MULTI-LEVEL INTEGER PROGRAMMING PROBLEM WITH MULTIPLE OBJECTIVES AT EACH LEVEL

Ritu Arora[1]*[1] and Kavita Gupta**
*Department of Mathematics, Keshav Mahavidyalaya, University of Delhi.
Delhi, India
**Department of Mathematics, Kirori Mal College, University of Delhi.
Delhi, India

ABSTRACT
A Multi-Level Programming Problem (MLPP) is a hierarchical optimization problem where the constraint region of the first level is implicitly determined by the other optimization problems. In this paper, an integer multi-level programming problem is considered. At each level, there are multiple objective functions which are linear fractional and the feasible region is assumed to be a convex polyhedron. Here, the variables are bounded. An algorithm is developed for ranking and scanning the set of feasible solutions. These ranked solutions are used to find the efficient solution of Multi- Level Linear Fractional Programming Problem (MLLFPP). An example is illustrated and solved using LINGO 17.

KEYWORDS: Linear fractional programming problem, integer programming, multi-level programming, efficient solution, bounded variables, multi-objective programming.

MSC: 90C10, 90C29, 90C32

RESUMEN
A multi-level integer linear fractional programming problem with bounded variables is considered. The multiple objective functions at each level are linear fractional. To find the set of efficient solutions for this multi-level programming problem, a mathematical model is evolved. This model scans the feasible region to find the efficient integral points. A solution procedure has been developed describing the above model. A numerical example is illustrated which is also solved by the software LINGO 17.

PALABRAS CLAVE: Problema de programación Lineal fraccional, programación entera , programación multi-nivel, eficiente solución, variables acotadas, programación multi-objectivo.

1. INTRODUCTION

A Multi-Level Programming Problem (MLPP) deals with decentralized planning problems with multiple decision makers in a multi-level or hierarchical organization where decisions have interacted with each other. Herein, at each level, attempts are made by each decision maker to optimize their objective functions. In this process, it is also affected by the actions of the other decision makers. Distinct solution methodologies for multi-level programming problem and its applications have been analyzed by various authors. Candler et al. [11] in 1981, discussed the role of multilevel programming in agricultural economics. Bard and Falk [4] in 1982, proposed an explicit solution to the multi-level programming problem. In 1988, Anandalingam [3] proposed a model of decentralized multi-level systems. In 1992 [7], Blair discussed the computational complexity of multi-level linear programs. Pramanik and Roy [28] in 2007, solved multilevel problems by fuzzy goal programming approach. In 2015, Liu and Yao [21] applied genetic algorithm to solve uncertain multilevel programming problem. In 2016, Kassa [18] gave a branch and bound multi-parametric

1 Corresponding Author E mail : ritunarora@yahoo.in , gupta_kavita31@yahoo.com

programming approach for multi-level optimization. The detailed literature review on bilevel, multi-level programming problems with their solution techniques can be found in [20].
There are many scenarios which require planning/ execution. These situations can be properly represented by a multi-objective programming model. Mathematically, a Multi-Objective Programming Problem (MOPP) is defined as,
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(MOPP):
$$\begin{aligned} &\text{Max } \{f_1(X) = Z_1\} \\ &\text{Max } \{f_2(X) = Z_2\} \\ &- \quad - \quad - \\ &\text{Max } \{f_K(X) = Z_K\} \\ &\text{subject to } X \in S, \end{aligned}$$

where S is a feasible set and $f_j(X)$; $\{j = 1, 2, ..., K\}$, be linear/non-linear.
In 1971, Benayoun et al. [6] described a solution technique for linear programming problems with multiple objective functions. Evans [13] in 1984 discussed the techniques for solving multi-objective mathematical programs. Models with fractional objectives are more expedient as they are relevant in production planning, financial and corporate planning, health care and so forth. Evaluation of economic activities require indices in the form of ratios, like profit/cost, inventory/sales, output/employees', etc. Kornbluth and Steuer [19] have been pioneers in this field. Nykowski et al. [24] proposed a compromise solution for the multiple objective linear fractional programming problem. In 1995, Gupta and Malhotra [17] proposed a cutting plane algorithmic approach for multi-criteria integer linear fractional programming problem. In order to solve fuzzy multiobjective linear fractional programming problem, a goal programming method was developed by Pal et al. [26] in 2003. Abo Sinna [1] in 2007 proposed a method for solving multi-level multi-objective problem with linear or non-linear constraints. Emam [12] in 2013 developed an interactive approach for solving bi-level integer multi-objective fractional programming problem. Mishra and Singh [23] in 2013 developed a linear fractional model for agricultural production system. Mehdi et.al [22] in 2014 proposed a method to generate the efficient set of a multi-objective integer linear fractional program by branch and cut concept. Osman [25] in 2016 gave a solution procedure for multi-level multi-objective fractional programming with fuzziness in the constraints.
Extensive work has been done on integer programming problems. Many cutting plane algorithms like Dantzig cut, Gomory cut, edge truncating cut etc. are used to solve such problems. The first cutting plane algorithm was developed by Gomory in 1958 [16] for the pure integer programming problem. In 1969, Geoffrian [15] proposed an implicit enumeration approach for integer programming. Fisher in 1981 [14], solved the integer programming problems by the lagrangian relaxation method. Alves and Climaco [2] in 2007 reviewed the methods for multi-objective integer and mixed integer programming problems. Integer Programming problems are of paramount importance in business and industry since they have many practical implications in the actuality. Paquay et.al [27] in 2016 in his paper dealt with real world applications in the three dimensional case. In 2017, Gustavo Braier [8] in 2017 developed an integer programming approach for recyclable waste collection.

2. CONVENTIONAL DEFINITION

The linear fractional bilevel programming problem with bounded variables is mathematically stated as,

(BLFPP):
$$\max_{X_1} Z_1(X_1, X_2) = \frac{c_{11}X_1 + c_{12}X_2 + \alpha_1}{d_{11}X_1 + d_{12}X_2 + \beta_1}$$
where X_2 solves
$$\max_{X_2} Z_2(X_1, X_2) = \frac{c_{21}X_1 + c_{22}X_2 + \alpha_2}{d_{21}X_1 + d_{22}X_2 + \beta_2}, \text{ for a given } X_1$$
subject to $A_1X_1 + A_2X_2 = b$
$$X_1 \in \mathbb{R}^{n_1}, \qquad X_2 \in \mathbb{R}^{n_2}$$
$$c_{11}^T, d_{11}^T, c_{21}^T, d_{21}^T \in \mathbb{R}^{n_1}, \; c_{12}^T, d_{12}^T, c_{22}^T, d_{22}^T \in \mathbb{R}^{n_2}$$
$$A_1 \in \mathbb{R}^{m \times n_1}, \; A_2 \in \mathbb{R}^{m \times n_2}, \quad b \in \mathbb{R}^m \text{ and } \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$$
Here, $(d_{11}X_1 + d_{12}X_2 + \beta_1) > 0$ and $(d_{21}, X_1 + d_{22}X_2 + \beta_2) > 0$; $\forall\ (X_1, X_2) \in S$,

where $S = \{(X_1, X_2) : A_1X_1 + A_2X_2 = b;\ l_1 \le X_1 \le u_1,\ l_2 \le X_2 \le u_2, (X_1, X_2)$ is an integer vector$\}$.
Here, l_1, u_1 and l_2, u_2 are lower and upper bounds for the upper level problem and lower level problem respectively. Also, S is non-empty and compact.

In the bilevel linear fractional programming problem defined above, each of the objective functions at both the levels are linear fractional. Therefore, they are both pseudoconcave and pseudoconvex and thus, its optimal solution will be at an extreme point of S. The optimality criterion for solving (BLFPP) with bounded variables is that $\Delta_j \ge 0$ for upper bounded non-basic variables and $\Delta_j \le 0$ for lower bounded non-basic variables, where $\Delta_j = Z_1(z_j^2 - d_j) - Z_2(z_j^1 - c_j)$. Kanti Swarup [29] and Bazarra [5] have explained the criterion for entering and departing of variables in linear fractional programming problem with bounded variables. V. Verma et al. [31] developed an algorithm to rank the integer feasible solutions of an integer linear fractional programming problem. Thirwani and Arora [30] gave an algorithm for solving integer linear fractional bilevel programming problem. Calvete and Gale [9] in 1998, proved that the optimal solutions of the bilevel programming problem in which the objective functions are quasi- concave and the constraint region of both the levels is a convex polyhedron can be found at the extreme point of the polyhedra. In 1999 [10], they proposed an enumerative algorithm that finds a global optimal solution to the bilevel linear/ linear fractional programming problem.

