Correction, Interrogation 1. Sujet A

Exo 1.

1. $\iint f(x,y)\,dx\,dy = 1$

$\int_0^1\int_0^1 \frac{1}{a\sqrt{xy}}\,dx\,dy = \frac{4}{a} = 1$, $a = 4$.

2. $f_X(x) = \int f(x,y)\,dy = \int_0^1 \frac{1}{4\sqrt{xy}} \mathbb{1}_{(0,1]}(x)\,dy = \frac{1}{4\sqrt{x}} \mathbb{1}_{(0,1]}(x) \int_0^1 y^{-\frac{1}{2}}\,dy$

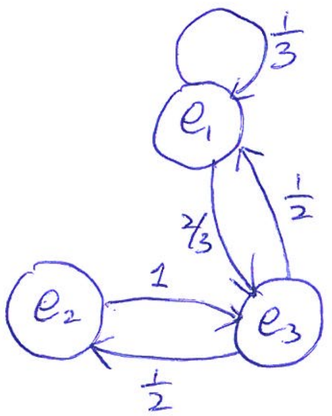
$= \frac{1}{2\sqrt{x}} \mathbb{1}_{(0,1]}(x)$

$f_Y(y) = \frac{1}{2\sqrt{y}} \mathbb{1}_{(0,1]}(y)$.

oui, parceque $f(x,y) = f_X(x) f_Y(y)$.

3. $f_{X|Y=y} = f_X(x)$

Exo 2.



1. $M = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ \frac{2}{3} & 1 & 0 \end{pmatrix}$, oui.

2. $M\mu = \mu$, $\mu = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$

$$\begin{cases} \frac{1}{3}u_1 + 0 + \frac{1}{2}u_3 = u_1 \\ 0 + 0 + \frac{1}{2}u_3 = u_2 \\ \frac{2}{3}u_1 + u_2 + 0 = u_3 \\ u_1 + u_2 + u_3 = 1 \end{cases} \Rightarrow \begin{cases} u_1 = \frac{1}{3} \\ u_2 = \frac{2}{9} \\ u_3 = \frac{4}{9} \end{cases}$$

3. ① Sur le graphe, on peut trouver qu'il existe deux chemins de longueur 2 qui va de e_1 à e_1. Ils sont $e_1 e_3 e_1$ et $e_1 e_1 e_1$.

② $M^2 = \begin{pmatrix} \frac{4}{9} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{2} & 0 \\ \frac{2}{9} & 0 & \frac{5}{6} \end{pmatrix}$, $M^2_{(1,1)} = \frac{4}{9} > 0$.

③ supposons $X_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\hat{X}_2 = M^2 X_0 = \begin{pmatrix} 1/9 \\ 0 \\ 0 \end{pmatrix}$. $P(\hat{X}_2 = e_1) = \frac{1}{9} > 0$

$P(X_3 = e_3 \mid X_1 = e_3) = M^2_{(3,3)} = \frac{5}{6}$.

4. L'état le plus probable est e_3 car $\max(u_1, u_2, u_3) = u_3$.

Exo 3.

1.

S \ N	0	1	2	marg. S
0	$\frac{1}{3}$	$\frac{1}{3}p$	$\frac{1}{3}p^2$	$\frac{1}{3}(1+p+p^2)$
1	0	$\frac{1}{3}(1-p)$	$\frac{2}{3}p(1-p)$	$\frac{1}{3}(1+p-2p^2)$
2	0	0	$\frac{1}{3}(1-p)^2$	$\frac{1}{3}(1-p)^2$
marg. N	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

2.

$S\|_{N=1}$	0	1	2
$\mathbb{P}_{S\|_{N=1}}$	p	$1-p$	0

$Ber(1-p)$ / $Bin(1, 1-p)$

$S\|_{N=2}$	0	1	2
$\mathbb{P}_{S\|_{N=2}}$	p^2	$2p(1-p)$	$(1-p)^2$

$Bin(2, 1-p)$

3. $\mathbb{E}(S|N=0)=0$, $\mathbb{E}(S|N=1) = 0\times p + 1\times(1-p) + 2\times 0 = 1-p$

$\mathbb{E}(S|N=2) = 0\times p^2 + 1\times 2p(1-p) + 2\times(1-p)^2 = 2(1-p)$

$\mathbb{E}(S) = \sum_{i=0}^{2} \mathbb{E}(S|N=i)\mathbb{P}(N=i) = \frac{1}{3}(0 + 1-p + 2(1-p)) = 1-p$

4. $\mathbb{E}(S|N=i) = i(1-p)$ car $\mathbb{P}_{S|_{N=i}} \sim Bin(i, 1-p)$

$\mathbb{E}(S) = \sum_{i=1}^{n} \mathbb{E}(S|N=i)\mathbb{P}(N=i) = \sum_{i=1}^{n} i(1-p)\mathbb{P}(N=i)$

$= (1-p)\sum_{i=1}^{n} i\,\mathbb{P}(N=i) = (1-p)\,\mathbb{E}(N) = (1-p)\frac{n+1}{2}$.

