a}) \cdot Nd(\boldsymbol{a}', \boldsymbol{a}). \qquad (3)$$

Finally, assume there is a $\lambda \in (0.5,1]$ threshold such that $\boldsymbol{a}'\boldsymbol{S}\boldsymbol{a}$ iff $\sigma(\boldsymbol{a}', \boldsymbol{a}) \geq \lambda$. Then for all $(\boldsymbol{a}', \boldsymbol{a}) \in \{A \times A\}$, $\sigma(\boldsymbol{a}', \boldsymbol{a})$ lets us set the following relations of preference between $\boldsymbol{a}'$ and $\boldsymbol{a}$.

Strict preference, P. P is an asymmetric binary relation that indicates that there is clear evidence justifying a significant preference in favor of one of two alternatives. P is denoted as $\boldsymbol{a}'P\boldsymbol{a}$: "$\boldsymbol{a}'$ is strictly preferred to $\boldsymbol{a}$". $\boldsymbol{a}'P\boldsymbol{a} \Leftrightarrow \sigma(\boldsymbol{a}',\boldsymbol{a}) \geq \lambda \wedge \sigma(\boldsymbol{a},\boldsymbol{a}') < 0.5$.
Indifference, I. I is a symmetric relation that indicates that there is clear evidence justifying an equivalence between two alternatives. I is denoted as $\boldsymbol{a}'I\boldsymbol{a}$: "$\boldsymbol{a}'$ is indifferent with $\boldsymbol{a}$". $\boldsymbol{a}'I\boldsymbol{a} \Leftrightarrow \sigma(\boldsymbol{a}',\boldsymbol{a}) \geq \lambda \wedge \sigma(\boldsymbol{a},\boldsymbol{a}') \geq \lambda$.
Weak preference, Q. Q is an asymmetric binary relation that indicates that there is clear evidence justifying a preference in favor of one of two alternatives. Nevertheless, this preference is not significant, so strict preference and indifference are indistinguishable. Q is denoted as $\boldsymbol{a}'Q\boldsymbol{a}$: "$\boldsymbol{a}'$ is weakly preferred to $\boldsymbol{a}$". $\boldsymbol{a}'Q\boldsymbol{a} \Leftrightarrow \sigma(\boldsymbol{a}',\boldsymbol{a}) \geq \lambda \wedge \sigma(\boldsymbol{a}',\boldsymbol{a}) > \sigma(\boldsymbol{a},\boldsymbol{a}') \wedge \neg\boldsymbol{a}'P\boldsymbol{a} \wedge \neg\boldsymbol{a}'I\boldsymbol{a}$.
Incomparability, R. R is a symmetric binary relation indicating that there are no clear evidence that justify any of the above relations. R is denoted as $\boldsymbol{a}'R\boldsymbol{a}$: "$\boldsymbol{a}'$ and $\boldsymbol{a}$ are incomparable". $\boldsymbol{a}'R\boldsymbol{a} \Leftrightarrow \sigma(\boldsymbol{a}',\boldsymbol{a}) < 0.5 \wedge \sigma(\boldsymbol{a},\boldsymbol{a}') < 0.5$.
Where $\wedge$ indicates conjunction and $\neg$ indicates negation.

2.2. Interval analysis theory

2.2.1. Interval arithmetic

An interval number is an imprecise quantity whose true value is contained in a range of numbers. For this reason, interval numbers are naturally expressed as range of numbers:

$$[x^-, x^+] = \{x \in \mathbb{R}: x^- \leq x \leq x^+\}.$$

The idea is simple. Each interval number represents a parameter. The true value of this parameter is unknown -due to problems of inaccuracy in the estimation methods, for instance- but we are certain that it is contained within a lower and an upper bound. It is possible to refer to any $r \in [x^-, x^+]$ as a realization of the interval number x.
If x and y are two interval numbers such that $x = [x^-, x^+]$ and $y = [y^-, y^+]$, then the theory of interval numbers sets the following arithmetic operations between them (see [10]).

$$x + y = [x^- + y^-, x^+ + y^+],$$
$$x - y = [x^- - y^+, x^+ - y^-],$$
$$x \cdot y = [\min\{x^-y^-, x^-y^+, x^+y^-, x^+y^+\}, \max\{x^-y^-, x^-y^+, x^+y^-, x^+y^+\}],$$
$$\frac{x}{y} = [x^-, x^+] \cdot \left[\frac{1}{y^-}, \frac{1}{y^+}\right],$$
$$x = y \Leftrightarrow x^- = y^- \text{ and } x^+ = y^+,$$
$$m(x) = \frac{1}{2}(x^- + x^+),$$
$$w(x) = \frac{1}{2}(x^+ - x^-).$$

2.2.2. Ordering of interval numbers

Several authors have worked on the ranking of interval numbers. For example, Ishibuchi and Tanaka in [6] established that $y \leq x \Leftrightarrow y^- \leq x^- \wedge y^+ \leq x^+$; and that $y < x \Leftrightarrow y \leq x \wedge y \neq x$. They also proposed a method to rank interval numbers through $m(x)$ and $w(x)$ in the following way. $y \leq x \Leftrightarrow m(y) \leq m(x) \wedge w(y) \geq w(x)$; and $y < x \Leftrightarrow y \leq x \wedge y \neq x$. Sengupta and Pal [13] also used $m(x)$ and $w(x)$ to rank interval numbers. They suggested a condition-based method to establish the *grade of acceptability* of the expression "the interval number y is inferior to the interval number x", $y \prec x$:

$$y \prec x \begin{cases} = 0 & m(y) = m(x), \\ \in (0,1) & m(y) < m(x) \wedge y^+ > x^-, \\ \geq 1 & m(y) < m(x) \wedge y^+ \leq x^- \end{cases}$$

and define it as

$$y \prec x = \frac{m(x) - m(y)}{w(x) + w(y)},$$

where $w(x) + w(y) \neq 0$.