In the present paper, a mathematical model is developed with a novel technique which disparate it from the methods proposed by the past authors. In this model, an algorithm is developed to solve a multi-level programming problem in which the variables are integers and bounded. At each level, multiple objective functions are considered which are linear fractional. The algorithm scrutinizes the set of feasible solutions to procure the efficient solution for the multi-level programming model. The method is elucidated with an example which is solved using computing software LINGO 17.

3. MATHEMATICAL FORMULATION

The Multi-Level Integer Linear Fractional Programming Problem with bounded variables is defined as,

(MLLFPP):

$$\underset{X_1}{\text{Max}}\, Z_1(X) = \text{Max}(g_{11}(X), g_{12}(X), \ldots, g_{1s_1}(X))$$

$$\underset{X_2}{\text{Max}}\, Z_2(X) = \text{Max}(g_{21}(X), g_{22}(X), \ldots, g_{2s_2}(X)) \quad \text{for a given } X_1$$

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$$\underset{X_n}{\text{Max}}\, Z_n(X) = \text{Max}(g_{n1}(X), g_{n2}(X), \ldots, g_{ns_n}(X)), \text{ for a given } (X_1, X_2, \ldots, X_{n-1})$$

where $X = (X_1, X_2, \ldots, X_n) \in S^*$.

Here, $S^* = \{X \mid AX = b \mid L \le X \le U\}$ is non-empty and bounded.

Define, $S_1^* = \{X \mid AX = b \mid L \le X \le U,\ X$ is an integer vector$\}$.

Clearly, $S_1^* \subseteq S$. We are interested in finding the solution of the problem in S_1^*.

(MLLFPP) can be written as

$$\text{Max}\, Z_i(X) = \text{Max}(g_{iu}(X)), \qquad i = 1, 2, \ldots n;\ u = 1, 2, \ldots s_i.$$

Here,

$$g_{1u}(X) = \frac{c_{u1}X_1 + c_{u2}X_2 + \ldots + c_{un}X_n + \alpha_{u1}}{d_{u1}X_1 + d_{u2}X_2 + \ldots + d_{un}X_n + \alpha_{u2}}, \quad u = 1, 2, \ldots, s_1$$

$$g_{2u}(X) = \frac{e_{u1}X_1 + e_{u2}X_2 + \ldots + e_{un}X_n + \beta_{u1}}{f_{u1}X_1 + f_{u2}X_2 + \ldots + f_{un}X_n + \beta_{u2}}, \quad u = 1, 2, \ldots, s_2$$

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$$g_{nu}(X) = \frac{q_{u1}X_1 + q_{u2}X_2 + \ldots + q_{un}X_n + \gamma_{u1}}{p_{u1}X_1 + p_{u2}X_2 + \ldots + p_{un}X_n + \gamma_{u2}}, \quad u = 1, 2, \ldots, s_n$$

Here, $c_{u1}, d_{u1}, X_1 \in \mathbb{R}^{n_1};\ u = 1,2,....,s_1$

$e_{u1}, f_{u1}, X_1 \in \mathbb{R}^{n_1};\ u = 1,2,....,s_2$

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$q_{u1}, p_{u1}, X_1 \in \mathbb{R}^{n_1},\ u = 1,2,....s_n$

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$c_{un}, d_{un}, X_n \in \mathbb{R}^{n_n};\ u = 1,2,....,s_1$

$e_{un}, f_{un}, X_n \in \mathbb{R}^{n_n};\ u = 1,2,....,s_2$

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$q_{un}, p_{un}, X_n \in \mathbb{R}^{n_n},\ u = 1,2,....s_n$

Also,

$$\left.\begin{array}{l}(d_{u1}X_1 + d_{u2}X_2 + ... + d_{un}X_n + \alpha_{u2}) > 0 \\ (f_{u1}X_1 + f_{u2}X_2 + ... + f_{un}X_n + \beta_{u2}) > 0 \\ . \quad . \quad . \quad . \quad . \\ (p_{u1}X_1 + p_{u2}X_2 + ... + p_{un}X_n + \gamma_{u2}) > 0\end{array}\right\} \forall\ (X_1, X_2, ..., X_n) \in S_1^*$$

$A_1 \in \mathbb{R}^{m\times n_1},\ A_2 \in \mathbb{R}^{m\times n_2}, ..., A_n \in \mathbb{R}^{m\times n_n}; \qquad b \in \mathbb{R}^{m\times 1}$

$\alpha_{u_1}, \alpha_{u_2} \in \mathbb{R}, \qquad u = 1,2,....,s_1$

$\beta_{u_1}, \beta_{u_2} \in \mathbb{R}, \qquad u = 1,2,....,s_2$

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$\gamma_{u_1}, \gamma_{u_2} \in \mathbb{R}; \qquad u = 1,2,...,s_n$

$L = (L_1, L_2, ..., L_n)^T, \qquad L_l \in \mathbb{R}^{n_l\times 1}, \qquad l = 1,2,...,n$

$U = (U_1, U_2, ..., U_n)^T,\ U_l \in \mathbb{R}^{n_l\times 1},\ \ l = 1,2,...,n$

The objective functions defined in the above problem at each level are linear fractional programming problems. The polyhedron S_1^* defined by the constraint region of the problem (MLLFPP) is assumed to be non-empty and compact.

Constraint region of the problem $Z_n(X)$ for given value of $(X_1,X_2,....,X_{n-1})$ is given by

$S_1^*(X_1, X_2, ...X_{n-1}) = \{X_n \mid A_nX_n \le A_1X_1 + A_2X_2 + ... + A_{n-1}X_{n-1},\ L_n \le X_n \le U_n, X_n \text{is an integer vctor}\}.$

The inducible region of (MLLFPP) is given by

$IR = \{(X_1, X_2, ..., X_n) \mid (X_1, X_2, ..., X_n) \in S_1^*, X_n \in M(X_1, X_2, ..., X_{n-1})\}$

where $M(X_1, X_2, ..., X_{n-1}) = \{X_n \mid X_n \in \arg\max\ Z_n(X_1, X_2, ..., X_n)\}$, is the rational reaction set of the follower's problem $Z_n(X)$, for given value of $(X_1,X_2,...,X_{n-1})$.

3.1 Definitions Used For Developing Algorithm

Definition (1): Feasible Solution for (MLLFPP)

A point $(X_1, X_2, ..., X_n)$ is called feasible for (MLLFPP) if $(X_1, X_2, ..., X_n) \in IR$.

Definition (2): Efficient Solution

A feasible solution $(\bar{X}_1, \bar{X}_2, ..., \bar{X}_n) \in IR$ is an efficient solution for (MLLFPP) if there is no $(X_1, X_2, ..., X_n) \in IR$ such that $Z_i(\bar{X}_1, \bar{X}_2, ..., \bar{X}_n) \le Z_i(X_1, X_2, ..., X_n)$ for $i = 1, 2,...,n$ and $Z_j(\bar{X}_1, \bar{X}_2, ..., \bar{X}_n) < Z_j(X_1, X_2, ..., X_n)$ for some $j \in \{1, 2,...,n\}$.

Definition (3): Efficient set

The set of all efficient solutions is denoted by (SE) and is called the efficient set.

3.2 Technique to solve the problem (MLLFPP) in S_1^*.

3.2.1 Consider the problem (MLLFPP) in S^*

For i ≥ 1 and K ≥ 1, let B_K be the basis matrix corresponding to the basic feasible solution X_{B_K}.

Suppose that the non-basis matrix is decomposed into N_K^1 and N_K^2, where

$N_K^1 = \{j \mid a_j^K \notin B_K$ and x_j^K is at its lower bound$\}$,

$N_K^2 = \{j \mid a_j^K \notin B_K$ and x_j^K is at its upper bound$\}$,

$I_K = \{t \mid a_t \in B_K\}$.

Further, $A_{N_K^1} = \{a_j^K \in A \mid j \in N_K^1\}$, $A_{N_K^2} = \{a_j^K \in A \mid j \in N_K^2\}$.

Let $X_{N_K^1} = \{x_j \mid j \in N_K^1\}$ be a vector of non-basic variables at their lower bounds and $X_{N_K^2} = \{x_j \mid j \in N_K^2\}$ be a vector of non-basic variables at their upper bounds respectively.

