A NEW STEFFENSEN-HOMEIER ITERATIVE METOD FOR SOLVING NONLINEAR EQUATIONS

Laurian-Ioan PIŞCORAN*, Dan MICLĂUŞ,
Technical University of Cluj Napoca, North University Center at Baia Mare, Romania.

ABSTRACT
In this paper we present a new family of efficient iterative methods, in order to approximate the simple roots of various nonlinear equations. By some numerical examples we test the accuracy of our methods making a comparative study with other well known iterative methods.

KEYWORDS: Steffensen type method; Homeier type method; Iterative methods.

MSC: 49M15.

RESUMEN
En este trabajo presentamos una nueva familia de métodos iterativos eficientes, para aproximar las raices simples de varias ecuaciones no lineales. Mediante algunos ejemplos numéricos, probamos la precisión de nuestros métodos haciendo un estudio comparativo con otros métodos iterativos bien conocidos.

PALABRAS CLAVE: Método de tipo Steffensen; Método de tipo Homeier; Métodos iterativos.

1. INTRODUCTION

One of the most important problem in all the history of mathematics was to solve the nonlinear equation $f(x) = 0$. We can not always find an exact solution to this equation, but we can obtain some approximative solutions using iterative methods. Newton's method is the best known iterative method for solving nonlinear equations, given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \cdots \tag{1.1}$$

which converges quadratically. In order to improve the order of convergence many researchers [1] [2], [3], [5], [6], [9] introduced and studied some modifications of Newton's iterative method, based especially on the expense of additional evaluations of the functions, derivatives and changes in the point of iterations. In this sense a modification was done by Homeier [5], which studied the following iterative method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\left(x_n - \frac{f(x_n)}{2f'(x_n)}\right)}, \tag{1.2}$$

*plaurian@yahoo.com

with cubic convergence. This method is suitable if the computation of the derivative has a similar or lower cost than that of the function itself. Although the Newton iterative method is the most used in solving nonlinear equations, there exists a disadvantage concerning the application, because it depends upon derivatives which are sometimes restricted in engineering. This disadvantage which appears in application of the Newton iterative method was eliminated by Steffensen [8]. He replaced the derivative $f'(x_n)$ from the relation (1.1) by forward-difference approximation

$$f'(x_n) \cong \frac{f\left(x_n + f\left(x_n\right)\right) - f\left(x_n\right)}{f\left(x_n\right)} \tag{1.3}$$

and got the famous Steffensen's iterative method

$$x_{n+1} = x_n - \frac{f^2\left(x_n\right)}{f\left(x_n + f(x_n)\right) - f\left(x_n\right)}, \tag{1.4}$$

free from any derivative of the function. The Steffensen iterative method (1.4) is also quadratically and requires two functional evaluations per iteration, but in contrast with the Newton method (1.1) is free from any derivative of the function. Following the idea of Steffensen, in many research articles have been developed and studied new derivative-free iterative methods, with the aim to improve the order of convergence.

The main focus of our paper is to present a new family of iterative methods depending on a real parameter, constructed as a linear combination of the Steffensen and the Homeier's method. These methods require two functional evaluations per iteration. We will prove that each family member converges quadratically. In the last part of this article we introduce another cubically convergent method and by some numerical examples we put in evidence the performance of them, making a comparative study with other well known iterative methods.

2. THE FAMILY OF ITERATIVE METHODS

Considering the nonlinear equation $f(x) = 0$, we assume that f has a simple root α and there exists always an initial guess sufficiently close to α. In the first part, we will transform the Homeier method (1.2) in a derivative-free iteration method. Taking the idea of Steffensen into account, we approximate the following term

$$f'\left(x_n - \frac{f\left(x_n\right)}{2f'(x_n)}\right) \cong f'\left(x_n - y_n\right) \cong \frac{f\left(x_n - y_n + f\left(x_n - y_n\right)\right) - f\left(x_n - y_n\right)}{f\left(x_n - y_n\right)}, \tag{2.1}$$

where

$$y_n = \frac{1}{2}\left(\frac{f^2(x_n)}{f\left(x_n + f\left(x_n\right)\right) - f\left(x_n\right)}\right). \tag{2.2}$$

