EXPONENTIATED NEW WEIBULL-PARETO DISTRIBUTION

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ABSTRACT
In this paper, we introduce a new continuous distribution called as the exponentiated new Weibull-Pareto (ENWP) distribution. Several properties of the ENWP distribution are derived such as Rényi entropy, reliability and hazard rate functions, and the moments. The maximum likelihood estimators of the unknown parameters are derived. The shape of the distribution is illustrated and some simulations are computed for different values of the distribution parameters

KEYWORDS: New Weibull-Pareto distribution; Maximum likelihood estimator; Renyi entropy, Moments; Exponentiated distribution.

MSC: 60E05

RESUMEN
En este paper, introducimos una nueva distribución continua llamada "exponentiated new Weibull-Pareto" (ENWP). Varias propiedades de la distribución ENWP son derivadas como las funciones entropía de Rényi, fiabilidad y la tasa de "hazard", y los momentos. Estimadores máximo verosímiles de los parámetros desconocidos se derivan. La forma de la distribución es ilustrada y algunas simulaciones son computadas para diferentes valores de los parámetros de la distribución.

PALABRAS CLAVE: Nueva distribución Weibull-Pareto; estimador Máximo Verosímil; entropía de Renyi, Momentos; "Exponentiated" distribución

1. INTRODUCTION

Statistical distributions are actual valuable in describing and expecting real world occurrences. Although many distributions have been established, there are always areas for developing distributions which are either more elastic or appropriate for specific real world situations. This has encouraged scientists looking for and developing new and more elastic distributions. Consequently, many new distributions have been developed and considered. Gupta et al. (1998) defined the exponentiated exponential distribution as a generalization of the exponential distribution.
For a given cumulative distribution function (CDF) $G(x)$, the exponentiated class of distribution is given by

$$F(x)=\left[G(x)\right]^{w},\ \ w>0, \tag{1}$$

where w is a shape parameter, and the corresponding probability density function (PDF) is

$$f(x)=w\,g(x)\left[G(x)\right]^{w-1},$$

where $G(x)$ and $g(x)$ are the CDF and PDF of the base distribution.

The exponentiated Weibull distribution in Mudholkar and Srivastava (1993) is one of the class of exponentiated distributions by taking $F(x)$ to be the CDF of a Weibull distribution, they studied the exponentiated Weibull distribution to evaluate bathtub failure data. Many scientists used the class of exponentiated distributions to generate new distributions. For instance, Nadarajah (2005) suggested the exponentiated Gumbel distribution. Tahmasebi and Jafari (2015) introduced a new class of distributions by compounding the exponentiated extended Weibull family and power series family. Nekoukhou1 and Bidram (2015) introduced the exponentiated discrete Weibull distribution, they defined a new generalization of the discrete Rayleigh distribution for the first time in the literature. Alzaghal et al. (2013) proposed new family of distributions called exponentiated *T-X* distribution, they discussed some of its properties and studied the three-

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parameter exponentiated Weibull exponential distribution. Salem and Selim (2014) studied the properties of the generalized Weibull-Exponential distribution and its applications. Gupta and Kundu (1999) studied generalized exponential distributions. Alzaatreh et al. (2013) suggested Weibull-Pareto distribution and investigated some of its applications. Some exponentiated distributions are proposed by Ali et al. (2007).

Let X be a random variable with new Weibull-Pareto distribution. Nasiru and Luguterah (2015) defined the CDF of the NWP distribution as

$$F(x;\psi,\eta,\varphi)=1-e^{-\varphi\left(\frac{x}{\eta}\right)^{\psi}}, \quad x,\eta,\psi,\varphi>0,$$

with corresponding PDF

$$f(x;\psi,\eta,\varphi)=\frac{\varphi\psi}{\eta}\left(\frac{x}{\eta}\right)^{\psi-1}e^{-\varphi\left(\frac{x}{\eta}\right)^{\psi}}.$$

The expected value and the variance of the NWP distribution are given by

$$E(X)=\frac{\eta}{\sqrt[\psi]{\varphi}}\Gamma\left(\frac{\psi+1}{\psi}\right) \text{ and } \operatorname{Var}(X)=2\frac{\eta}{\sqrt[\psi]{\varphi^2}}\Gamma\left(\frac{\psi+2}{\psi}\right)-\left[\frac{\eta}{\sqrt[\beta]{\varphi}}\Gamma\left(\frac{\psi+1}{\psi}\right)\right]^2, \text{ respectively.}$$

The survival function hazard functions of the NWP distribution are defined as $S(x;\psi,\eta,\varphi)=e^{-\varphi\left(\frac{x}{\eta}\right)^{\psi}}$ and $R(x;\psi,\eta,\varphi)=\frac{\varphi\psi}{\eta^{\psi}}x^{\psi-1}$. This paper develops an exponentiated new Weibull-Pareto distribution by replacing the CDF $G(x)$ in Equation (1) by the CDF of the NWP distribution, $F(x;\psi,\eta,\varphi)$.