For K ≥ 1, we have

$$B_K X_{B_K} + N_K^1 X_{N_K^1} + N_K^2 X_{N_K^2} = b$$

This implies $X_{B_K} + (B_K^{-1} N_K^1) X_{N_K}^1 + (B_K^{-1} N_K^2) X_{N_K^2} = B_K^{-1} b$ (1)

This implies $X_{B_K} + \sum_{j \in N_K^1} y_{K_j} x_j^K + \sum_{j \in N_K^2} y_{K_j} x_j^K = B_K^{-1} b$ (2)

3.2.2 The problem (MLLFPP) at the first level in S^*.

It is defined as

$$\text{(MLLFPP1):} \quad \underset{X_1}{\text{Max}}\, Z_1(X) = (g_{11}(X), g_{12}(X), \ldots, g_{1s_1}(X))$$

$$\text{subject to } X = (X_1, X_2, \ldots, X_n) \in S^*.$$

Here, (MLLFPP1) is a multi-objective programming problem where each objective function is a linear function.

3.3.3. Solving each of the objective function

$g_{1u}(X);\ u = 1, 2, \ldots, s_1$ w.r.t. the basis B_k.

For K ≥ 1 the value of the objective function corresponding to the basis B_K is given by

$$g_{1u}(X) = \frac{(C_{B_K})_{u1} X_{B_K} + (C_{N_K^1})_{u1} X_{N_K^1} + (C_{N_K^2})_{u1} X_{N_K^2} + \alpha_{u1}}{(D_{B_K})_{u1} X_{B_K} + (D_{N_K^1})_{u1} X_{N_K^2} + (D_{N_K^2})_{u1} X_{N_K^2} + \alpha_{u2}} = \frac{N(X)}{D(X)} \text{ (say)} \tag{3}$$

Consider the numerator N(X) in equation (3), using equation (1), it can be re-written as

$$N(X) = (C_{B_K})_{u1}[B_K^{-1}b - (B_K^{-1}N_K^1)X_{N_K^1} - (B_K^{-1}N_K^2)X_{N_K^2}] + (C_{N_K^1})_{u1} X_{N_K^1} + (C_{N_K^2})_{u1} X_{N_K^2} + \alpha_{u1}$$

$$= (C_{B_K})_{u_1} B_K^{-1} b + [(C_{N_K^1})_{u1} - (C_{B_K})_{u_1} B_K^{-1} N_K^1)] X_{N_K^1} + [(C_{N_K^2})_{u1} - (C_{B_K})_{u1} B_K^{-1} N_K^2] X_{N_K^2} + \alpha_{u1}$$

$$= (C_{B_K})_{u1} B_K^{-1} b - \sum_{j \in N_K^1} (z_j^{u1} - c_j)_K x_{j_K} - \sum_{j \in N_K^2} (z_j^{u1} - c_j)_K x_{j_K} + \alpha_{u1} \tag{4}$$

Similarly, D(X) in equation (3) can be rewritten as

$$D(X) = (D_{B_K})_{u1} B_K^{-1} b - \sum_{j \in N_K^1} (z_j^{u1} - d_j)_K x_{j_K} - \sum_{j \in N_K^2} (z_j^{u1} - d_j)_K x_{j_K} + \alpha_{u2} \tag{5}$$

Suppose that we have a current basic feasible solution,

$$X_{B_K}^0 = (x_{j_k}^0), \text{ where } x_{j_k}^0 = l_{j_k}, j_k \in N_k^1 \text{ and } x_{j_k}^0 = u_{j_k}, j_k \in N_K^2.$$

Therefore, improved objective function value for u ≥ 1 is given by

$$g_{1u}(X^0_{B_K}) = \frac{(C_{B_K})_{u1} B_K^{-1} b - \sum_{j \in N_K^1} (z_j^{u1} - c_j)_K l_{j_K} - \sum_{j \in N_K^2} (z_j^{u1} - c_j)_K u_{j_K} + \alpha_{u1}}{(D_{B_K})_{u1} B_K^{-1} b - \sum_{j \in N_K^1} (z_j^{u1} - d_j)_K l_{j_K} - \sum_{j \in N_K^2} (z_j^{u1} - d_j)_K u_{j_K} + \alpha_{u2}} \tag{6}$$

We have, $$g_{1u}(X^0_{B_K}) = \frac{N(X^0_{B_K})_{u1}}{D(X^0_{B_K})_{u1}}, \quad u = 1, 2, ..., s_1. \tag{7}$$

3.2.4 To find a new feasible solution for each objective function

Take $g_{iu}(X)$; u = 1, 2, ...s, of the problem (MLLFPP1).

We have, $AX^0_{B_K} = b$

That is, $B_K X^0_{B_K} + \sum_{j \in N_K^1} a_{j_K} x_{j_K} + \sum_{j \in N_K^2} a_{j_K} x_{j_K} = b$

Therefore, corresponding to the current basic feasible solution,

$x^0_{j_K} = l_{j_K}$, $j_K \in N_K^1$ and $x^0_{j_K} = u_{j_K}, j_K \in N_K^2$, we have

$$\sum_{t \in I_K} b_t x^0_{B_t} + \sum_{j \in N_K^1} a_{j_K} l_{j_K} + \sum_{j \in N_K^2} a_{j_K} u_{j_K} = b \tag{8}$$

Suppose a non-basic variable x_{r_K} at its lower bound undergoes a change ϕ_r^K, where $\phi_r^K > 0$. We have from equation (8),

$$\sum_{t \in I_K} b_t x^0_{B_t} + \sum_{j \in N_K^1} a_{j_K} l_{j_K} + \sum_{j \in N_K^2} a_{j_K} u_{j_K} + \phi_r^K a_{r_K} - \phi_r^K a_{r_K} = b$$

This implies $\sum_{t \in I_K} b_t x^0_{B_t} + \sum_{j \in N_K^1} a_{j_K} l_{j_K} + \sum_{j \in N_K^2} a_{j_K} u_{j_K} + \phi_r^K a_{r_K} - \phi_r^K \left[\sum_{t \in I_K} y^K_{t_r} b_t \right] = b$

That is, $$\sum_{t \in I_K} [x^0_{B_t} - \phi_r^K y_{t_K}] b_t + \sum_{j \in N_K^1 | \{r\}} a_{j_K} l_{j_K} + a_{r_K}(l_{r_K} + \phi_r^K) + \sum_{j \in N_K^2 |} a_{j_K} u_{j_K} = b \tag{9}$$

Equation (9) gives the new solution, $\hat{X}_K = (\hat{x}_{j_K})$, where

$$\begin{cases} \hat{x}_{t_K} = x^0_{t_K} - \phi_r^K y^K_{t_r}, & \forall t \in I_K \\ \hat{x}_{r_K} = l_{r_K} + \phi_r^K, & \\ \hat{x}_{j_K} = x^0_{j_K}, & j_K \in N_K^1 \cup N_K^2 \setminus \{r\} \end{cases} \tag{10}$$

The objective function value of the problem $g_{1u}(X)$; u = 1, 2, ..., s_1 corresponding to a new feasible solution $\hat{X}_K$ is given by

$$\begin{aligned} N(\hat{X}_K)_{u1} &= (C_{\hat{B}_K})_{u1}(B_K^{-1} b) - \sum_{j \in N_K^1 \setminus \{r\}} (z_j^{u1} - c_j)_K l_{j_K} - (z_r^{u1} - c_r)_K (l_{r_K} + \phi_r^K) - \sum_{j \in N_K^2} (z_j^{u1} - c_j)_K u_{j_K} + \alpha_{u1} \\ &= (C_{\hat{B}_k})_{u1}(B_K^{-1} b) - \left[\sum_{j \in N_K^1} (z_j^{u1} - c_j)_K l_{j_K} - (z_r^{u1} - c_r)_K l_{r_K} \right] - \phi_r^K (z_r^{u1} - c_r)_K - \sum_{j \in N_K^2} (z_j^{u1} - c_j)_K u_{j_K} + \alpha_{u1} \\ &= \left[(C_{\hat{B}_k})_{u1}(B_K^{-1} b) - \sum_{j \in N_K^1 \setminus \{r\}} (z_j^{u1} - c_j)_K l_{j_K} - \sum_{j \in N_K^2} (z_j^{u1} - c_j)_K u_{j_K} + \alpha_{u1} \right] - \phi_r^K (z_r^{u1} - c_r)_K \\ &= N(X_K^0) - \phi_r^K (z_r^{u1} - c_r)_K . \end{aligned}$$

$$\left. \begin{aligned} \text{Therefore,} \quad N(\hat{X}_K) &= N(X_K^0) - \phi_r^K (z_r^{u_1} - c_r)_K \\ \text{Similarly,} \quad D(\hat{X}_K) &= D(X_K^0) - \phi_r^K (z_r^{u_1} - d_r)_K \end{aligned} \right] \tag{11}$$