Later, Shi et al. in [14] defined a different way to rank interval numbers through a so called grey possibility function -given that it was proposed in the context of grey theory (see [8]), an extension of interval theory. This grey possibility function is defined as

$$p(y \leq x) = \frac{\max\{0, d - max\{0, y^+ - x^-\}\}}{d}$$

where $d = (y^+ - y^-) + (x^+ - x^-) > 0$.

The latter is the most broadly mentioned in the literature.

All these ways to compare and rank interval numbers are highly interesting and intuitively appealing; nevertheless, all of them suffer of at least one of the following problems.

- There are some cases when it is not possible to rank interval numbers.
- They concentrate in the midpoint of the interval, ignoring most of the uncertainty within the intervals.
- They do not consider areas of certainty; that is, areas where it is sure that an interval number is strictly lesser or greater than another. We speak about this in the next section.

In order to take into account these situations when comparing alternatives, we propose a different possibility function that allows us to establish the parameters of the outranking approach as interval numbers. We suppose that due to imprecision/vagueness the parameters of the outranking method (i.e., performance of the alternatives, weights, preference, indifference, veto and pre-veto thresholds) are defined as interval numbers. Therefore, we use the basic terminology of the classic outranking theory combined with the interval numbers theory. Let us start by defining the marginal information of the outranking method.

3. INTERVAL-BASED OUTRANKING APPROACH

In this section, the generalization of the classic outranking approach is presented. This section proceeds by generalizing each component of the classic outranking to an interval based approach.

3.1. Interval based marginal concordance

If $g_j(x) = [x_j^+, x_j^-]$ and $g_j(y) = [y_j^+, y_j^-]$ are the performances of alternatives x and y in the j^{th} criterion, and $p_j = [p_j^+, p_j^-]$ and $q_j = [q_j^+, q_j^-]$ are the preference and indifference thresholds in the same criterion, then we can identify three conditions that can occur when comparing $g_j(x)$ and $g_j(y)$ incorporating these thresholds:

i) $x_j^+ \leq (y_j^- - p_j^+)$.

ii) $x_j^- \geq (y_j^+ - q_j^-)$.

iii) None of the above conditions are fulfilled.

Only when the first condition is met we can be sure that there will *never* be a realization of x that is at least as good as y in the j^{th} criterion. Thus, the level of concordance of this criterion with the expression xSy must be zero if and only if the upper bound of $g_j(x)$ is lesser than or equal to the lower bound of $g_j(y)$ when the highest possible realization of p_j is considered. More formally:

$$c_j(x, y) = 0 \Leftrightarrow x_j^+ \leq (y_j^- - p_j^+).$$

Similarly, only when the second condition is met we can be sure that x will be *always* at least as good as y in the j^{th} criterion. So the level of concordance of this criterion must be one if and only if the lower bound of $g_j(x)$ is greater than or equal to the upper bound of $g_j(y)$ when the lowest possible realization of q_j is considered. More formally:

$$c_j(x, y) = 1 \Leftrightarrow x_j^- \geq (y_j^+ - q_j^-).$$

In the other hand, when none of the above conditions are met then it is true that $(y_j^- - p_j^+) < (y_j^+ - q_j^-)$ and there will be *some* realizations for which the expression "x is at least as good as y in the j^{th} criterion" is met. In order to estimate the proportion of realizations for which this expression might be true we propose the following procedure.

Let $r_j^- = (y_j^- - p_j^+)$ and $r_j^+ = (y_j^+ - q_j^-)$. Then, when comparing $g_j(x) = [x_j^+, x_j^-]$ and $r_j = [r_j^-, r_j^+]$, exactly one of the following scenarios will be true:

(1) $(x_j^- < r_j^-)$ and $(x_j^+ < r_j^+)$.

(2) $(x_j^- < r_j^-)$ and $(x_j^+ \geq r_j^+)$.

(3) $\left(x_j^- \geq r_j^-\right)$ and $\left(x_j^+ < r_j^+\right)$.
(4) $\left(x_j^- \geq r_j^-\right)$ and $\left(x_j^+ \geq r_j^+\right)$.

Scenario (1). When $\left(x_j^- < r_j^-\right)$ and $\left(x_j^+ < r_j^+\right)$, the only zone where x can be at least as good as y in the j^{th} criterion is when there are realizations of $g_j(x)$ and r_j between r_j^- and x_j^+. The proportion of realizations in this zone is given by

$$\delta_{j(1)} = \frac{x_j^+ - r_j^-}{r_j^+ - x_j^-}.$$

This means that the proportion of realizations in the j^{th} criterion that are in favor of xSy will be no more than $\delta_{j(1)}$. Doubtless, it is also possible that none of these realizations is in favor of the expression. So the marginal concordance in the Scenario (1) must be between zero and $\delta_{j(1)}$. More formally:

$$c_{j(1)}(x,y) = \left[0, \delta_{j(1)}\right].$$

Scenario (2). When $\left(x_j^- < r_j^-\right)$ and $\left(x_j^+ \geq r_j^+\right)$, we can be sure that x is at least as good as y in the j^{th} criterion in the following proportion.