5. $\mathbb{E}(S) = (1-p)\,\mathbb{E}(N) = \lambda(1-p)$.

Correction, Interrogation 1. Sujet B

Exo 1.

1. $\iint f(x,y)\,dx\,dy = 1$

$$\int_0^2\int_0^2 \frac{a}{\sqrt{xy}}\,dx\,dy = 8a = 1, \quad a = \frac{1}{8}.$$

2. $$f_X(x) = \int f(x,y)\,dy = \int_0^2 \frac{1}{8\sqrt{xy}}\mathbb{1}_{(0,2]}(x)\,dy = \frac{1}{8\sqrt{x}}\mathbb{1}_{(0,2]}(x)\int_0^2 y^{-\frac{1}{2}}\,dy$$

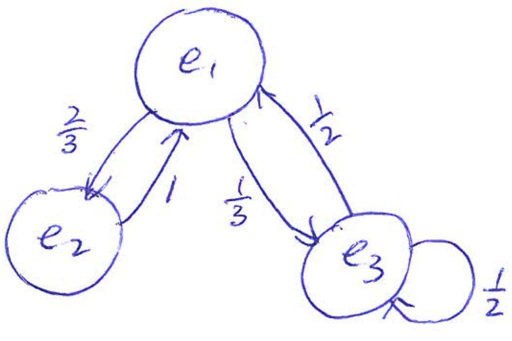
$$= \frac{\sqrt{2}}{4\sqrt{x}}\mathbb{1}_{(0,2]}(x)$$

$$f_Y(y) = \frac{\sqrt{2}}{4\sqrt{y}}\mathbb{1}_{(0,2]}(y).$$

oui, parce que $f(x,y) = f_X(x)f_Y(y)$.

3. $f_{X|Y=y} = f_X(x)$

Exo 2.



1. $$M = \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{2} \end{pmatrix}, \quad \text{oui.}$$

2. $M\mu = \mu$, $\mu = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$

$$\begin{cases} 0 + 1 \times u_2 + \frac{1}{2} u_3 = u_1 \\ \frac{2}{3} u_1 + 0 + 0 = u_2 \\ \frac{1}{3} u_1 + 0 + \frac{1}{2} u_3 = u_3 \\ u_1 + u_2 + u_3 = 1 \end{cases} \Rightarrow \begin{cases} u_1 = \frac{3}{7} \\ u_2 = \frac{2}{7} \\ u_3 = \frac{2}{7} \end{cases}$$

3. ① Sur le graphe, on peut trouver qu'il existe deux chemins de longueur 2 qui va de e_3 à e_3. Ils sont $e_3 e_1 e_3$ et $e_3 e_3 e_3$.

② $M^2 = \begin{pmatrix} \frac{5}{6} & 0 & \frac{1}{4} \\ 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{5}{12} \end{pmatrix}$, $M^2_{(\cancel{1}3,\cancel{1}3)} = \frac{5}{\cancel{6}12} > 0$.

$\mathbb{P}(X_3 = e_1 \mid X_1 = e_1) = M^2(1,1) = \frac{5}{6}$.

4. L'état le plus probable est e_1 car $\max(u_1, u_2, u_3) = u_1$.

Exo 3.

1.

S \ N	0	1	2	marg. S
0	$\frac{1}{4}$	$\frac{1}{2}(1-p)$	$\frac{1}{4}(1-p)^2$	$\frac{1}{4}(4-4p+p^2)$
1	0	$\frac{1}{2}p$	$\frac{2}{4}(1-p)p$	$\frac{1}{2}(2p-p^2)$
2	0	0	$\frac{1}{4}p^2$	$\frac{1}{4}p^2$
marg. N	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	1

B ②

2.

$S\|_{N=1}$	0	1	2
$\mathbb{P}_{S\|N=1}$	$1-p$	p	0

$Ber(p)$ / $Bin(1, p)$

$S\|_{N=2}$	0	1	2
$\mathbb{P}_{S\|N=2}$	$(1-p)^2$	$2p(1-p)$	p^2

$Bin(2, p)$

3. $\mathbb{E}(S|N=0)=0$, $\mathbb{E}(S|N=1) = 0\times(1-p)+1\times p = p$

$$\mathbb{E}(S|N=2) = 0\times(1-p)^2 + 1\times 2p(1-p) + 2p^2 = 2p$$

$$\mathbb{E}(S) = \sum_{i=0}^{2} \mathbb{E}(S|N=i)\,\mathbb{P}(N=i) = p\times\frac{1}{2} + 2p\times\frac{1}{4} = p$$

4. $\mathbb{E}(S|N=i) = ip$ car $\mathbb{P}_{S|N=i} \sim Bin(i, p)$

$$\mathbb{E}(S) = \sum_{i=0}^{n} \mathbb{E}(S|N=i)\,\mathbb{P}(N=i) = \sum_{i=0}^{n} ip\,\mathbb{P}(N=i)$$

$$= p\sum_{i=0}^{n} i\,\mathbb{P}(N=i) = p\,\mathbb{E}(N) = npq.$$

5. $\mathbb{E}(S) = p\,\mathbb{E}(N) = \lambda p.$