Therefore, $g_{1u}(\hat{X}_K) = \dfrac{N(\hat{X}_K)_{u1}}{D(\hat{X}_K)_{u1}} = \dfrac{N(X_K^0)_{u1} - \phi_r^K (z_r^{u1} - c_r)_K}{D(X_K^0)_{u1} - \phi_r^K (z_r^{u1} - d_r)_K}$, u = 1,2, ..., s_1 (12)

The new solution will be a feasible extreme point solution, provided

$$\phi_r^K = \mathrm{Min}\left\{(u_r - l_r)_K, \left(\frac{x_{Bt}^K - l_{B_t}}{(y_{t_j})_k}\middle|(y_{t_j})_K > 0,\ t \in I_K\right), \left(\frac{u_{Bt}^K - x_{B_t}^K}{(y_{t_j})_k}\middle|(y_{t_j})_K < 0,\ t \in I_K\right)\right\}.$$

The change in the value of the objective function $g_{1u}(X)$ is given by

$$g_{1u}(\hat{X}_K) - g_{1u}(X_K^0) = \frac{N(\hat{X}_K)_{u1}}{D(\hat{X}_K)_{u1}} - \frac{N(X_K^0)_{u1}}{D(X_K^0)_{u1}}; \quad u = 1,2,\ldots,s_1$$

$$= \frac{N(X_K^0)_{u1} - \phi_r^K (z_r^{u1} - c_r)_K}{D(X_K^0)_{u1} - \phi_r^K (z_r^{u1} - d_r)_K} - \frac{N(X_K^0)_{u1}}{D(X_K^0)_{u1}}$$

$$= \frac{\phi_r^K [N(X_K^0)_{u1}(z_r^{u1} - d_r) - D(X_K^0)_{u1}(z_r^{u1} - c_r)_K]}{D(X_K^0)_{u1}[D(X_K^0)_{u1} - \phi_r^K (z_r^{u1} - d_r)_K]}$$

$$\therefore\ g_{1u}(\hat{X}_K) - g_{1u}(X_K^0) = \frac{\phi_r^K (\Delta_{r_K})_{u1}}{D(X_K^0)_{u1}[D(X_K^0)_{u1} - \phi_r^K (z_r^{u1} - d_r)_K]}; \quad u = 1,2,\ldots,s_1 \tag{13}$$

where $(\Delta_{r_K})_{u1} = N(X_K^0)_{u1}(z_r^{u1} - d_r)_K - D(X_K^0)_{u1}(z_r^{u1} - c_r)_K$ (14)

Similarly, if variable $x_{r_K} = u_{r_K}$ undergoes a change, then the new solution $\hat{X}_K = (\hat{x}_{j_K})$ is defined as

$$\begin{cases} \hat{x}_{t_K} = x_{t_K}^0 + y_{t_r}^K \phi_r^K, \ \forall\, t \in I_K \\ \hat{x}_{r_K} = u_{r_K} - \phi_r^K \\ \hat{x}_{j_K} = x_{j_K}^0, \qquad \forall\, j_K \in N_K^1 \cup N_K^2 \setminus \{r\} \end{cases} \tag{15}$$

The objective function value corresponding to a new feasible solution $\hat{X}_K$ is given by

$$g_{1u}(\hat{X}_K) = \frac{N(X_K^0)_{u1} + \phi_r^K (z_r^{u1} - c_r)_K}{D(X_K^0)_{u1} + \phi_r^K (z_r^{u1} - d_r)_K} \tag{16}$$

The new solution will be a feasible extreme point solution, provided

$$\phi_r^K = \mathrm{Min}\left\{(u_r - l_r)_K, \left(\frac{x_{B_t}^K - l_{B_t}}{-(y_{t_j})_K}\middle|(y_{t_j})_K < 0,\ t \in I_K\right), \left(\frac{u_{B_t}^K - x_{B_t}^K}{(y_{t_j})_K}\middle|(y_{t_j})_K > 0,\ t \in I_K\right)\right\}.$$

The change in the value of the objective function (MLLFPP1) is given by

$$g_{1u}(\hat{X}_K) - g_{1u}(X_K^0) = \frac{-\phi_r^K (\Delta_{r_K})_{u1}}{D(X_K^0)_{u1}[D(X_K^0)_{u1} + \phi_r^K (z_r^{u1} - d_r)_K]} \tag{17}$$

Thus, we conclude that the non-basic variable x_{r_K} enters the basis which gives maximum improvement in the value of the objective function.

3.2.5. To find the integral solution for the objective function $g_{1u}(X)$

Take u = 1, 2,...., s_1

Define, $\Delta_{j_K} = N(X_K)(z_{j_K} - d_{j_K}) - D(X_K)(z_{j_K} - c_{j_K})$

$$J_1^K = \{j \mid j \in N_K^1 \text{ such that } \Delta_{j_K} = 0\}$$

$$J_2^K = \{j \mid j \in N_K^2 \text{ such that } \Delta_{j_K} = 0\}$$

$$T_1^K = \{j \mid j \in N_K^1 \text{ such that } \Delta_{j_K} \neq 0\}$$

$$T_2^K = \{j \mid j \in N_K^2 \text{ such that } \Delta_{j_K} \neq 0\}$$

In (MLLFPP1) problem, each of the objective function is linear fractional. For the problem $g_{1u}(X_K)$, (u = 1, 2, ..., s_1) any basic feasible solution for which $(\Delta_{j_K}) \le 0\ \forall\ j \in N_K^1$ and $(\Delta_{j_K}) \ge 0\ \forall\ j \in N_K^2$ is a locally optimal solution.

Since each of the objective function $g_{1u}(X)$, (u = 1, 2, ...,s_1) in (MLLFPP1) problem is quasi-monotone and is maximised over a compact polyhedron, every locally optimal solution will also be a globally optimal solution. An optimal integer solution can be obtained by repeated application of mixed integer cut in the simplex table. This helps in finding the optimal feasible solution of each objective function $g_{1u}(X)$, (u= 1, 2, ...,s_1).

4. THEOREMS

Theorem 4.1: Let $(X_K)_\omega$ (K ≥ 1); $\omega \in \{1, 2, ...s_1\}$ be an integer feasible solution for the problem $g_{1\omega}(X)$, where $g_{1\omega}(X)$ is one of the objective function of the problem (MLLFPP1). Then, all integer feasible solutions in S_1^* for $g_{1\omega}(X)$ yielding value lesser than $g_{1\omega}(X_K)$ (K ≥ 1), $\omega \in \{1, 2, ..., s_1\}$ lies in the closed half space

$$\sum_{j \in T_1^K} (x_j - l_j)_\omega - \sum_{j \in T_2^K} (u_j - x_j)_\omega \ge 1 \tag{18}$$

Proof: Let $(X_K)_\omega$ (K ≥ 1) be an integer feasible solution for the problem $g_{1\omega}(X)$ $\omega \in \{1,2, ..., s_1\}$. We need to show that all integer feasible solutions which give value less than $(X_K)_\omega$ for $g_{1\omega}(X)$ lies in the closed half space, given by equation (18).

Let $B_{K\omega}$ be the basis matrix corresponding to ($X_{B_{K\omega}}$).