$$\mu_{j(2)} = \frac{x_j^+ - r_j^-}{x_j^+ - x_j^-}.$$

While the following proportion express the quantity of realizations where x can be at least as good as y in the j^{th} criterion:

$$\delta_{j(2)} = \frac{r_j^+ - r_j^-}{x_j^+ - x_j^-}.$$

So the proportion of realizations that are in favor of xS_jy is between $\mu_{j(2)}$ and $(\mu_{j(2)} + \delta_{j(2)})$ and the marginal concordance in this scenario is

$$c_{j(2)}(x,y) = \left[\mu_{j(2)}, (\mu_{j(2)} + \delta_{j(2)})\right].$$

The marginal concordance for the scenarios three and four are defined in a very similar way. In general, the marginal credibility of the j^{th} criterion being in concordance with xSy is calculated by

$$c_j(x,y) = \begin{cases} [0,0] & x_j^+ \leq y_j^- - p_j^+, \\ [1,1] & x_j^- \geq y_j^+ - p_j^-, \\ \left[\Delta_{\mathrm{j}}, \Delta_{\mathrm{j}} + \delta_{\mathrm{j}}\right] & \text{otherwise} \end{cases} \tag{4}$$

where $\Delta_{\mathrm{j}} = \left(\mu_{\mathrm{j}}^- \beta_{\mathrm{j}}^- + \mu_{\mathrm{j}}^+ \beta_{\mathrm{j}}^+\right)$,

$$\mu_{\mathrm{j}}^- = \frac{x_j^- - r_j^-}{\delta_s^+ - \delta_i^-},$$

$$\mu_{\mathrm{j}}^+ = \frac{x_j^+ - r_j^+}{\delta_s^+ - \delta_i^-},$$

$$\beta_{\mathrm{j}}^- = \begin{cases} 0 & r_j^- \geq x_j^-, \\ 1 & \text{otherwise.} \end{cases}$$

$$\beta_{\mathrm{j}}^+ = \begin{cases} 0 & r_j^+ \geq x_j^+, \\ 1 & \text{otherwise.} \end{cases}$$

$$\delta_{\mathrm{j}} = \frac{\delta_s^- - \delta_i^+}{\delta_s^+ - \delta_i^-},$$

$$\delta_i^- = \min\left\{x_j^-, r_j^-\right\},$$
$$\delta_i^+ = \max\left\{x_j^-, r_j^-\right\},$$
$$\delta_s^- = \min\left\{x_j^+, r_j^+\right\},$$
$$\delta_s^+ = \max\left\{x_j^+, r_j^+\right\}.$$

3.2. Interval-based marginal discordance

Let $s_j^- = \left(y_j^- - v_j^+\right)$ and $s_j^+ = \left(y_j^+ - u_j^-\right)$. If $v_j = \left[v_j^-, v_j^+\right]$ and $u_j = \left[u_j^-, u_j^+\right]$ are the veto and pre-veto thresholds in the j^{th} criterion, then similarly as above and following the simplification of Mousseau and Dias [9], the marginal discordance of the j^{th} criterion with xSy is defined as

$$d_j(x,y) = \begin{cases} [1,1] & x_j^+ \leq y_j^- - v_j^+, \\ [0,0] & x_j^- \geq y_j^+ - u_j^-, \\ [\Gamma_j, \Gamma_j + \gamma_j] & \text{otherwise} \end{cases} \quad (5)$$

where $\Gamma_j = (\eta_j^- \alpha_j^- + \eta_j^+ \alpha_j^+)$,

$$\eta_j^- = \frac{s_j^- - x_j^-}{\gamma_s^+ - \gamma_i^-},$$
$$\eta_j^+ = \frac{s_j^+ - x_j^+}{\gamma_s^+ - \gamma_i^-},$$
$$\alpha_j^- = \begin{cases} 1 & s_j^- > x_j^-, \\ 0 & \text{otherwise.} \end{cases}$$
$$\alpha_j^+ = \begin{cases} 1 & s_j^+ > x_j^+, \\ 0 & \text{otherwise.} \end{cases}$$

$$\gamma_j = \frac{\gamma_s^- - \gamma_i^+}{\gamma_s^+ - \gamma_i^-},$$

$$\gamma_i^- = \min\{x_j^-, s_j^-\},$$
$$\gamma_i^+ = \max\{x_j^-, s_j^-\},$$
$$\gamma_s^- = \min\{x_j^+, s_j^+\},$$
$$\gamma_s^+ = \max\{x_j^+, s_j^+\}.$$

3.3. Interval-based concordance index

Considering a problem with n criteria, the concordance index associated with xSy is obtained by

$$c(x,y) = \sum_{j=1}^{n} w_j c_j(x,y). \quad (6)$$

With $c(x,y) \in [0,1]$. The importance that the DM assigns to the j^{th} criterion is represented by $\omega_j = [\omega_j^-, \omega_j^+]$. With the purpose of ensuring consistency in the affirmation $c(x,y) \in [0,1]$, we need to be sure that $\sum_{j=1}^{n} \omega_j = 1$. Nevertheless, the uncertainty contained in ω_j (and represented as an interval number) makes this a complicated work. Hence, we establish a procedure of *whitening*[2] in the following way.

$$w_j^- = w_j^+ = \frac{\omega_j^- + \omega_j^+}{\sum\omega_j^- + \sum\omega_j^+}. \quad (7)$$

3.4. Interval-based non-discordance index

The non-discordance level associated with xSy is calculated as

$$Nd(x,y) = \min_{j \in C(yPx)} \{[1,1] - d_j(x,y)\}. \quad (8)$$

Where $C(yPx) = \{j: x_j^- \leq y_j^+ - p_j^-\}$.