The results of this article are as follows. Section 2 deals with the CDF and probability density function of the ENWP distribution. The moments of the ENWP distribution including the rth moment, mean, and variance, as well as the coefficients of skewness, kurtosis and the coefficient of variation are derived in Section 3. In Section 4, the Rényi entropy of the ENWP discussed in Section 5. The distribution of order statistics, quantile function and the reliability analysis of the ENWP distribution are presented in Section 6. Finally, the paper is concluded in Section 7.

2. THE SUGGESTED EXPONENTIATED NEW WEIBULL-PARETO DISTRIBUTION

A random variable X is said to have an exponentiated new Weibull-Pareto distribution, denoted by $X:\ ENWP(x;\delta,\theta,\beta,w)$ if its PDF is given by

$$f(x;\delta,\theta,\beta,w)=\frac{w\delta\beta}{\theta^{\beta}}x^{\beta-1}e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{w-1}, x>0,\beta,\delta,\theta,w>0, \tag{2}$$

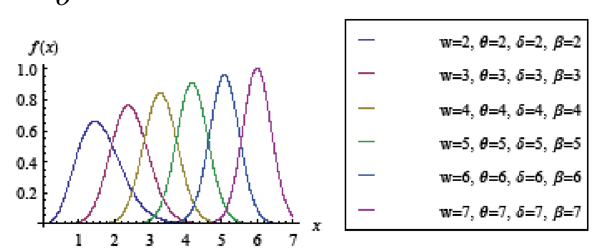

FIGURE 1: Plots of the PDF of ENWP distribution for some parameters values

where θ is a scale parameter, w and β are shape parameters. The corresponding CDF of the ENWP distribution is defined as

$$F(x;\delta,\theta,\beta,w)=\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{w}, \quad x>0,\beta,\delta,\theta,w>0. \tag{3}$$

The shapes of PDF and CDF of the ENWP distribution for different values of the distribution parameters are illustrated in Figures (1) and (2), respectively.

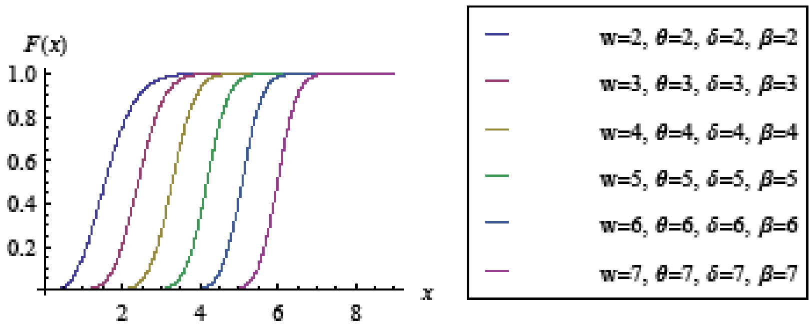

FIGURE 2: Plots of the CDF of ENWP distribution for some parameters values

Based on Figure (1), it can be noted that the ENWP distribution is skewed for $\delta = \beta = w = \theta = 2$, while it will be semi symmetric as the distribution parameters values are increasing.

3. MOMENTS

This section presents the *r*th moment, mean, variance, coefficient of skewness, coefficient of kurtosis, coefficient of variation, and the moment generating function (mgf) of the ENWP distribution.

Theorem (1): Let $X : ENWP(x;\delta,\theta,\beta,w)$, then the *r*th moment of X is given by

$$E\left(X^r\right) = w\theta^r \delta^{-\frac{r}{\beta}} \Gamma\left(\frac{r}{\beta}+1\right) \sum_{i=0}^{w-1} \binom{w-1}{i} (-1)^i \left(\frac{1}{1+i}\right)^{\frac{r}{\beta}+1}. \tag{4}$$

Proof: From the PDF of the ENWP distribution in (3), the *r*th moment of X can be obtain as

$$\begin{aligned} E\left(X^r\right) &= \int_0^\infty x^r f(x)dx \\ &= \int_0^\infty x^r \left[\frac{w\delta\beta}{\theta^\beta} x^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^\beta} \left(1-e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\right)^{w-1}\right] dx \\ &= \frac{w\delta\beta}{\theta^\beta} \int_0^\infty \left[x^{r+\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^\beta} \left[1-e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\right]^{w-1}\right] dx. \end{aligned}$$

Let $-\delta\left(\frac{x}{\theta}\right)^\beta = \frac{-\delta}{\theta^\beta} x^\beta$, $u = \frac{\delta}{\theta^\beta} x^\beta$, $u\frac{\theta^\beta}{\delta} = x^\beta$, $x = \left(\frac{\theta^\beta}{\delta}\right)^{\frac{1}{\beta}} u^{\frac{1}{\beta}}$, $dx = \delta^{-\frac{1}{\beta}} u^{\frac{1}{\beta}-1} \frac{\theta}{\beta} du$.