We have $\quad AX_{B_{K\omega}} = b$

That is,
$$B_{K\omega}X_{B_{K\omega}} + \sum_{j \in N_K^1} a_j x_j + \sum_{j \in N_K^2} a_j x_j = b \tag{19}$$

Suppose that corresponding to the current feasible solution, we have $x_{j_K} = l_{j_K}, j_K \in N_{K\omega}^1$ and $x_{j_K} = u_{j_K}, j_K \in N_{K\omega}^2$. Therefore, from (19), we have

$$B_{K\omega}X_{B_{K\omega}} + \sum_{j \in N_K^1} a_{j_K} l_{j_K} + \sum_{j \in N_K^2} a_{j_K} u_{j_K} = b \tag{20}$$

For some $r \in T_1^{K\omega}$, $a_{r_K} = \sum_{t \in I_{K\omega}} y_{t_r}^K b_r$, where $I_{K\omega} = \{t \mid a_t \in B_{K\omega}\}$. Choose a scalar $\phi_r^K > 0$, equation (20) becomes

$$\sum_{t \in I_{K\omega}} b_t x_{B_t}^K + \sum_{j \in N_K^1} a_{j_K} l_{j_K} + \sum_{j \in N_K^2} a_{j_K} u_{j_K} + \phi_r^K a_{r_K} - \phi_r^K a_{r_K} = b$$

That is,

$$\sum_{t \in I_{K\omega}} [x_{B_t}^K - \phi_r^K y_{t_r}^K] b_t + \sum_{j \in N_K^1 \setminus \{r\}} a_{j_K} l_{j_K} + a_{r_K}(l_{r_K} + \phi_r^K) + \sum_{j \in N_K^2} a_{j_K} u_{j_K} = b \tag{21}$$

Equation (21) gives a new basic feasible solution for the objective function $g_{1\omega}(X)$. It is given by

$$X_K^1 = \begin{cases} x_{B_t}^1 = x_{B_t}^K - \phi_r^K y_{t_r}^K, & \forall t \in I_{K\omega} \\ x_{r_K}^1 = l_{r_K} + \phi_r^K, & \text{for } r \in T_1^{K\omega} \\ x_{j_K}^1 = l_{j_K}, & \forall j \in N_{K\omega}^1 \setminus \{r\} \\ x_{j_K}^1 = u_{j_K}, & \forall j \in N_{K\omega}^2 \end{cases} \tag{22}$$

Here, $x_{j_K}^1 = l_{j_K}, \forall j \in N_{K\omega}^1 \setminus \{r\}$ and $x_{j_K}^1 = u_{j_K}, \forall\ j \in N_{K\omega}^2$ are integers. Therefore, for $X_{K\omega}^1$ to be an integer solution, it is required that ϕ_r^K should be a positive integer, so that $x_{K\omega}^1 = l_{r_K} + \phi_r^K$, for $r \in T_1^{K\omega}$ is also an integer. It is required that $\phi_r^K y_{t_r}^K, \forall t \in I_{K\omega}$ is an integer, so that $x_{B_t}^1 = x_{B_t}^K - \phi_r^K y_{t_r}^K, \forall t \in I_{K\omega}$ is an integer. Besides this, $x_{B_t}^1$ and $x_{r_K}^1$ should lie between the specified bounds, that is,

$$l_{B_t} \le x_{B_t}^1 \le u_{B_t} \quad \forall t \in I_{K\omega},$$

and $\quad l_{r_K} \le x^1_{r_K} \le u_{r_K} \quad \forall r \in T_1^{K\omega}$

This implies $\quad l_{r_K} \le l_{r_K} + \phi_r^K \le u_{r_K}$, that is, $\phi_r^K \le u_{r_K} - l_{r_u}$ for $r \in T_1^{K\omega}$ (23)

Again, we have $l_{B_t} \le x^1_{B_t} \le u_{B_t}\ \forall t \in I_{K\omega}$, that is,

$$l_{B_t} \le x^K_{B_t} - \phi_r^K y^K_{t_r} \le u_{B_t} \qquad \forall\ t \in I_{K\omega}.$$

Three different cases arise depending on the value of $y^K_{t_r}$.

Case (1) : If $y^K_{t_r} = 0$, then $\phi_r^K y^K_{t_r} = 0$.

This implies $l_{B_t} \le x^K_{B_t} \le u_{B_t} \quad \forall\ t \in I_{K\omega}$.

The condition is satisfied.

Case (2) : If $y^K_{t_r} < 0$, then $(-\phi_r^K y^K_{t_r}) > 0$.

This implies $(x^K_{B_t} - \phi_r^K y^K_{t_r})$ is a positive integer which cannot exceed its upper bound that is,

$$x^K_{B_t} - \phi_r^K y^K_{t_r} \le u_{B_t} \quad \forall\ t \in I_{K\omega}$$

or $$\phi_r^K \le \frac{u_{B_t} - x^K_{B_t}}{-y^K_{t_r}} \quad \forall\ t \in I_{K\omega} \tag{24}$$

Case (3): If $y^K_{t_r} > 0$, then $-(\phi_r^K y^K_{t_r}) < 0$

Thus, we have $(x^K_{B_t} - \phi_r^K y^K_{t_r})$ which cannot be less than its lower bound, that is,

$$l_{B_t} \le x^K_{B_t} - \phi_r^K y^K_{t_r} \quad \forall t \in I_{K\omega}$$

or $$\phi_r^K \le \frac{x^K_{B_t} - l_{B_t}}{y^K_{t_r}} \quad \forall t \in I_{K\omega} \tag{25}$$

Thus, from (23), (24) and (25), we get ϕ_r^K can assume any possible value given by

$$\phi_r^K = \text{Min}\left\{(u_{r_K} - l_{r_K}), \left(\frac{x^K_{B_t} - l_{B_t}}{y^K_{t_r}} : y^K_{t_r} > 0,\ t \in I_{K\omega}\right), \left(\frac{u_{B_t} - x^K_{B_t}}{-y^K_{t_r}} : y^K_{t_r} < 0,\ t \in I_{K\omega}\right)\right\},$$

The change in the value of the objective function $g_{1\omega}(X)$ is given by

$$g_{1\omega}(X'_K) - g_{1\omega}(X_K) = \frac{\phi_r^K (\Delta_{r_K})_\omega}{D(X_K)_\omega [D(X_K)_\omega - \phi_r^K (z_r^u - d_r)_{K\omega}]}; \quad \omega \in \{1, 2, \ldots, s_1\}$$

where $(\Delta_{r_K})_\omega = N(X_K)_\omega (z_r^\omega - d_r)_{K\omega} - D(X_K)_\omega (z_r^\omega - c_r)_{K\omega}$

Since $(\Delta_{r_K})_\omega < 0$ for $r \in T_1^K$, therefore, $g_{1\omega}(X'_K) < g_{1\omega}(X_K)$.

Thus, all other integer feasible solutions which can be derived from $(X_K)_u$ by moving in the direction of $r \in T_1^K$ will give value lower than $g_{1\omega}(X_K)$ and lie in the closed half space

$$\sum_{r \in T_1^K} (x_{j_K} - l_{j_K}) \ge 1 \tag{26}$$

Similarly, it can be shown that all integer feasible solutions which can be derived from $(X_K)_\omega$ by moving in the direction of $r \in T_2^K$ will give value lesser than $g_{1\omega}(X_K)$ and lie in the closed half space

$$\sum_{r \in T_2^K} (u_{j_K} - x_{j_K}) \ge 1 \tag{27}$$

From (26) and (27), we have $\sum_{r \in T_1^K} (x_{j_K} - l_{j_K}) + \sum_{r \in T_2^K} (u_{j_K} - x_{j_K}) \ge 1$.

Note: The result in Theorem (1) is proved for $g_{1\omega}(X)$, one of the objective function of the problem (MLLFPP1). It holds for each objective function at the first level $g_{1u}(X)$, u = 1, 2, ..., s_1. The result can be proved for each of the objective function of the problem (MLLFPP), $g_{iu}(X);\ i = 1, 2, ..., n;\ u = 1, 2, ..., s_i$.

Definition 4.1: Edge - An edge E_r^K for some $\{r\} \in N_K^1$ incident at an integer feasible solution (X_K) is defined as

$$E_r^K : \begin{cases} x_t = x_{t_K} - \phi_r^K (y_{t_{r_K}}), & t \in I_K \\ x_{j_K} = l_{t_K} + \phi_r^K, & \{r\} \in N_K^1 \\ x_{j_K} = l_{j_K}, & j \in N_K^1 \setminus \{r\} \\ x_{j_K} = u_{j_K}, & j \in N_K^2 \end{cases} \tag{28}$$

where

$$0 \le \phi_r^K \le \mathrm{Min}\left\{ (u_{r_K} - l_{r_K}), \left(\frac{x_{B_t}^K - l_{B_t}^K}{(y_{t_j})_K} : (y_{t_j})_K > 0, t \in I_K \right), \left(\frac{u_{B_t}^K - x_{B_t}^t}{-(y_{t_j})_K} : (y_{t_j})_K < 0, t \in I_K \right) \right\} \tag{29}$$