Finally, we define the concept of a minimum among a set of interval numbers as follows. Let $\mathcal{B}$ be a set of interval numbers, $b^* \in \mathcal{B}$ is the minimum of $\mathcal{B}$, denoted by $\min\{\mathcal{B}\}$, if and only if $p(b^* \geq b) \leq 0.5$ for all $b \in \mathcal{B}$. $p(\cdot)$ is calculated as in Eq. (11).

3.5. Interval-based outranking

Consequently, we obtain the credibility of xSy in the classic way:

$$\sigma(x,y) = C(x,y) \cdot Nd(x,y). \quad (9)$$

[2] It is not uncommon to find whitening procedures in the interval theory. These procedures define the realization that represents an interval number as a real number. See for example [11].

Moreover, we say that "x is at least as good as y" if and only if the possibility that   is greater than or equal to an uncertain cutting level λ is at least 0.5. More formally:

$$xSy \Leftrightarrow p(\sigma(x,y) \geq \lambda) \geq 0.5. \tag{10}$$

Where $p(\cdot)$ can be based in any of the possibility functions mentioned before. Nonetheless, given the discussion in Section 2.2, and based in (4) and (5), we propose to obtain the possibility that an interval number, $\otimes_1 = [\otimes_1^-, \otimes_1^+]$, is greater than or equal to another interval number, $\otimes_2 = [\otimes_2^-, \otimes_2^+]$, in the following way.

$$p(\otimes_1 \geq \otimes_2) = \begin{cases} 0 & \otimes_1^+ < \otimes_2^-, \\ 1 & \otimes_1^- \geq \otimes_2^+, \\ \dfrac{(2P+\rho)}{2} & \text{otherwise.} \end{cases} \tag{11}$$

Where

$P = (\mu^-\beta^- + \mu^+\beta^+)$,

$$\mu^- = \frac{\otimes_1^- - \otimes_2^-}{\rho_s^+ - \rho_i^-},$$

$$\mu^+ = \frac{\otimes_1^+ - \otimes_2^+}{\rho_s^+ - \rho_i^-},$$

$$\beta^- \begin{cases} 0 & \otimes_2^- - \otimes_1^-, \\ 1 & \text{otherwise.} \end{cases}$$

$$\beta^+ \begin{cases} 0 & \otimes_2^+ - \otimes_1^+, \\ 1 & \text{otherwise.} \end{cases}$$

$\rho = \dfrac{\rho_s^- - \rho_i^+}{\rho_s^+ - \rho_i^-}$,

$$\rho_i^- = \min\{\otimes_1^-, \otimes_2^-\},$$

$$\rho_i^+ = \max\{\otimes_1^-, \otimes_2^-\},$$

$$\rho_s^- = \min\{\otimes_1^+, \otimes_2^+\},$$

$$\rho_s^+ = \min\{\otimes_1^+, \otimes_2^+\}.$$

3.6. Axiomatic structure

To show the plausibility of the procedure, it is necessary to establish some minimal requirements. First, it is natural to suppose that all the parameters based on the possibility function defined in (11) must result in values between zero and one. Second, given that we are looking for a generalization of the outranking approach, both the generalization and the classic outranking must provide the same result when the parameters are the same. Third, if $g_j(x) = [x_j^-, x_j^+]$ increases and/or $g_j(y) = [y_j^-, y_j^+]$ decreases, we can expect the credibility of the asseveration xSy to increase. Conversely, if $g_j(x)$ decreases and/or $g_j(y)$ increases, we can expect the credibility of the asseveration xSy to decrease. We express all this in the form of the following axioms.

Axiom I. Range

The bounds of $c_j(x,y)$, $d_j(x,y)$, $C(x,y)$, $Nd(x,y)$ and $\sigma(x,y)$ are in [0,1].

Axiom II. Generalization

The interval based outranking must be reduced to the classic outranking when the parameters (performance of the alternatives, weights, preference, indifference, veto, pre-veto thresholds and lambda cutting level) are real numbers.

Axiom III. Monotonicity

a) If $g_j(x) = [x_j^-, x_j^+]$ (i.e., x_j^- and/or x_j^+) increases, or if $g_j(y) = [y_j^-, y_j^+]$ (i.e., y_j^- and/or y_j^+) decreases, then xSy must be more credible (i.e., $\sigma(x,y)$ must increase).

b) If $g_j(x) = [x_j^-, x_j^+]$ decreases or if $g_j(y) = [y_j^-, y_j^+]$ increases, then xSy must be less credible.

It can be demonstrated that the proposed procedure meets the axioms. See Appendices A and B for proofs.

4. EXPERIMENTS AND RESULTS

4.1 Representativeness of $c_j(x,y)$ and $d_j(x,y)$.

The first experiment was carried out with the intention of discovering the representativeness of $c_j(x,y)$; namely, how well it can characterize the expression xSy. In this context, Table 1 shows the results of a Montecarlo simulation when comparing ten pairs of alternatives. This simulation randomly generates the parameters of the interval based outranking setting them as intervals. After that, we generate a realization of each parameter by obtaining a real number in the given interval. This allow us to establish if xSy is true for the parameters generated. We do the latter procedure for 1000 times and obtain the average of times xSy is true.
The second column of Table 1 contains an interval number representing the performance of x while the third column contains the performance of y. The bounds of both $g_j(x)$ and $g_j(y)$ are generated in the interval [200000, 2000000]. The fourth column consists of the preference and indifference thresholds. The fifth column has the average of times that xSy was met in the simulation, given as a proportion. Finally, the sixth column is the level of marginal concordance, $c_j(x,y)$, as defined in (4).

Table 1. Simulation of $g_j(x) \geq g_j(y) - q_j$.