Then,

$$\begin{aligned} E\left(X^r\right) &= \frac{w\delta\beta}{\theta^\beta} \delta^{-\frac{1}{\beta}} u^{\frac{1}{\beta}-1} \frac{\theta}{\beta} \int_0^\infty \left\{\left[\theta\delta^{-\frac{1}{\beta}} u^{\frac{1}{\beta}}\right]^{r+\beta-1} e^{-u} \left(1-e^{-u}\right)^{w-1}\right\} du \\ &= w\theta^r \delta^{-\frac{r}{\beta}} \int_0^\infty u^{\frac{r}{\beta}} e^{-u} \left(1-e^{-u}\right)^{w-1} du. \end{aligned}$$

Now, $\left(1-e^{-u}\right)^{w-1} = \sum_{i=0}^{w-1} \binom{w-1}{i} (-1)^i (e^{-u})^i$. Therefore,

$$E\left(X^r\right) = w\theta^r \delta^{-\frac{r}{\beta}} \int_0^\infty \left[u^{\frac{r}{\beta}} e^{-u} \sum_{i=0}^{w-1} \binom{w-1}{i} (-1)^i e^{-iu}\right] du$$

$$= w\theta^r \delta^{-\frac{r}{\beta}} \sum_{i=0}^{w-1} \binom{w-1}{i} (-1)^i \int_0^\infty u^{\frac{r}{\beta}} e^{-u(1+i)} du$$

$$= w\theta^r \delta^{-\frac{r}{\beta}} \sum_{i=0}^{w-1} \binom{w-1}{i} (-1)^i \int_0^\infty u^{\frac{r}{\beta}} e^{-u(1+i)} du.$$

Let $y = u(1+i), u = \frac{y}{1+i}$, $du = \frac{dy}{1+i}$. Then,

$$E(X^r) = w\theta^r \delta^{-\frac{r}{\beta}} \sum_{i=0}^{w-1} \binom{w-1}{i} (-1)^i \int_0^\infty \left(\frac{y}{1+i}\right)^{\frac{r}{\beta}} e^{-y} \frac{1}{1+i} dy$$

$$= w\theta^r \delta^{-\frac{r}{\beta}} \sum_{i=0}^{w-1} \binom{w-1}{i} (-1)^i \left(\frac{1}{1+i}\right)^{\frac{r}{\beta}+1} \int_0^\infty y^{\frac{r}{\beta}} e^{-y} dy$$

$$= w\theta^r \delta^{-\frac{r}{\beta}} \sum_{i=0}^{w-1} \binom{w-1}{i} (-1)^i \left(\frac{1}{1+i}\right)^{\frac{r}{\beta}+1} \Gamma\left(\frac{r}{\beta}+1\right).$$

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Remark:

Note that, when $w = 1$, the *r*th moment will be reduced to $E(X^r) = \theta^r \delta^{-\frac{r}{\beta}} \Gamma\left(\frac{r}{\beta}+1\right)$.

The coefficients of skewness (Sk), kurtosis (Ku) and coefficient of variation (CV) of a random variable, respectively, are defined as

$$Sk = \frac{E(X^3) - 3\mu\sigma^2 - \mu^3}{\sigma^3}, \quad Ku = \frac{E(X^4) - 4\mu E(X^3) + 6\mu^2\sigma^2 + 3\mu^4}{\sigma^4}, \text{ and } CV = \frac{\sigma}{\mu},$$

where

$$\mu = E(X) = w\theta \delta^{-\frac{1}{\beta}} \Gamma\left(\frac{1}{\beta}+1\right) \sum_{i=0}^{w-1} \binom{w-1}{i} (-1)^i \left(\frac{1}{1+i}\right)^{\frac{1}{\beta}+1}, \tag{5}$$

and

$$\sigma^2 = w\theta^2 \delta^{-\frac{2}{\beta}} \Gamma\left(\frac{2}{\beta}+1\right) \sum_{i=0}^{w-1} \binom{w-1}{i} (-1)^i \left(\frac{1}{1+i}\right)^{\frac{2}{\beta}+1} - \left[w\theta \delta^{-\frac{1}{\beta}} \Gamma\left(\frac{1}{\beta}+1\right) \sum_{i=0}^{w-1} \binom{w-1}{i} (-1)^i \left(\frac{1}{1+i}\right)^{\frac{1}{\beta}+1}\right]^2 \tag{6}$$

$$E(X^2) = w\theta^2 \delta^{-\frac{2}{\beta}} \Gamma\left(\frac{2+\beta}{\beta}\right) \sum_{i=0}^{w-1} \binom{w-1}{i} (-1)^i \left(\frac{1}{1+i}\right)^{\frac{2}{\beta}+1}, \tag{7}$$

$$E(X^3) = w\theta^3 \delta^{-\frac{3}{\beta}} \Gamma\left(\frac{3+\beta}{\beta}\right) \sum_{i=0}^{w-1} \binom{w-1}{i} (-1)^i \left(\frac{1}{1+i}\right)^{\frac{3}{\beta}+1}, \tag{8}$$

$$E(X^4) = w\theta^4 \delta^{-\frac{4}{\beta}} \Gamma\left(\frac{4+\beta}{\beta}\right) \sum_{i=0}^{w-1} \binom{w-1}{i} (-1)^i \left(\frac{1}{1+i}\right)^{\frac{4}{\beta}+1}. \tag{9}$$