Definition 4.2: An edge E_r^K for some $\{r\} \in N_K^2$ incident at an integer feasible solution (X_K) is defined as

$$E_r^K : \begin{cases} x_t = x_{t_K} + \phi_r^K (y_{t_{jk}}), & t \in I_K \\ x_{r_k} = u_{r_K} - \phi_r^K, & \{r\} \in N_K^2 \\ x_{j_K} = l_{j_K}, & j \in N_K^1 \\ x_{j_K} = u_{j_K}, & j \in N_K^2 \setminus \{r\} \end{cases} \tag{30}$$

where

$$0 \le \phi_r^K \le \mathrm{Min}\left\{ (u_{r_K} - l_{r_K}), \left(\frac{x_{B_t}^K - l_{B_t}^K}{-(y_{t_j})_K} : (y_{t_j})_K < 0, t \in I_K \right), \left(\frac{u_{B_t}^K - x_{B_t}^K}{(y_{t_j})_K} : (y_{t_j})_K > 0, t \in I_K \right) \right\} \tag{31}$$

Theorem 4.2: Edge Truncating Cut [ETC]: An integer feasible solution of the objective function $g_{1\omega}(X)$, $\omega \in \{1, 2, ..., s_1\}$ for the problem (MLLFPP1), not lying at an edge E_r^K, $r \in T_1^K$ of the truncated region, through an integer point, say $(X_K)_\omega$, lies in the closed half space

$$\sum_{j \in N_K^1 \setminus \{r\}} (x_{j_K} - l_{j_K}) + \sum_{j \in N_K^2} (u_{j_K} - x_{j_K}) \ge 1 \tag{32}$$

Proof: Let $(X_K^*)_u = (x_{j_K}^*)_u$ be an integer feasible solution of the objective function $g_{1\omega}(X)$ such that $(X_{j_K}^*)_\omega$ does not lie in the closed half space. Then, $(X_{j_K}^*)_\omega$ must lie in the open half space, given by

$$\sum_{j \in N_K^1 \setminus \{r\}} (x_{j_K} - l_{j_K}) + \sum_{j \in N_K^2} (u_{j_K} - x_{j_K}) < 1$$

Since $(X_{j_K}^*)_\omega$ is an integer feasible solution of $g_{1\omega}(X)$, $\omega \in \{1, 2, ..., s_1\}$, which is lying in the open half space, therefore,

$$(X_K^*)_\omega = \begin{cases} x_{B_{t_K}}^* = x_{B_{t_K}} - \phi_r^K y_{t_{r_K}}, & \forall\, t \in I_{K\omega} \\ x_{r_k}^* = l_{r_K} + \phi_r^K, & r \in N_K^1 \\ x_{j_K}^* = l_{j_K}, & \forall\, j \in N_K^1 \setminus \{r\} \\ x_{j_K}^* = u_{j_K}, & \forall\, j \in N_K^2. \end{cases}$$

We have, $x^*_{j_k} = l_{j_K}$ $\forall j \in N^1_K \setminus \{r\}$; $x^*_{j_K} = u_{j_K}$ $\forall$ $j \in N^2_K$ and either $x^*_{j_K} = l_{j_K}$ or $x^*_{r_k} > l_{r_K}$.

Case (1): Suppose that $x^*_{r_k} = l_{r_K}$.

We have, $x^*_{j_k} = l_{j_K}$ $\forall j \in N^1_K$ and $x^*_{j_K} = u_{j_K}$ $\forall$ $j \in N^2_K$.

$\therefore$ $(N^1_K \setminus \{r\}) \cup \{r\} \cup N^2_K = N^1_K \cup N^2_K$.

This means that index set of non-basic variables corresponding to $(X^*_K)_\omega$ is same as $(X_K)_\omega$. This implies that $(X^*_K)_\omega = (X_K)_\omega$. But this is not possible, since $(X_K)_\omega$ lies on edge E^K_r, whereas $(X^*_K)_\omega$ does not. Hence $x^*_{r_K} \neq l_{r_K}$.

Case (2): Suppose that $x^*_{r_k} > l_{r_K}$.

We have, $x^*_{r_k} = l_{r_K} + \phi^K_r$ where $\phi^K_r > 0$ is a scalar. This means that $(X^*_K)_\omega$ lies in the direction of vector $a_r, r \in N^1_K$. Since $(X^*_K)_\omega$ is a positive integer satisfying $x^*_{r_K} = l_{r_K} + \phi^K_r$ and

$$\phi^K_r \le \mathrm{Min}\left\{(u_{r_K} - l_{r_K});\left(\frac{x^K_{B_t} - l^K_{B_t}}{(y_{t_j})_K} : (y_{t_j})_K > 0, t \in I_{K\omega}\right), \left(\frac{u^K_{B_t} - x^K_{B_t}}{-(y_{t_j})_K} : (y_{t_j})_K < 0, t \in I_{K\omega}\right)\right\}$$

Thus, $(X^*_K)_\omega$ lies on an edge E^K_r in the direction of vector $a_r, r \in N^1_K$. But this is a contradiction to our assumption that $(X^*_K)_\omega$ does not lie on edge E^K_r. Therefore, we have either $x^*_{j_K} > l_{jK}$ for at least one $j \in N^1_K \setminus \{r\}$ or $x^*_{j_K} < u_{j_K}$ for some $j \in N^2_K$.

If $x^*_{j_K} > l_{j_K}$ for at least one $j \in N^1_K \setminus \{r\}$ this implies that $(x^*_{j_K} - l_{j_K}) > 0$ or $(x^*_{j_K} - l_{j_K}) \ge 1$ for at least one $j \in N^1_K \setminus \{r\}$. If $x^*_{j_K} < u_{j_K}$ for some $j \in N^2_K$, then $(x^*_{j_K} - u_{j_K}) < 0$ or $(x^*_{j_K} - u_{j_K}) \le 1$ for some $j \in N^2_K$, that is, $(u_{j_K} - x^*_{j_K}) \ge 1$ for some $j \in N^2_K$. Thus, integer feasible solution $(X^*_K)_\omega$ not lying on edge E^K_r, lies in the closed half space

$$\sum_{j \in N^K_1 \setminus \{r\}} (x_{j_K} - l_{j_K}) + \sum_{j \in N^2_K} (u_{j_K} - x_{j_K}) \ge 1$$

Proposition 4.1: For K ≥ 1, all integer feasible solutions alternate to $(X_K)_\omega$, $\omega \in \{1, 2, ..., s_1\}$ depends on whether $\phi^K_r < 1$ or $\phi^K_r \ge 1$.

Proof: Let $(X_K)_\omega$ be an integer feasible solution for $g_{1\omega}(X)$, one of the objective functions of the problem at the first level (MLLFPP1).

Let $(X_{K'})_\omega (K' \ge 1)$ be its Kth best integer feasible solution. Let $A^{K'}_j$ denote the set of integer feasible solutions alternate to $(X_{K'})_\omega$ on an edge $E^{K'}_r$. The alternate solution to $X_{K'}$ if it exists is obtained by moving along the edge $E^{K'}_r$ for some $r \in J^{K'}_1 \cup J^{K'}_2$. Suppose that for some $r \in J^{K'}_1 \cup J^{K'}_2$, $K' \ge 1$, $\phi^{K'}_r < 1$. Then, there is no eligible directions incident at the integer feasible solution $(X_{K'})_\omega$. Hence, there is no integer feasible solution on the edge $E^{K'}_r$. This edge $E^{K'}_r$ is truncated by applying ETC.

Let $\phi^{K'}_r \ge 1$ for some $r \in J^{K'}_1 \cup J^{K'}_2$. Since $\phi^{K'}_r$ and $\phi^{K'}_r y^{K'}_{t_r}$ are integers for all $t \in I_{k'_\omega}$, therefore, by moving along the edge $E^{K'}_r$, a solution alternate to $(X_{K'})_\omega$ is obtained. After obtaining all integer feasible solutions on the edge $E^{K'}_r$, this edge is truncated using ETC. Thus, an optimal feasible solution for $g_{1\omega}(X)$ is obtained over the truncated region. It is either an integer feasible solution alternate to $(X_{K'})_\omega$ or the next best integer solution $(X_{K'+1})_\omega$ or a non-integer point. Therefore, by repeated application of ETC and the mixed-integer cuts, whole feasible region for the integer solution at each level is scanned.

If after applying ETC's the solution at any level is infeasible, the objective function $g_{1\omega}(X)$ is infeasible. Thus, the process terminates.

For the problem $g_{1\omega}(X)$, $\omega \in \{1, 2, ..., s_1\}$, the procedure for finding the integer solution moves from one extreme point to another which are finite in number, therefore, the procedure for finding the optimal solution for $g_{1\omega}(X)$ terminates in a finite number of steps.