#	$g_j(x)$	$g_j(y)$	$p = q$	xSy(%)	c_j
1	[618338,693580]	[682800,1957644]	[74204,99868]	0.04	[0.03,0.09]
2	[236824,309613]	[331958,1743575]	[66143,85200]	0.01	[0.0,0.04]
3	[972268,1435535]	[206448,822273]	[82607,99958]	0.01	[1.0,1.0]
4	[256424,407511]	[1760913,1816184]	[23591,31959]	0.0	[0.0,0.0]
5	[552062,1094819]	[1187692,1554793]	[73324,75213]	0.0	[0.0,0.0]
6	[1217054,1645877]	[1298739,1520698]	[31388,69912]	0.66	[0.37,0.97]
7	[401371,896272]	[256296,1236920]	[61801,73764]	0.47	[0.22,0.72]
8	[1137064,1177550]	[228306,1637655]	[28573,41934]	0.68	[0.67,0.7]
9	[302704,1888840]	[1309211,1364147]	[49413,54092]	0.38	[0.36,0.4]
10	[750555,890113]	[588318,1899060]	[50358,83857]	0.22	[0.18,0.29]

We can see from Table 1 that in all the cases the proportion of times that xSy was met in the simulation is contained in the interval $c_j(x,y)$. This means that $c_j(x,y)$ is representative of xSy in the experiments. Of course, the representation of $c_j(x,y)$ as interval number introduces uncertainty in the form of intervals. Nevertheless, the average uncertainty of $c_j(x,y)$ (i.e., the average of $c_j^+(x,y) - c_j^-(x,y)$ in the 10 results) is 0.138. Which implies a relatively small uncertainty produced by the procedure.
Similar results and conclusions are reached when the experiment is carried out on the marginal discordance, $d_j(x,y)$., and the veto threshold.

Table 2. Simulation of $g_j(x) \leq g_j(y) - p_j$

#	$g_j(x)$	$g_j(y)$	v	Vetoes(%)	d_j
1	[618338,693580]	[682800,1957644]	[264883,714793]	62	[0.58,0.62]
2	[236824,309613]	[331958,1743575]	[234180,236000]	87	[0.85,0.9]
3	[972268,1435535]	[206448,822273]	[383639,630008]	0	[0.0,0.0]
4	[256424,407511]	[1760913,1816184]	[161413,473087]	100	[1.0,1.0]
5	[552062,1094819]	[1187692,1554793]	[161747,299559]	94	[0.75,1.0]
6	[1217054,1645877]	[1298739,1520698]	[587911,683753]	0	[0.0,0.0]
7	[401371,896272]	[256296,1236920]	[280628,740543]	12	[0.04,0.39]
8	[1137064,1177550]	[228306,1637655]	[294673,960914]	9	[0.08,0.1]
9	[302704,1888840]	[1309211,1364147]	[120731,495715]	45	[0.32,0.59]
10	[750555,890113]	[588318,1899060]	[660875,875828]	23	[0.23,0.32]

The uncertainty produced by the procedure in this case is 0.107 that is also small.

4.2 Effectiveness of the novel possibility function

The following experiment consists in the comparison of the possibility function introduced in Eq. (11) and the grey possibility function proposed by Shi et al. in [14]. The goal of the comparison consists in determining the effectiveness of both functions. This effectiveness is given by the accuracy of the functions to determine the real possibility of a given interval number, $\otimes_1$, to be greater than or equal to another interval number, $\otimes_2$. In order to obtain the real possibility of this event, we perform a Montecarlo Simulation where the bounds of $\otimes_1$

and $\otimes_2$ are randomly generated. After that, we obtain a realization of each interval number, $r_1 \in \otimes_1$ and $r_2 \in \otimes_2$, by randomly generating a real number between the corresponding bounds. This allow us to compare both realizations and determine if $r_1 \geq r_2$. We do the latter procedure for 1000 times and obtain the average of times the condition is met. In the following Table 3, we show the comparison of 20 pairs of interval numbers. The second and third columns of this table show the values of $\otimes_1$ and $\otimes_2$, respectively. The fourth column represents the percentage of occasions on which $r_1 \geq r_2$ was met. The fifth column is the result of the grey possibility function proposed by Shi et al., it is expressed as a percentage. Finally, the sixth column is the result of the function $p(\cdot)$ introduced in Eq. (11), it is also expressed as a percentage.

Table 3. Simulation of $\otimes_1 \geq \otimes_2$ and effectiveness comparison of the possibility functions.

#	$\otimes_1$	$\otimes_2$	$\otimes_1 \geq \otimes_2$	*Shi*	$p(\cdot)$
1	[1798500,1939100]	[1071144,1943146]	91	86	91
2	[470863,1846082]	[275526,1722951]	60	56	60
3	[708197,771719]	[1859749,1903846]	0	0	0
4	[1389670,1654481]	[861928,919101]	100	100	100
5	[358787,1872206]	[252562,689336]	91	83	90
6	[573798,1090306]	[1451372,1558608]	0	0	0
7	[1428154,1701322]	[466982,1774124]	83	78	84
8	[579318,1095713]	[327245,1998346]	30	35	31
9	[1427873,1913307]	[486138,1963600]	80	73	80
10	[649815,959503]	[963838,1516812]	0	0	0
11	[964247,1126687]	[883856,1637738]	21	27	21
12	[1200674,1251574]	[726528,1394759]	74	73	75
13	[1567043,1872689]	[245668,1122706]	100	100	100
14	[409322,842992]	[776113,1263268]	1	7	4
15	[776113,1263268]	[534932,839668]	74	66	75
16	[282385,1887176]	[912509,1463213]	43	45	44
17	[621620,1583679]	[703136,1518226]	48	50	49
18	[337598,961952]	[404442,804258]	57	54	57
19	[251129,1584712]	[1165244,1833464]	10	21	13
20	[1266448,1817200]	[1198515,1870849]	50	51	51

In the 20 comparisons, it is true that $p(\cdot)$ has an accuracy that is not worse than that of the grey possibility function proposed in [14]. Furthermore, $p(\cdot)$ has a better accuracy in 13 comparisons (i.e., 65%). Finally, from this experiment we can conclude that $p(\cdot)$ tends to overestimate the possibility of $\otimes_1 \geq \otimes_2$.