Using the relations (2.1) and (2.2), the Homeier method (1.2) can be presented as

$$x_{n+1} = x_n - \frac{f(x_n)f(x_n - y_n)}{f\left(x_n - y_n + f\left(x_n - y_n\right)\right) - f\left(x_n - y_n\right)}, \tag{2.3}$$

a method free from any derivative of the function. Now, taking a real parameter $a \in [0,1]$ we propose a linear combination of the methods (1.2) and (2.3), given by

$$x_{n+1} = x_n - a\frac{f^2(x_n)}{f\left(x_n + f\left(x_n\right)\right) - f\left(x_n\right)} - (1-a)\frac{f\left(x_n\right)}{f'\left(x_n - \frac{f(x_n)}{2f'(x_n)}\right)}. \tag{2.4}$$

For brevity, we will call the relation (2.4) the Steffensen-Homeier method (shortly S.H.M.).

Remark 2.1 *Obviously, when $a = 0$ we get Homeier iterative method* (1.2)*, respectively for $a = 1$ we get the Steffensen iterative method* (1.4)*. Otherwise, the family of iterative methods* (2.4) *is interesting and attractive.*

In the literature, there exists an extension due to Potra and Pták [7], that may be rewritten as an iterative method

$$x_{n+1} = x_n - \frac{f(x_n) + f\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)}{f'(x_n)},$$

that converges cubically in some neighborhood of the root α. Called in [1] the "two step method", for a quite long time, this was the only known method which converge cubically apart form the methods that involve higher-order derivatives. Making some computational tests, we introduce the following cubically convergent method

$$x_{n+1} = x_n - \frac{f(x_n) + f\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)}{f'(x_n)} + \frac{f\left(x_n - \frac{f(x_n)}{f'(x_n)}\right) \cdot f(x_n)}{f'\left(x_n - \frac{f(x_n)}{f'(x_n)}\right) + f'(x_n)}, \tag{2.5}$$

called modified Potra-Pták method (shortly P.P.M.).

3. NUMERICAL RESULTS

In order to test the performance of these new methods proposed for study, we recall some basic definitions.

Definition 3.1 [9] *Let $\alpha \in \mathbb{R}$ and $x_n \in \mathbb{R}$, with $n = 0, 1, 2, \cdots$. Then, the sequence (x_n) converge to α, if*

$$\lim_{n\to\infty} |x_n - \alpha| = 0.$$

In addition, if there exists a constant $c \geq 0$, an integer $n_0 \geq 0$ and $p \geq 0$, such that for all $n > n_0$ holds

$$|x_{n+1} - \alpha| \leq c|a_n - \alpha|^p,$$

then (x_n) converge to α with q-order at least p. If $p = 2$ or $p = 3$, the convergence is q-quadratic or q-cubic respectively.

Definition 3.2 [9] *Let α be a root of the function f and suppose that x_{n-1}, x_n and x_{n+1} are three consecutive iterations closer to the root α. Then, the computation order of convergence ρ can be approximated using the formula*

$$\rho \approx \frac{\log |(x_{n+1} - \alpha)/(x_n - \alpha)|}{\log |(x_n - \alpha)/(x_{n-1} - \alpha)|}. \tag{3.1}$$

We compare the classical Newton method (1.1), the Steffensen free derivative method (1.4), the Steffensen-Homeier method (2.4) and the modified Potra-Pták method (2.5) respectively, concerning the approximation accuracy of the root α for different functions. All the computations are performed using Mathematica 7 software with 16 significant digits. We choose for numerical tests various values for the parameter a and the following functions:

$$f_1(x) = x^3 - 13,$$

with solution $x^* = 2.3513346877207577$;

$$f_2(x) = 3x^2 + e^x - 2,$$

with solution $x^* = 0.40718983301936296$, and

$$f_3(x) = \cos x - \tfrac{3}{4},$$

with solution $x^* = 0.7227342478134157$.

Table 1: First comparison of various iterative methods.