Note: The above results have been proved for each $g_{1\omega}(X)$; $\omega \in \{1, 2, ..., s_1\}$, one of the objective function of the problem (MLLFPP1). These results also hold for each of the objective functions ($g_{1\omega}(X)$; $i = 1, ..., n$, $u = 1, 2, ..., s_i$) at K-levels of the problem (MLLFPP).

5. ALGORITHM FOR SOLVING MULTI-LEVEL INTEGER PROGRAMMING PROBLEM WITH MULTIPLE OBJECTIVES AT EACH LEVEL

Step 1 Set $i = 1$, $K = 1$.

Step 2 Set $u = 1, 2, ..., s_i$.

Step 3 Consider $Z_i(X_K)$. Let $(X_K)^r_{i,u}$ be an optimal solution of $g_{iu}(X)$, ($u = 1, 2, ..., s_i$).
If $(X_K)^r_{i,u}$ is an integer solution for $g_{1u}(X)$, go to step 4.
Otherwise, apply the mixed integer cut to find the integer solution of $g_{1u}(X)$.

Step 4 Consider $Z_{i+1}(X_K)$. Solve $g_{i+1,u}(X)$, $u = 1, 2, ..., s_{i+1}$.
Let its optimal integer solution be $(\tilde{X}_K)^r_{i+1,u}, u = 1, 2, ..., s_{i+1}$.
If $(X_K)^r_{i,u} = (\tilde{X}_K)^r_{i+1,u}$, go to step 6 or to step 9.
Otherwise, set $(J^K)_{i,u} = (J^K_{1,r})_{i,u} \cup (J^K_{2,r})_{i,u}$. Go to step 5.

Step 5 If $(J^K)_{i,u} = \phi$, introduce the cut given by equation (18) into the optimal table of $(X_K)^r_{i,u}$.
Go to step 8.
If $(J^K)_{i,u} \neq \phi$ choose $j \in J^K$ for which $(\phi^K_j)_{i,u} \geq 1$ and determine all the integer solutions along the edge $(E^K_j)_{i,u}$. Formulate the set $(A^K_j)^r_{i,u}$, that is the set of integer feasible solutions alternate to $(X_K)^r_{i,u}$ on the edge $(E^K_j)^r_{i,u}$. Go to step 7.
If $(\phi^K_j)_{i,u} < 1$, for $j \in (J^K)_{i,u}$, choose any $\{j\}$ and go to step 7.

Step 6 Formulate the set $(A^K_j)^r_{i+1,u}$.
If $(A^K_j)^r_{i,u} \cap (A^K_j)^r_{i+1,u} \neq \phi$ i.e. for some j, $(X^K_j)_{i,u} = (X^K_j)_{i+1,u}$; go to step 9. Otherwise, go to step 7.
If $(A^K_j)^r_{i,u} \cap (A^K_j)^r_{i+1,u} = \phi$. Go to step 9.

Step 7 Truncate the edge $(E^K_j)_{i,u}$ by applying the cut

$$\sum_{j \in N^1_K \setminus \{r\}} (x_{j_K} - l_{j_K})_u + \sum_{j \in N^2_K} (u_{j_K} - x_{j_K})_u \geq 1 \qquad \{j\} \in T^1_K$$

or $$\sum_{j \in N^1_K} (x_{j_K} - l_{j_K})_u + \sum_{j \in N^2_K \setminus \{r\}} (u_{j_K} - x_{j_K})_u \geq 1 \qquad \{j\} \in T^2_K$$

Step 8 If the problem so obtained is infeasible, go to step 11. Otherwise, set $r = r + 1$. Go to step 3.

Step 9 Find the efficient feasible solution for the problem $Z_i(X_K)$.
Set $i = i + 1$. Go to step 2.

Step 10 Formulate the set of efficient feasible solutions at every level of the problem (MLLFPP).
From the set of efficient feasible solutions so formed at each level of the problem, formulate the set of efficient solutions (SE) for the problem (MLLFPP).

Step 11 (MLLFPP) is infeasible.

6. ILLUSTRATIVE EXAMPLE

Consider the Multi-Level Integer Linear Fractional Programming Problem with Bounded Variables

(MLLFPP): $\underset{x_1}{\text{Max}}\, Z_1(x_1, x_2, x_3, x_4) = \text{Max}(g_{11}(X),\ g_{12}(X))$

$$g_{11}(X) = \frac{2x_1 + 3x_2 + 2}{x_1 + 3x_4 + 4}, \qquad g_{12}(X) = \frac{x_1 + x_3 + 5}{x_3 + x_4 + 3}$$

where (x_2, x_3, x_4) solves

$$\underset{x_2, x_3, x_4}{\text{Max}}\ Z_2(x_1, x_2, x_3, x_4) = \text{Max}(g_{21}(X),\ g_{22}(X))$$

$$g_{21}(X) = \frac{x_1 + 2x_2 + x_4 + 5}{x_2 + 2x_3 + 3}, \qquad g_{22}(X) = \frac{x_1 + 2x_2 + x_3 + 2}{x_2 + x_4 + 5}$$

subject to

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &\le 13 \\ 2x_2 + x_3 + x_4 &\le 10 \\ x_2 + 5x_3 &\le 11 \\ -x_1 + 4x_2 + x_3 + x_4 &\le 15 \\ x_1 + x_2 + x_4 &\le 12 \end{aligned} \tag{33}$$

where $0 \le x_1 \le 4$, $0 \le x_2 \le 2$, $1 \le x_3 \le 3$, $0 \le x_4 \le 5$ x_1, x_2, x_3, x_4 are integers.

Solution: Consider the problem at first level w.r.t. the constraints (33), defined as (MLLFPP1).

(MLLFPP1): $\text{Max}\ Z_1(x_1, x_2, x_3, x_4) = \text{Max}\ (g_{11}(X), g_{12}(X))$

subject to

$$\begin{aligned} 2x_1 + x_2 + 3x_3 + x_5 &= 13 \\ 2x_2 + x_3 + x_4 + x_6 &= 10 \\ x_2 + 5x_3 + x_7 &= 11 \\ -x_1 + 4x_2 + x_3 + x_4 + x_8 &= 15 \\ x_1 + x_2 + x_4 + x_9 &= 12 \end{aligned} \tag{34}$$

where $0 \le x_1 \le 4,\ 0 \le x_2 \le 2,\ 1 \le x_3 \le 3,\ 0 \le x_4 \le 5,\ 0 \le x_5, x_6, x_7, x_8, x_9 \le \infty.$

Solving $\text{Max}\, g_{11}(X) = \dfrac{2x_1 + 3x_2 + 2}{x_1 + 3x_4 + 4}$

subject to the constraints (34).

At lower bound, we have $x_1 = 0$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$, $x_5 = 10$, $x_6 = 9$, $x_7 = 6$, $x_8 = 14$, $x_9 = 12$.

				ℓ	ℓ	ℓ	ℓ					
			$c_j \to$	2	3	0	0	0	0	0	0	0
			$d_j \to$	1	0	0	3	0	0	0	0	0
C_B	D_B	V_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
0	0	x_5	10	2	1	3	0	1	0	0	0	0
0	0	x_6	9	0	2	1	1	0	1	0	0	0
0	0	x_7	6	0	1	5	0	0	0	1	0	0
0	0	x_8	14	−1	4	1	1	0	0	0	1	0
0	0	x_9	12	1	1	0	1	0	0	0	0	1
$N_1(X) = 2$		$z_j^{11} - c_j \to$		−2	−3	0	0	0	0	0	0	0
$D_1(X) = 4$		$z_j^{12} - d_j \to$		−1	0	0	−3	0	0	0	0	0
		$\Delta_j \to$		6	12	0	−6	0	0	0	0	0

Table 1: Solution table for the problem $g_{11}(X)$

Entering Variable: x_2

Departing Variable: $\Delta_2 = \text{Min}(\gamma_1, \gamma_2, u_2 - \ell_2)$,

where $\gamma_1 = \text{Min}\left(\dfrac{x_{B_t} - l_{B_t}}{y_{t_r}} : y_{t_r} > 0\right)$ and $\gamma_2 = \text{Min}\left(\dfrac{u_{B_t} - x_{B_t}}{-y_{t_r}} : y_{t_r} < 0\right)$

$\therefore \qquad \Delta_2 = \text{Min}\left(10, \frac{9}{2}, 6, \frac{14}{4}, 12, 2\right) = 2$