5. CONCLUSIONS

We have proposed a novel way to deal with uncertainty and vagueness in the definition of all the outranking's parameters. We do it by setting the values of parameters as interval numbers instead of real numbers. This let us consider all the possible values that a parameter might have, and for which we cannot find an exact value. The procedure proposed needs a way of ordering interval numbers. In this sense, we have also presented a novel way to determine the possibility degree that an interval number is greater than or equal to another interval number. Through this possibility function, the model is able to obtain an interval-based credibility index of the outranking.

The experiments carried out show that the concordance and discordance indices are actually representative of the outranking relation and the veto situations, respectively. These experiments also show that the possibility function presented is effective determining the grade in which an interval number is greater than or equal to another. The effectiveness of the possibility function proposed here is superior to that of the function proposed by Shi et al. [14] in the experiments carried out.

Finally, we have shown that the proposed model based on interval analysis theory is a generalization of the classic outranking, that the model meets several intuitive requirements and that it can effectively represent the asseveration "alternative $\boldsymbol{a}'$ is at least as good as alternative $\boldsymbol{a}$".

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APPENDIX A

Proof of Axiom I fulfillment

We show that each of the parameters obtained in the method proposed are, as intuitively supposed, in $[0,1]$.

Lemma 1. The bounds of $c_j(x,y)$, c_j^- and c_j^+, are in $[0,1]$.

Proof. Eq. (4) defines $c_j(x,y)$ by means of three conditions. If the first condition is satisfied, then $c_j^- = c_j^+ = 0$; if the second condition is satisfied, then $c_j^- = c_j^+ = 1$. In the third condition, a division is used to obtain the value of $c_j(x,y)$. The denominator of this division is always $(\delta_s^+ - \delta_i^-)$; while the numerator will always be between zero and $(\delta_s^+ - \delta_i^-)$. Thus, c_j^- and c_j^+, will always be in the $[0,1]$ range.

Lemma 2- The bounds of $d_j(x,y)$, d_j^- and d_j^+, are in $[0,1]$.

Proof. Eq. (5) defines $d_j(x,y)$ by means of three conditions. If the first condition is satisfied, then $d_j^- = d_j^+ = 1$; And if the second condition is satisfied, then $d_j^- = d_j^+ = 0$. In the third condition, a division is used to obtain the value of $d_j(x,y)$. The denominator of this division is always $(\gamma_s^+ - \gamma_i^-)$; while the numerator will always be between zero and $(\gamma_s^+ - \gamma_i^-)$. Thus, d_j^- and d_j^+, will always be in $[0,1]$.

Lemma 3- The bounds of $c(x,y)$, c^- and c^+, are in $[0,1]$.

Proof. Eq. (6) defines $c(x,y)$ as the sum of the product between $c_j(x,y)$ and w_j. Due to the definition of w_j in (7), we are sure that $w_j^- = w_j^+ \in [0,1]$. We know that $\sum_{j=1}^{K} w_j^- = \sum_{j=1}^{K} w_j^+ = 1$. In addition, we know the bounds of $c_j(x,y)$ are in $[0,1]$. (See proof of Lemma

1.) Given the definition of addition of interval numbers, know that the smallest value that c^- and c^+ can take is zero and will occur when $c_j^- = c_j^+ = 0; j = 1, \cdots, K$. While the largest value that c^- and c^+ can take is one and will take it when $c_j^- = c_j^+ = 1; j = 1, \cdots, K$.
Lemma 4- The bounds of $Nd(x, y)$, Nd^- and Nd^+, are in $[0,1]$.
Proof. $Nd(x, y)$ is defined in (8) as a function of the difference $[1,1] - d_j(x, y)$. If we assume that d_j^- and d_j^+ are between $[0,1]$ (see proof of Lemma 2), then the upper and lower bounds of $Nd(x, y)$ will always be in $[0,1]$.
Lemma 5- The bounds of $\sigma(x, y)$, σ^- and σ^+, are in $[0,1]$.
Proof. $\sigma(x, y)$ was defined in (9) as the product of $Nd(x, y)$ and $c(x, y)$. And, according to Lemmas 3 and 4, the bounds of $Nd(x, y)$ and $c(x, y)$ are in $[0,1]$. Therefore, the bounds of $\sigma(x, y)$ will also always be in $[0,1]$.

APPENDIX B

Proof of Axiom II fulfillment

Theorem 1 The interval based outranking method proposed in this work is a generalization of the classic outranking approach.

Proof. To prove that the interval based outranking proposed generalizes the classic outranking approach, we show that each component of the first is a generalization of the corresponding component of the second. For reasons of clarity, in this section we refer to the components of the interval based outranking by means of the classic identifier of grey numbers, $\otimes$. Hereafter, we assume that the bounds of the parameters defined as interval numbers in the interval-based outranking coincide with the parameters of the classic outranking.