The standard deviation of the ENWP distribution is

$$\sigma = \theta \sqrt{w\delta^{-\frac{2}{\beta}}\Gamma\left(\frac{2}{\beta}+1\right)\sum_{i=0}^{w-1}\binom{w-1}{i}(-1)^i\left(\frac{1}{1+i}\right)^{\frac{2}{\beta}+1} - \left[w\delta^{-\frac{1}{\beta}}\Gamma\left(\frac{1}{\beta}+1\right)\sum_{i=0}^{w-1}\binom{w-1}{i}(-1)^i\left(\frac{1}{1+i}\right)^{\frac{1}{\beta}+1}\right]^2}, \qquad (10)$$

TABLE 1: The mean, variance, CV, Sk, and Ku of the ENWP distribution with $\theta = \delta = \beta = 2$ for different values of *w*

w	Mean	Variance	CV	Sk	Ku
1	1.25331	0.429204	0.522723	0.631111	17.6157
2	1.62040	0.374299	0.377561	0.507910	23.1828
3	1.82486	0.336542	0.317899	0.491493	29.1834
4	1.96364	0.310773	0.283896	0.495244	34.7221
5	2.06753	0.291996	0.261359	0.504028	39.7908
6	2.14998	0.277580	0.245053	0.513891	44.4548
7	2.21803	0.266073	0.232560	0.523602	48.7765
8	2.27576	0.256613	0.222593	0.532771	52.8075
9	2.32579	0.248655	0.214402	0.541298	56.5889
10	2.36983	0.241835	0.207512	0.549187	60.1535
15	2.53345	0.218087	0.184333	0.580811	75.5138
20	2.64425	0.203430	0.170571	0.603520	88.0312
25	2.72741	0.193169	0.161146	0.620862	98.6846
30	2.79366	0.185434	0.154142	0.634717	108.008
40	2.89531	0.174280	0.144188	0.656853	123.619

From Tables (1) and (2) it can be seen that the ENWP distribution is moderately skewed for large values of the distribution parameters. For fixed values of $\theta = 3$, $\delta = 4$, $\beta = 5$ as the value of *w* is increasing, the mean, skewness and kurtosis values are increasing while the variance and coefficient of variation values are decreasing.

TABLE 2: The mean, variance, CV, Sk, and Ku of the ENWP distribution with $\theta = 3$, $\delta = 4, \beta = 5$ for different values of *w*

w	Mean	Variance	CV	SK	Ku
1	2.08753	0.22863	0.22905	-0.25411	55.102
2	2.35775	0.13795	0.15753	-0.16773	157.699
3	2.48643	0.10478	0.13019	-0.08302	274.110
4	2.56725	0.08738	0.11514	-0.01966	394.650
5	2.62485	0.07651	0.10538	0.02889	515.844
6	2.66900	0.06899	0.09841	0.06749	636.146
7	2.70449	0.06344	0.09313	0.09914	754.815
8	2.73396	0.05913	0.08895	0.12573	871.492
9	2.75906	0.05569	0.08553	0.14852	986.017
10	2.78083	0.05285	0.08267	0.16836	1098.340
15	2.85924	0.04373	0.07314	0.23983	1628.120
20	2.91032	0.03863	0.06754	0.28592	2111.190
25	2.94769	0.03528	0.06372	0.31916	2556.230
30	2.97690	0.03287	0.06091	0.34475	2970.090
40	3.02085	0.02924	0.05661	0.85408	15108.800

Theorem (2): The moment generating function of ENWP distribution is given by

$$E\left(e^{tX}\right)=\sum_{k=0}^{\infty}\sum_{i=0}^{w-1}\binom{w-1}{i}(-1)^i\left(\frac{1}{1+i}\right)^{\frac{k}{\beta}+1}\frac{t^k w\theta^k\delta^{-\frac{k}{\beta}}}{k!}\Gamma\left(\frac{k}{\beta}+1\right). \tag{11}$$

Proof: The moment generating function of a random variable X is defined as

$$E\left(e^{tX}\right)=\int_0^{\infty}e^{tx}f(x)dx \text{ and since } e^{tx}=\sum_{k=0}^{\infty}\frac{(tx)^k}{k!}=\sum_{k=0}^{\infty}\frac{t^k x^k}{k!}.$$

Then,

$$\begin{aligned}E\left(e^{tX}\right)&=\int_0^{\infty}e^{tx}f(x)dx=\int_0^{\infty}\sum_{k=0}^{\infty}\frac{t^k x^k}{k!}f(x)dx\\&=\int_0^{\infty}\sum_{k=0}^{\infty}\frac{t^k x^k}{k!}f(x)dx=\sum_{k=0}^{\infty}\frac{t^k}{k!}\int_0^{\infty}x^k f(x)dx\\&=\sum_{k=0}^{\infty}\frac{t^k}{k!}\int_0^{\infty}x^k f(x)dx=\sum_{k=0}^{\infty}\frac{t^k}{k!}E\left(X^k\right).\end{aligned}$$