	Iteration	Computational order of convergence	x^*
$f_1(x)$, $x_0 = 2$			
N.M.	5	1.9998904733219338	2.3513346877207577
S.M.	12	1.9998047666530478	2.3513346877207577
P.P.M.	6	3.0061667895051634	2.3513346877207577
S.H.M. $a = 0.25$	6	not defined	2.3513346877207577
S.H.M. $a = 0.5$	7	not defined	2.3513346877207577
S.H.M. $a = 0.9$	10	1.999525849639129	2.3513346877207577
$f_2(x)$, $x_0 = 1$			
N.M.	6	2.000145683279451	0.40718983301936296
S.M.	11	2.0220075093269851	0.40718983301936296
P.P.M.	5	2.975655626748209	0.40718983301936296
S.H.M. $a = 0.25$	6	not defined	0.40718983301936296
S.H.M. $a = 0.5$	7	not defined	0.40718983301936296
S.H.M. $a = 0.9$	10	1.9979810608619408	0.40718983301936296
$f_3(x)$, $x_0 = 1$			
N.M.	4	2.00169107783203	0.7227342478134157
S.M.	4	1.9972122898758495	0.7227342478134157
P.P.M.	4	3.0494217115526516	0.7227342478134157
S.H.M. $a = 0.25$	4	not defined	0.7227342478134157
S.H.M. $a = 0.5$	4	not defined	0.7227342478134157
S.H.M. $a = 0.9$	5	1.9976487679658457	0.7227342478134157

Legend: N.M. – Newton method (1.1); S.M. – Steffensen method (1.4); P.P.M. – modified Potra-Pták method (2.5); S.H.M. – Steffensen-Homeier method (2.4).

Analyzing the results in the Table 1, we can see that the modified Potra-Pták iterative method is comparable with classical Newton method in the approximation of the solutions of tested functions. For this reason, we continue the evaluation of these two recalled iterative methods by taking some well known test functions used by Weerakon and Fernando in [9]. These test functions are given below:

$$g_1(x) = x^3 + 4x^2 - 10,$$

with solution $x^* = 1.365230013414097$;

$$g_2(x) = \sin^2 x - x^2 + 1,$$

with solution $x^* = 1.404491648215341$;

$$g_3(x) = x^2 - e^x - 3x + 2,$$

with solution $x^* = 0.2575302854398607$;

$$g_4(x) = \cos x - x,$$

with solution $x^* = 0.7390851332151607$;

$$g_5(x) = (x-1)^3 - 1,$$

with solution $x^* = 2$;

$$g_6(x) = x^3 - 10,$$

with solution $x^* = 2.154434690031884$;

$$g_7(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5,$$

with solution $x^* = -1.207647827130919$;

$$g_8(x) = e^{x^2+7x-30} - 1,$$

with solution $x^* = 3$.

Table 2: Second comparison of various iterative methods.

Function	Root α	Initial guess x_0	N.M. it.	P.P.M. it.
$g_1(x) = x^3 + 4x^2 - 10$	1.365230013414097	1	5	3
$g_2(x) = \sin^2 x - x^2 + 1$	1.404491648215341	1	5	5
$g_3(x) = x^2 - e^x - 3x + 2$	0.2575302854398607	0	4	3
$g_4(x) = \cos x - x$	0.7390851332151607	1	4	3
$g_5(x) = (x-1)^3 - 1$	2	2.3	4	4
$g_6(x) = x^3 - 10$	2.154434690031884	2	4	4
$g_7(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5$	-1.207647827130919	-1	5	4
$g_8(x) = e^{x^2+7x-30} - 1$	3	3.1	5	5

A thorough investigation of the results presented in the Table 1 and Table 2, leads to the fact that the modified Potra-Pták method (2.5) is comparable or even better in evaluations as the classical Newton method. This fact underline the importance of the method (2.5) and for this reason we will dedicate it the next section.

4. CONVERGENCE ANALYSIS

Theorem 4.1 *Let $\alpha \in \mathbb{R}$ be a simple root of the function $f : D \subset \mathbb{R} \to \mathbb{R}$, where D is an open interval. Assume that f has derivatives up to the third order in D and the initial guess x_0 is sufficiently close to α, then the modified Potra-Pták iterative method* (2.5) *has the third order convergence and satisfies the following error equation*

$$e_{n+1} = 3c_2(c_2+1)e_n^3 + O\left(e_n^4\right), \tag{4.1}$$

where $e_n = x_n - \alpha$ and $c_j = \frac{1}{j!}\frac{f^{(j)}(\alpha)}{f'(\alpha)}$, $j = 1, 2, 3, \cdots$.