Lemma 6- The uncertain marginal concordance level, $\otimes c_j(x, y)$, is a generalization of the classic concordance index, $c_j(x, y)$.
Proof. We will rely on the definition of white number of the grey theory to prove this Lemma. Specifically, we want to prove that $\otimes c_j(x, y) = [c_j(x, y), c_j(x, y)]$.
In the classic approach, we have g_j, p_j, q_j, v_j all in the reals; and the concordance index, $c_j(x, y)$, is defined as

$$c_j = \begin{cases} 0 & g_j(x) \le g_j(y) - p_j, \\ 1 & g_j(x) \ge g_j(y) - q_j, \\ \dfrac{(g_j(x) - g_j(y) + p_j)}{p_j - q_j} & \text{otherwise.} \end{cases}$$

While Eq. (4) defines $\otimes c_j(x, y)$ in a very similar way. (Eq (4) is shown here again.)

$$\otimes c_j(x, y) = \begin{cases} [0,0] & x_j^+ \le y_j^- - p_j^+, \\ [1,1] & x_j^- \ge y_j^+ - p_j^-, \\ [\Delta_j, \Delta_j + \delta_j] & \text{otherwise} \end{cases} \quad (4)$$

It is possible to show that $\otimes c_j(x, y)$ defined this way is a generalization of $c_j(x, y)$, if it is true that
$\otimes g_j(x) = [g_j(x), g_j(x)]$ (i.e., $x_j^- = x_j^+ = g_j(x)$),
$\otimes g_j(y) = [g_j(y), g_j(y)]$,
$\otimes p_j = [p_j, p_j]$,
$\otimes q_j = [q_j, q_j]$,
First, since $x_j^+ = g_j(x)$, $y_j^- = g_j(y)$ and $p_j^+ = p_j$; then, provided that $g_j(x) \le (g_j(y) - p_j)$ is true, it will also be true that $x_j^+ \le (y_j^- - p_j^+)$. So whenever $c_j(x, y)$ equals zero, $\otimes c_j(x, y)$ will also be zero[3].
Second, because $x_j^- = g_j(x)$, $y_j^+ = g_j(y)$ and $q_j^- = q_j$; then, provided that $g_j(x) \ge (g_j(y) - q_j)$ is true, it will also be true that $x_j^- \ge (y_j^+ - q_j^-)$. So whenever $c_j(x, y)$ equals one, $\otimes c_j(x, y)$ will also be one.
Finally, the third condition of (4) occurs when $x_j^+ > (y_j^- - p_j^+)$ and $x_j^- < (y_j^+ - q_j^-)$. This coincides with Scenario (3) of the Section Interval based marginal concordance. Therefore, $\otimes c_j(x, y)$ is defined in this case as

$$\otimes c_j(x, y) = [\tau, \tau],$$

where

$$\tau = \frac{x_j^- - (y_j^- - p_j^+)}{(y_j^+ - q_j^-) - (y_j^- - p_j^+)} = \frac{x_j^- - y_j^- + p_j^+}{p_j^+ - q_j^-}.$$

Which coincides with the third condition of $c_j(x, y)$, since the original assumption was $x_j^- = g_j(x)$, $y_j^- = g_j(y)$, $p_j^+ = p_j$, and $q_j^- = q_j$. Thus, we can say that it will always be true that $\otimes c_j(x, y) = [c_j(x, y), c_j(x, y)]$ when the parameters are white numbers; and the proof that $\otimes c_j(x, y)$ is a generalization of $c_j(x, y)$ has ended.

Lemma 7- The interval based marginal discordance level $\otimes d_j(x, y)$, is a generalization of the classic discordance index, $d_j(x, y)$.
Proof. In the classic approach, the discordance index, $d_j(x, y)$, is defined as

$$d_j(x, y) = \begin{cases} 1 & g_j(x) \le g_j(y) - v_j, \\ 0 & g_j(x) \ge g_j(y) - p_j, \\ \dfrac{g_j(y) - g_j(x) - p_j}{v_j - p_j} & \text{otherwise.} \end{cases}$$

[3] When the upper and lower bounds of an interval number are equal, it is called a white number. This is because there is no uncertainty about what the actual value of the interval number is and it can be considered as equal to a real number.

While Eq. (5) defines in a very similar way. Eq. (5) is shown here again.

$$\otimes d_j(x,y) = \begin{cases} [1,1] & x_j^+ \le y_j^- - v_j^+, \\ [0,0] & x_j^- \ge y_j^+ - u_j^-, \\ [\Gamma_j, \Gamma_j + \gamma_j] & \text{otherwise} \end{cases} \quad (5)$$

It is possible to show that $\otimes d_j(x,y)$ defined this way is a generalization of $d_j(x,y)$, if it is true that
$\otimes g_j(x) = [g_j(x), g_j(x)]$ (i.e., $x_j^- = x_j^+ = g_j(x)$),
$\otimes g_j(y) = [g_j(y), g_j(y)]$,
$\otimes p_j = [p_j, p_j]$,
$\otimes v_j = [v_j, v_j]$,
First, since $x_j^+ = g_j(x)$, $y_j^- = g_j(y)$ and $v_j^+ = v_j$; then, provided that $g_j(x) \le (g_j(y) - v_j)$ is true, it will also be true that $x_j^+ \le (y_j^- - v_j^+)$. So whenever $d_j(x,y)$ equals zero, $\otimes d_j(x,y)$ will also be one.
Second, because $x_j^- = g_j(x)$, $y_j^+ = g_j(y)$ and $p_j^- = p_j$; then, provided that $g_j(x) \ge (g_j(y) - p_j)$ is true, it will also be true that $x_j^- \ge (y_j^+ - p_j^-)$. So whenever $d_j(x,y)$ equals one, $\otimes d_j(x,y)$ will also be zero.
Finally, the third condition of (5) occurs when $x_j^+ > (y_j^- - v_j^+)$ and $x_j^- < (y_j^+ - p_j^-)$. This coincides with Scenario (3) of the Section Interval based marginal discordance. Therefore, $\otimes d_j(x,y)$ is defined in this case as

$$\otimes d_j(x,y) = [\phi, \phi]$$

Where

$$\phi = \frac{(y_j^+ - p_j^-) - x_j^-}{(y_j^+ - p_j^-) - (y_j^- - v_j^+)} = \frac{y_j^+ - x_j^- - p_j^-}{v_j^+ - p_j^-}.$$