From (2), we get

$$\begin{aligned}E\left(e^{tX}\right)&=\sum_{k=0}^{\infty}\frac{t^k}{k!}w\theta^k\delta^{-\frac{k}{\beta}}\sum_{i=0}^{w-1}\binom{w-1}{i}(-1)^i\left(\frac{1}{1+i}\right)^{\frac{k}{\beta}+1}\Gamma\left(\frac{k}{\beta}+1\right)\\&=\sum_{k=0}^{\infty}\sum_{i=0}^{w-1}\binom{w-1}{i}(-1)^i\left(\frac{1}{1+i}\right)^{\frac{k}{\beta}+1}\frac{t^k w\theta^k\delta^{-\frac{k}{\beta}}}{k!}\Gamma\left(\frac{k}{\beta}+1\right).\end{aligned}$$

Remark: The characteristic function, $\phi(it)=E\left(e^{itX}\right)$, of the ENWP distribution is

$$\phi(t)=\sum_{k=0}^{\infty}\sum_{i=0}^{w-1}\binom{w-1}{i}(-1)^i\left(\frac{1}{1+i}\right)^{\frac{k}{\beta}+1}\frac{(it)^k w\theta^k\delta^{-\frac{k}{\beta}}}{k!}\Gamma\left(\frac{k}{\beta}+1\right).$$

4. RÉNYI ENTROPY

The entropy of a random variable X is a measure of variation of the uncertainty. A large entropy value indicates greater uncertainty in the data. The Rényi entropy (1961) is defined as

$$\Upsilon_R(p)=\frac{1}{1-p}\log\left(\int_0^{\infty}f(x)^p dx\right), \text{ where } p>0 \text{ and } p\neq 0. \tag{12}$$

The Rényi entropy of the ENWP random variable X is given in the following theorem.

Theorem (3): The Rényi entropy of the ENWP random variable X is defined as

$$\Upsilon_R(p)=\frac{1}{1-p}\left\{\begin{aligned}&\left[p\log(w\delta)\right]+(1-p)\left[\log\left(\frac{\theta}{\beta}\right)\right]-\left(\frac{p(\beta-1)+1}{\beta}\right)\log(\delta)+\\&\log\left[\sum_{i=0}^{p(w-1)}\binom{p(w-1)}{i}(-1)^i\left(\frac{1}{p+i}\right)^{\frac{p\beta-p+1}{\beta}}\Gamma\left(\frac{p\beta-p+1}{\beta}\right)\right]\end{aligned}\right\}. \tag{13}$$

Proof:

$$\Upsilon_R(p) = \frac{1}{1-p}\log\left(\int_0^\infty f(x)^p\,dx\right)$$

$$= \frac{1}{1-p}\log\left\{\int_0^\infty\left[\frac{w\delta\beta}{\theta^\beta}x^{\beta-1}e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\left(1-e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\right)^{w-1}\right]^p dx\right\}$$

$$= \frac{1}{1-p}\log\left\{\int_0^\infty\left(\frac{w\delta\beta}{\theta^\beta}\right)^p\left(x^{\beta-1}\right)^p\left(e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\right)^p\left[\left(1-e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\right)^{w-1}\right]^p dx\right\}.$$

Now, let $-\delta\left(\frac{x}{\theta}\right)^\beta = \frac{-\delta}{\theta^\beta}x^\beta$, $u = \frac{\delta}{\theta^\beta}x^\beta$, $u\frac{\theta^\beta}{\delta} = x^\beta$, $x = \left(\frac{\theta^\beta}{\delta}\right)^{\frac{1}{\beta}}u^{\frac{1}{\beta}}$, $dx = \delta^{-\frac{1}{\beta}}u^{\frac{1}{\beta}-1}\frac{\theta}{\beta}du$.

Thus,

$$\Upsilon_R(p) = \frac{1}{1-p}\log\left(\frac{w\delta\beta}{\theta^\beta}\right)^p\left(\frac{\theta^\beta}{\delta}\right)^{p\frac{\beta-1}{\beta}}\delta^{-\frac{1}{\beta}}\frac{\theta}{\beta}\left\{\int_0^\infty u^{p\left(\frac{\beta-1}{\beta}\right)+\frac{1}{\beta}-1}e^{-pu}\left[\left(1-e^{-u}\right)^{p(w-1)}\right]du\right\}.$$