Proof. In order to establish the order of convergence of the appropriate method we will use the MAPLE software. Let $\alpha \in \mathbb{R}$ be a simple root of the functions f, (i.e. $f(\alpha) = 0$, $f'(\alpha) \neq 0$) and $e_n = x_n - \alpha$ is the error in the n^{th} iterate. It is known (see [4]) that the fixed point iteration

$$x_{n+1} = F(x_n), \quad \text{for} \quad n \geq 0$$

is of convergence order q, if F is sufficiently many times differentiable on an interval containing the root α, which is a fixed point for F and satisfies

$$F'(\alpha) = F''(\alpha) = \cdots = F^{(q-1)}(\alpha) = 0, \quad \text{and} \quad F^{(q)}(\alpha) \neq 0. \tag{4.2}$$

Running the appropriate statements in MAPLE of the iteration function F with the fixed point α, for the modified iterative method (2.5), we get

$$F(\alpha) = \alpha, \quad F'(\alpha) = F''(\alpha) = 0, \quad F'''(\alpha) = 3\left(\frac{f''(\alpha)}{f'(\alpha)}\right)^2 + 3\left(\frac{f''(\alpha)}{f'(\alpha)}\right). \tag{4.3}$$

Because $F'''(\alpha) \neq 0$ provided that $3\left(\frac{f''(\alpha)}{f'(\alpha)}\right)^2 + 3\left(\frac{f''(\alpha)}{f'(\alpha)}\right) \neq 0$ and the modified iterative method (2.5) has the order of convergence three. Based on the above result (4.3) the error equation (4.1) can be obtained by using Taylor series. Expanding $F(x_n)$ around $x = \alpha$, we get

$$x_{n+1} = F(x_n) = F(\alpha) + F'(\alpha)(x_n - \alpha) + \frac{F''(\alpha)}{2!}(x_n-\alpha)^2 + \frac{F'''(\alpha)}{3!}(x_n-\alpha)^3 + O\left(x_n-\alpha\right)^4. \tag{4.4}$$

From the relation (4.3) and (4.4) yields

$$x_{n+1} = \alpha + 3\left(\frac{f''(\alpha)}{f'(\alpha)}\right)^2 + 3\left(\frac{f''(\alpha)}{f'(\alpha)}\right) = 3c_2(c_2+1)e_n^3 + O\left(e_n^4\right), \tag{4.5}$$

where $e_n = x_n - \alpha$ and $c_2 = \frac{1}{2!}\frac{f''(\alpha)}{f'(\alpha)}$. Thus $e_{n+1} = 3c_2(c_2+1)e_n^3 + O\left(e_n^4\right)$.

5. CONCLUSIONS

In this paper we successfully introduced a new family of efficient iterative methods. We have established by some numerical examples that the presented methods are comparable or even better as the classical Newton iterative method and the Steffensen iterative method.

ACKNOWLEDGMENT

The results presented in this paper were obtained with the support of the Technical University of Cluj-Napoca through the research Contract no. 2011/12.07.2017, Internal Competition CICDI-2017. The authors want to express their sincere gratitude to the reviewers of this paper for the comments and suggestions which improved the quality of the paper.

RECEIVED: FEBRUARY, 2018.
REVISED: JUNE, 2018.

REFERENCES

[1] AMAT S., BUSQUIER S., GUTIERREZ J.M., (2003): Geometric constructions of iterative functions to solve nonlinear equations, **J. Comput. Appl. Math.**, 157, 197–205.

[2] BAHGAT M.S.M., (2012): New two-step iterative methods for solving non-linear equations, **J. Math. Research**, 3(4): 128–131.

[3] CORDERO A., HUESO J.L., MARTINEZ E., TORREGROSSA J.R., VINDEL P., (2012): Steffensen type methods for solving nonlinear equation, **J. Comput. Appl. Math.**, 236, 3058–3064.

[4] GAUTSCHI W., (1997): Numerical analysis: An Introduction, Birkhauser, Boston.

[5] HOMEIER H.H.H., (2003): A modified Newton method for rootfinding with cubic convergence, **J. Comput. Appl. Math.**, 15,: 227–230.

[6] HOMEIER H.H.H.,(2005): On Newton-type methods with cubic convergence, **J. Comput. Appl. Math.**, 176: 425–432.

[7] POTRA F.A., PTÁK V.,(1984): Nondiscrete induction and iterative processes, in: Research Notes in Mathematics, 103, Pitman, Boston.

[8] STEFFENSEN J.F., (1933): Remarks on iteration, **Skand. Aktuarietidskrift** 16, 64–72.

[9] WEERAKOON S., FERNANDO T.G.I., (2000): A variant of Newton's method with accelerated third-order convergence, **Appl. Math. Lett.**, 13, 87–93.