$\therefore \qquad x_2 \to l_2 + \Delta_2 = 0 + 2 = 2$

Corresponding changes in the values of x_i's is given by $X_B = \hat{b} - y_2\Delta_2$.

$$\begin{bmatrix} x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} 10 \\ 9 \\ 6 \\ 14 \\ 12 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 2 \\ 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 4 \\ 6 \\ 10 \end{bmatrix}$$

$$\therefore\ g_{11}(X) = \frac{N_1(X)}{D_1(X)} = \frac{8}{4} = 2$$

The optimal table for the problem $g_{11}(X)$ is given by

				ℓ	u	ℓ	ℓ					
			$c_j \to$	2	3	0	0	0	0	0	0	0
			$d_j \to$	1	0	0	0	0	0	0	0	0
C_B	D_B	V_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
0	0	x_5	8	2	1	3	0	1	0	0	0	0
0	0	x_6	5	0	2	1	1	0	1	0	0	0
0	0	x_7	4	0	1	5	0	0	0	1	0	0
0	0	x_8	6	−1	4	1	1	0	0	0	1	0
0	0	x_9	10	1	1	0	1	0	0	0	0	1
$N_1(X) = 8$		$z_j^{11} - c_j \to$		−2	−3	0	0	0	0	0	0	0
$D_1(X) = 4$		$z_j^{12} - d_j \to$		−1	0	0	−3	0	0	0	0	0
		$\Delta_j \to$		0	12	0	−24	0	0	0	0	0

Table 2: Optimal Table for the problem $g_{11}(X)$

Here, $\Delta_j \le 0$ for lower bounded non-basic variables and $\Delta_j \ge 0$ for upper bounded non-basic variables.

$\therefore$ Optimal solution for $g_{11}(X)$ is $(X_1)^1_{1,1} = (0, 2, 1, 0)$.

Putting $(x_1)^1_{1,1} = 0$ in $g_{21}(X)$ and solving as explained above, we find that $(X_1)^1_{2,1} = (0, 2, 1, 5)$.

$\therefore\ (X_1)^1_{1,1} \ne (X_1)^1_{2,1}$.

Formulate the set $(J^1)_{1,1} = \{j \in N^1_K : \Delta_j = 0\} = \{1, 3\}$, from Table 2.

Take j = 1. We have $0 < \phi^1_1 \le \min(4, 4, 0, 6),$, i.e. $0 < \phi^1_1 \le 4$.

Since ϕ^1_1 is an integer $\therefore$ it can assume values 4, 3, 2, 1. Using equation (22), for the values of ϕ^1_1 = 4, 3, 2, 1, the corresponding solutions of $g_{11}(X)$ are given by

$(X_2)^1_{1,1} = (4, 2, 1, 0)$, $(X_3)^1_{1,1} = (3, 2, 1, 0)$, $(X_4)^1_{1,1} = (2, 2, 1, 0)$, $(X_5)^1_{1,1} = (1, 2, 1, 0)$. Again if we take j = 3, then $0 < \phi^1_3 \le \frac{4}{5} < 1$.

Therefore, no alternate feasible solution exists corresponding to this edge.

Applying the cut $\sum_{j \in N^1_K \setminus \{r\}} (x_{jK} - l_{jK}) + \sum_{j \in N^2_K} (u_{jK} - x_{jK}) \ge 1$ in the optimal table of $g_{11}(X)$, the solution so obtained is $(X_6)^1_{1,1} = (1, 2, 1, 0)$. Thus, by applying the algorithm for the various values of $g_{11}(X)$, the corresponding values of $g_{21}(X)$ and $g_{22}(X)$ are tabulated as below:

	$g_{11}(X)$	$g_{21}(X)$	$g_{22}(X)$
$(X_1)^1_{1,1} =$	(0, 2, 1, 0)	(0, 2, 1, 5)	(0, 2, 1, 0)
$(X_2)^1_{1,1} =$	(4, 2, 1, 0)	(4, 0, 1, 5)	(4, 2, 1, 0)
$(X_3)^1_{1,1} =$	(3, 2, 1, 0)	(3, 0, 1, 5)	(3, 2, 1, 0)

$(X_4)^1_{1,1}$ =	(2, 2, 1, 0)	(2, 0, 1, 5)	(2, 2, 1, 0)
$(X_5)^1_{1,1}$ =	(1, 2, 1, 0)	(1, 0, 1, 5)	(1, 2, 1, 0)
$(X_6)^1_{1,1}$ =	(1, 2, 1, 0)	(1, 0, 1, 5)	(1, 2, 1, 0)
$(X_7)^2_{1,1}$ =	(0, 0, 1, 0)	(0, 2, 1, 5)	(0, 2, 1, 0)
$(X_8)^2_{1,1}$ =	(0, 0, 1, 2)	(0, 2, 1, 5)	(0, 2, 1, 3)
$(X_9)^2_{1,1}$ =	(0, 2, 1, 5)	(0, 2, 1, 5)	(0, 2, 1, 0)

Table 3: Evaluation of $g_{21}(X)$ and $g_{22}(X)$ corresponding to the values of $g_{11}(X)$

From above, we obtain the set of efficient feasible solution is given by $(SE)_1 = \{(4, 2, 1, 0)\}$ corresponding to which $Z_1(X) = 2$ and $Z_2(X) = 2.5$.

Again, applying the algorithm to $g_{12}(X)$, the table so formed is given below:

	$g_{12}(X)$	$g_{21}(X)$	$g_{22}(X)$
$(X_1)^1_{1,2}$ =	(4, 0, 1, 0)	(4,0, 1, 5)	(4, 0, 1, 0)
$(X_2)^1_{1,2}$ =	(4, 0, 1, 5)	(4, 0, 1, 5)	(4, 2, 1, 0)
$(X_3)^2_{1,2}$ =	(3, 0,1, 0)	(3, 0, 1, 5)	(3, 2, 1, 0)
$(X_4)^3_{1,2}$ =	(2, 2, 1, 0)	(2, 0, 1, 5)	(2, 2, 1, 0)
$(X_5)^3_{1,2}$ =	(2, 2, 1, 5)	(2, 0, 1, 5)	(2, 2, 1, 0)
$(X_6)^3_{1,2}$ =	(3, 0, 1, 5)	(3, 0,1, 5)	(3, 0, 1, 0)
$(X_7)^3_{1,2}$ =	(1, 0, 2, 5)	(1, 0, 1, 5)	(1, 2, 1, 0)
$(X_8)^3_{1,2}$ =	(4, 0, 1, 5)	(4, 0, 1, 5)	(4, 2, 1, 0)
$(X_9)^3_{1,2}$ =	(4, 2, 1, 0)	(4, 0, 1, 5)	(4, 2, 1, 0)
$(X_{10})^3_{1,2}$ =	(4, 2, 1, 5)	(4, 0, 1, 5)	(4, 2, 1, 0)

Table 4: Evaluation of $g_{21}(X)$ and $g_{22}(X)$ corresponding to the values of $g_{12}(X)$

From above, the set of efficient feasible solutions is given by $(SE)_2 = \{(4, 2, 1, 0)\}$, corresponding to which $Z_1(X) = 2$ and $Z_2(X) = 2.5$

Thus, the set of efficient feasible solutions for the problem (MLLFPP) is given by

$(SE) = (SE)_1 \cup (SE)_2 = \{(4,2,1,0)\}$.

The above problem is solved using LINGO 17. The set of efficient solution for the problem (MLLFPP) so obtained is (4, 2, 1, 0).

7. CONCLUSION

The proposed algorithm scans the feasible region of multi-level programming problem with multi-objectives at each level (MLLFPP). The scanning is done to find the efficient integral solutions of (MLLFPP) problem. The portion of the feasible region which contains no integer feasible solution is removed by the edge truncating cut. The algorithm scrutinizes the edges in such a manner that edges once removed cannot reappear. The problem (MLFPP) is also solved using LINGO17. With the computing software, the following observations are noted:

For the objective function $g_{11}(X)$, total solver iterations: 21, Elapsed runtime in seconds: 0.19

For the objective function $g_{12}(X)$, total solver iterations: 26, Elapsed runtime in seconds: 0.22

After putting the value of x_1=0 from the upper level problem in the lower level problem and solving, we get

For the objective function $g_{21}(X)$, total solver iterations: 20, Elapsed runtime in seconds: 0.11

For the objective function $g_{22}(X)$, total solver iterations: 20, Elapsed runtime in seconds: 0.12

Thus, we observe that the computing software LINGO 17 supports the calculations and the convergence time of the algorithm is apparently reducing.

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