But $\left(1-e^{-u}\right)^{p(w-1)} = \sum_{i=0}^{p(w-1)}\binom{p(w-1)}{i}(-1)^i(e^{-u})^i$. Then,

$$\Upsilon_R(p) = \frac{1}{1-p}\log\left(\frac{w\delta\beta}{\theta^\beta}\right)^p\left(\frac{\theta^\beta}{\delta}\right)^{p\frac{\beta-1}{\beta}}\delta^{-\frac{1}{\beta}}\frac{\theta}{\beta}\sum_{i=0}^{p(w-1)}\binom{p(w-1)}{i}(-1)^i\int_0^\infty u^{p\left(\frac{\beta-1}{\beta}\right)+\frac{1}{\beta}-1}e^{-u(p+i)}du.$$

let $y = u(p+i)$, $\frac{dy}{p+i} = du$, $\frac{y}{p+i} = u$. Therefore,

$$\Upsilon_R(p) = \frac{1}{1-p}\log\left(\frac{w\delta\beta}{\theta^\beta}\right)^p\left(\frac{\theta^\beta}{\delta}\right)^{p\frac{\beta-1}{\beta}}\delta^{-\frac{1}{\beta}}\frac{\theta}{\beta}\sum_{i=0}^{p(w-1)}\binom{p(w-1)}{i}(-1)^i\left(\frac{1}{p+i}\right)^{p\left(\frac{\beta-1}{\beta}\right)+\frac{1}{\beta}-1}\frac{1}{p+i}\int_0^\infty y^{p\left(\frac{\beta-1}{\beta}\right)+\frac{1}{\beta}-1}e^{-y}dy$$

$$= \frac{1}{1-p}\log\left(\frac{w\delta\beta}{\theta^\beta}\right)^p\left(\frac{\theta^\beta}{\delta}\right)^{p\frac{\beta-1}{\beta}}\delta^{-\frac{1}{\beta}}\frac{\theta}{\beta}\sum_{i=0}^{p(w-1)}\binom{p(w-1)}{i}(-1)^i\left(\frac{1}{p+i}\right)^{p\left(\frac{\beta-1}{\beta}\right)+\frac{1}{\beta}-1}\frac{1}{p+i}\Gamma\left[p\left(\frac{\beta-1}{\beta}\right)+\frac{1}{\beta}-1+1\right]$$ Now,

$$= \frac{1}{1-p}\log\left\{\left(\frac{w\delta\beta}{\theta^\beta}\right)^p\left(\frac{\theta^\beta}{\delta}\right)^{p\frac{\beta-1}{\beta}}\delta^{-\frac{1}{\beta}}\frac{\theta}{\beta}\sum_{i=0}^{p(w-1)}\binom{p(w-1)}{i}(-1)^i\left(\frac{1}{p+i}\right)^{p\left(\frac{\beta-1}{\beta}\right)+\frac{1}{\beta}-1+1}\Gamma\left[p\left(\frac{\beta-1}{\beta}\right)+\frac{1}{\beta}\right]\right\}.$$

$$\left(\frac{w\delta\beta}{\theta^\beta}\right)^p\left(\frac{\theta^\beta}{\delta}\right)^{\frac{p(\beta-1)}{\beta}}\left(\frac{1}{\delta}\right)^{\frac{1}{\beta}}\frac{\theta}{\beta} = w^p\delta^p\frac{\beta^p}{\theta^{p\beta}}\left(\theta^\beta\right)^{\frac{p(\beta-1)}{\beta}}\left(\frac{1}{\delta}\right)^{\frac{p(\beta-1)}{\beta}}\left(\frac{1}{\delta}\right)^{\frac{1}{\beta}}\frac{\theta}{\beta}$$

$$= w^p\delta^p\frac{\theta^{p\beta-p+1}}{\theta^{p\beta}}\left(\frac{1}{\delta}\right)^{\frac{p(\beta-1)+1}{\beta}}\frac{\beta^p}{\beta}$$

$$= w^p\delta^p\theta^{1-p}\left(\frac{1}{\delta}\right)^{\frac{p(\beta-1)+1}{\beta}}\beta^{p-1}.$$

Therefore,

$$\Upsilon_R(p)=\frac{1}{1-p}\log\left\{w^p\delta^p\theta^{1-p}\beta^{p-1}\left(\frac{1}{\delta}\right)^{\frac{p(\beta-1)+1}{\beta}}\sum_{i=0}^{p(w-1)}\binom{p(w-1)}{i}(-1)^i\left(\frac{1}{p+i}\right)^{\frac{p\beta-p+1}{\beta}}\Gamma\left(\frac{p\beta-p+1}{\beta}\right)\right\}$$

$$=\frac{1}{1-p}\left\{\begin{array}{l}p\left[\log(w)+\log(\delta)\right]+(1-p)\left[\log(\theta)-\log(\beta)\right]-\left(\frac{p(\beta-1)+1}{\beta}\right)\log(\delta)\\+\log\left[\sum_{i=0}^{p(w-1)}\binom{p(w-1)}{i}(-1)^i\left(\frac{1}{p+i}\right)^{\frac{p\beta-p+1}{\beta}}\Gamma\left(\frac{p\beta-p+1}{\beta}\right)\right]\end{array}\right\}$$

$$=\frac{1}{1-p}\left\{\begin{array}{l}p\log(w\delta)+(1-p)\log\left(\frac{\theta}{\beta}\right)-\left(\frac{p(\beta-1)+1}{\beta}\right)\log(\delta)\\+\log\left[\sum_{i=0}^{p(w-1)}\binom{p(w-1)}{i}(-1)^i\left(\frac{1}{p+i}\right)^{\frac{p\beta-p+1}{\beta}}\Gamma\left(\frac{p\beta-p+1}{\beta}\right)\right]\end{array}\right\}.$$