This coincides with the third condition of $d_j(x,y)$, since $x_j^- = g_j(x)$, $y_j^+ = g_j(y)$ and $p_j^- = p_j$, and $v_j^+ = v_j$.
Therefore, we can say that $\otimes d_j(x,y) = [d_j(x,y), d_j(x,y)]$; and the proof that $\otimes d_j(x,y)$ is a generalization of $d_j(x,y)$ has ended.

Lemma 8- The interval based concordance index, $\otimes c(x,y)$, is a generalization of the classic concordance index, $c(x,y)$.
Proof. The importance of the j^{th} criterion is represented in the classic outranking as w_j. While in the interval based outranking, this importance is given by $\otimes \omega_j = [\omega_j^-, \omega_j^+]$. Equation (7) performs the following normalization of $\otimes \omega_j$.

$$w_j^- = w_j^+ = \frac{\omega_j^- + \omega_j^+}{\sum\omega_j^- + \sum\omega_j^+}. \quad (7)$$

Such that the normalized importance of the criteria in the interval based outranking is satisfied by $\sum_{j=1}^{K} \otimes w_j = [1,1]$.
On the other hand, the concordance level of xSy is calculated in Eq. (6) as

$$\otimes c(x,y) = \sum_{j=1}^{n} \otimes w_j \otimes c_j(x,y). \quad (6)$$

Suppose $\otimes \omega_j = w_j$ (i.e., $\omega_j^- = \omega_j^+ = w_j$); then, $\otimes w_j = w_j$ (i.e., $w_j^- = w_j^+ = w_j$). Thus, if $\otimes c_j(x,y)$ generalizes to $c_j(x,y)$, then $c(x,y)$ is a specificity of $\otimes c(x,y)$.

Lemma 9- The interval based non-discordance index, $\otimes Nd(x,y)$, is a generalization of the classic non-discordance index, $Nd(x,y)$.
Proof. The non-discordance index of xSy, $\otimes Nd(x,y)$, is calculated in (8) as

$$\otimes Nd(x,y) = \min_{j \in \otimes C(yPx)} \{[1,1] - \otimes d_j(x,y)\}. \quad (8)$$

Where $\otimes C(yPx) = \{j: x_j^- \le y_j^+ - p_j^-\}$.
While $Nd(x,y)$ is defined as

$$Nd(x,y) = \min_{j \in C(yPx)} \{1 - d_j(x,y)\}.$$

Where $C(yPx) = \{j: g_j(x) \le g_j(y) - p_j\}$.
Now assume that
$\otimes g_j(x) = [g_j(x), g_j(x)]$ (i.e., $x_j^- = x_j^+ = g_j(x)$),
$\otimes g_j(y) = [g_j(y), g_j(y)]$,
$\otimes p_j = [p_j, p_j]$,
$\otimes d_j(x,y) = [d_j(x,y), d_j(x,y)]$,
Given that $x_j^- = g_j(x)$, $y_j^+ = g_j(y)$ and $p_j^- = p_j$, then it follows $\otimes C(yPx) = C(yPx)$.
Assuming that $\otimes d_j(x,y)$ is a generalization of $d_j(x,y)$, then we can ensure that $\otimes Nd(x,y)$ is indeed a generalization of $Nd(x,y)$.

Lemma 10- The uncertain degree of credibility of the assertion xSy, $\otimes \sigma(x,y)$, is a generalization of the classic degree of credibility, $\sigma(x,y)$.
Proof. The definition of $\otimes \sigma(x,y)$ in (9) is

$$\otimes \sigma(x,y) = \otimes C(x,y) \cdot \otimes Nd(x,y). \quad (9)$$

Since Lemmas 8 and 9 demonstrate that $\otimes Nd(x,y)$ and $\otimes C(x,y)$ are generalizations of $Nd(x,y)$ and $C(x,y)$, respectively, the proof that $\otimes \sigma(x,y)$ is a generalization of $\sigma(x,y)$ is trivial. Namely, we need to show that the multiplication of two white numbers (interval numbers with equal upper and lower dimensions) is equivalent to the multiplication of two real numbers. Therefore, we omit this demonstration.

Lemma 11- The interval based outranking relation, $\otimes$ S, is a generalization of the classic outranking relation, S.
Proof. In (10), $\otimes$ S is defined as

$$x \otimes Sy \Leftrightarrow p(\otimes\, \sigma(x, y) \geq \otimes\, \lambda) \geq 0.5. \quad (10)$$

It can be shown that $p(\otimes\, \sigma(x, y) \geq \otimes\, \lambda) = 1$ only when $\sigma \geq \lambda$, and $p(\otimes\, \sigma(x, y) \geq \otimes\, \lambda) = 0$ otherwise. Hence, according to (10), $x \otimes Sy$ if and only if $\sigma \geq \lambda$ is true. This matches the definition of xSy. Therefore, S is indeed a specialization of $\otimes$ S.
At this point, we have shown that each component of the proposed generalization of the outranking approach is a generalization of the corresponding component in the classic approach. Therefore, the proof of Theorem 1 is ended.