5. MAXIMUM LIKELIHOOD ESTIMATION

In this section we discuss the maximum likelihood estimation (MLE's) of the ENWP distribution parameters. Let $X_1,X_2,...,X_n$ be a random sample of size *n* from the ENWP distribution with unknown parameters w, θ, β and δ. The likelihood function of the ENWP distribution is

$$\mathrm{l}(\theta,\beta,\delta,w)=\prod_{i=1}^{n}f(x)=\prod_{i=1}^{n}\left(\frac{w\delta\beta}{\theta^{\beta}}\right)\prod_{i=1}^{n}\left(x_i^{\beta-1}\right)\prod_{i=1}^{n}e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\prod_{i=1}^{n}\left[1-e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\right]^{w-1} \quad (14)$$

The log likelihood function of a sample of size *n*, $\Re=\log\mathrm{l}(x)$ can be written as

$$\Re=n\log\left(\frac{w\delta\beta}{\theta^{\beta}}\right)+(\beta-1)\sum_{i=1}^{n}\log x_i-\frac{\delta}{\theta^{\beta}}\sum_{i=1}^{n}x_i^{\beta}+\sum_{i=1}^{n}\log\left(1-e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\right)^{w-1}.$$

Now, to find the MLE's of the distribution parameters, we may maximize (14) directly with respect to w, θ, β and δ or by solving the non-linear normal equations.

$$\frac{\partial\Re}{\partial\theta}=-n\,w\delta\beta\theta^{-1}+\frac{\delta\beta}{\theta^{\beta+1}}\sum_{i=1}^{n}x_i^{\beta}+(w-1)\delta\beta\theta^{-\beta-1}\sum_{i=1}^{n}\frac{x_i^{\beta}e^{\frac{-\delta x_i^{\beta}}{\theta^{\beta}}}}{1-e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}, \quad (15)$$

$$\frac{\partial\Re}{\partial\delta}=\frac{n}{\delta}-\frac{1}{\theta^{\beta}}\sum_{1}^{n}x_i^{\beta}+(w-1)\sum_{i=1}^{n}\frac{1}{1-e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}\left(\frac{x_i}{\theta}\right)^{\beta}e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}, \quad (16)$$

$$\frac{\partial\Re}{\partial\beta}=n\frac{1-\beta^2\theta^{-1}}{\beta}+\sum_{i=1}^{n}\log(x_i)-\frac{\delta}{\theta^{\beta}}\sum_{1}^{n}x_i^{\beta}\ln(x_i)+\frac{\delta\ln(\theta)}{\theta^{\beta}}\sum_{1}^{n}x_i^{\beta}$$
$$+(w-1)\sum_{i=1}^{n}\frac{\delta\left(\frac{x_i}{\theta}\right)^{\beta}\ln\left(\frac{x_i}{\theta}\right)e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}{1-e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}} \quad (17)$$

$$\frac{\partial \Re}{\partial w} = \frac{n}{w} + \sum_{i=1}^{n} \ln\left(1 - e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\right). \tag{18}$$

The estimates of the unknown parameters can be found by setting the Equations (15)-(18) to zero and solving them simultaneously gives the MLE of $\hat{w}$, $\hat{\theta}$, $\hat{\beta}$ and $\hat{\delta}$, respectively.

6. ORDER STATISTICS, QUANTILE FUNCTION, AND RELIABILITY ANALYSIS

Let $X_{(1:m)}, X_{(2:m)}, ..., X_{(m:m)}$ be the order statistics of the random sample $X_1, X_2, ..., X_m$ selected from a pdf $f(x)$ and cdf $F(x)$. The PDF of the jth order statistics $X_{(j:m)}$ is defined as

$$f_{(j:m)}(x) = \frac{n!}{(j-1)!(n-j)!}[F(x)]^{j-1}[1-F(x)]^{n-j} f(x) \text{ for } j = 1, 2, ..., n. \tag{19}$$

From (1) and (2) we have the PDF of the ith ENWPD random variable $X_{(j:m)}$ as und by setting Equations (15)-(18) to zero and solving them simultaneously gives the MLE

$$f_{ENWP(j:m)}(x) = \frac{n!}{(j-1)!(n-j)!} \frac{w\delta\beta}{\theta^{\beta}} x^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} \left(1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)^{wi-1} \left[1 - \left(1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)^{w}\right]^{n-i}. \tag{20}$$

Therefore, from $f_{(j)}(x)$ the PDF of the smallest order statistics $X_{(1)} = Min\{X_1, X_2, ..., X_n\}$ is

$$f_{ENWP(1:m)}(x) = \frac{nw\delta\beta}{\theta^{\beta}} x^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} \left(1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)^{w} \left[1 - \left(1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)^{w}\right]^{n-1}. \tag{21}$$

and the PDF of the largest order statistics $X_{(n)} = Max\{X_1, X_2, ..., X_n\}$ has the form

$$f_{ENWP(n)}(x) = \frac{nw\delta\beta}{\theta^{\beta}} x^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} \left(1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)^{wn-1}. \tag{22}$$

Theorem (4): The quantile function of the ENWP distribution is defined as

$$F^{-1}(u) = \left[-\frac{\theta}{\delta} \ln\left(1 - u^{\frac{1}{w}}\right)\right]^{\frac{1}{\beta}}, \quad 0 \le u \le 1. \tag{23}$$

Proof: The proof comes directly by inverting the CDF given in Equation (3).

Simulating the ENWP distribution is directly. If *U* is uniformly distributed random variable on the interval (0,1), then the random variable $X = F^{-1}(u)$ follows the ENWP distribution given in Equation (3).

The three quartiles of the ENWP distribution are given by

$$Q_1 = \left[-\frac{\theta}{\delta} \ln\left(1 - 0.25^{\frac{1}{w}}\right)\right]^{\frac{1}{\beta}}, \quad Q_2 = \left[-\frac{\theta}{\delta} \ln\left(1 - 0.5^{\frac{1}{w}}\right)\right]^{\frac{1}{\beta}}, \text{ and } Q_3 = \left[-\frac{\theta}{\delta} \ln\left(1 - 0.75^{\frac{1}{w}}\right)\right]^{\frac{1}{\beta}}.$$

The reliability function $R(t)$ is the probability of an item not failing prior to a time *t*. It is defined as

$$R_{ENWP}(t) = 1 - F_{ENWP}(t)$$
$$= 1 - \left(1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)^{w}.$$

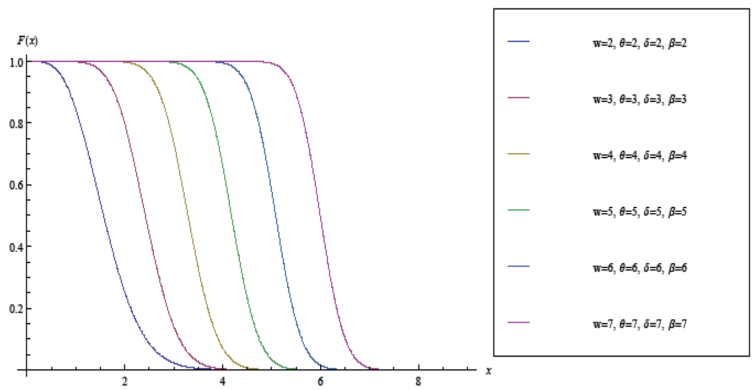

FIGURE 3: The reliability function of the ENWP distribution when $w = \delta = \beta = \theta$

Remarks:

1). $R_{ENWP}(0) = 1$.

2). $R_{ENWP}(x)$ is a decreasing function in *x, w*, β and θ.

3). $R_{ENWP}(x)$ is an increasing function in δ .

The hazard rate function of the ENWP distribution is defined as

$$h_{ENWP}(t) = \frac{f_{ENWP}(t)}{1-F_{ENWP}(t)}$$

$$= \frac{w\delta\,\beta t^{\beta-1}}{\theta^{\beta}} \frac{\left(1-e^{-\delta\left(\frac{t}{\theta}\right)^{\beta}}\right)^{w-1}}{1-\left(1-e^{-\delta\left(\frac{t}{\theta}\right)^{\beta}}\right)^{w}} e^{-\delta\left(\frac{t}{\theta}\right)^{\beta}}.$$

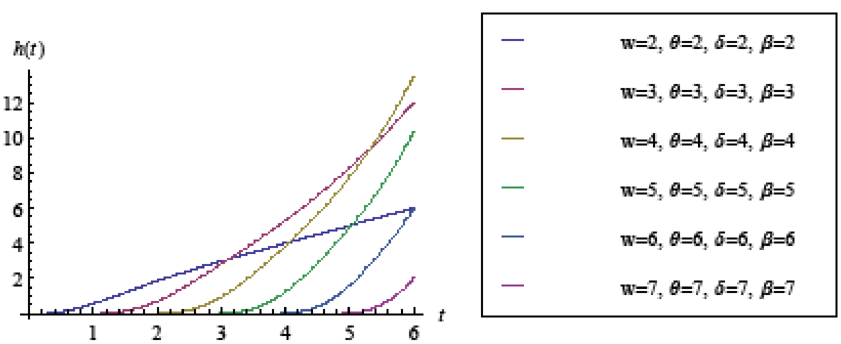

FIGURE 4: The hazard function of the ENWP distribution when $w = \delta = \beta = \theta$

7. CONCLUSIONS

In this paper, the exponentiated new Weibull-Pareto distribution is suggested. This distribution is suggested to provide more flexibility in modeling real data. Some statistical properties of the exponentiated new Weibull-Pareto distribution are proved such as the moments, the PDF of the order statistics of the distribution, the hazard rate and reliability functions. The maximum likelihood estimators of the unknown parameters of the ENWP distribution are provided and the Renyi entropy is presented and proved.

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