Correction, Interrogation 2. Sujet A

Exo 1.

1. $L(\lambda) = \prod_{i=1}^{n} \mathbb{P}(X = X_i) = (1-\frac{1}{\lambda})^{\Sigma(X_i - 1)} (\frac{1}{\lambda})^n = (\frac{\lambda-1}{\lambda})^{\Sigma X_i - n} \lambda^{-n}$

$H(\lambda) = \ln L(\lambda) = (\Sigma X_i - n)(\ln(\lambda-1) - \ln\lambda) - n\ln\lambda$

$$\frac{dH(\lambda)}{d\lambda} = (\Sigma X_i - n)\left(\frac{1}{\lambda-1} - \frac{1}{\lambda}\right) - \frac{n}{\lambda} = 0 \Rightarrow \hat{\lambda}_{MV} = \frac{1}{n}\Sigma X_i = \overline{X_n}$$

$$\frac{d^2H(\lambda)}{d\lambda^2} = (\Sigma X_i - n)\left(-(\lambda-1)^{-2} + \lambda^{-2}\right) + n\lambda^{-2} = \frac{n\lambda^2 - 2\lambda\Sigma X_i + \Sigma X_i}{\lambda^2(\lambda-1)^2}$$

Si $\lambda = \frac{1}{n}\Sigma X_i$, alors $n\lambda^2 - 2\lambda\Sigma X_i + \Sigma X_i = \Sigma X_i\left(1 - \frac{\Sigma X_i}{n}\right) < 0$

puisque $X_i \geq 1$ ($\Rightarrow \Sigma X_i \geq n \Rightarrow \frac{\Sigma X_i}{n} \geq 1$).

Donc $\frac{d^2H(\lambda)}{d\lambda^2} < 0$ pour $\lambda = \overline{X_n}$.

2. D'après le théorème central limite, si $\mathbb{E}(X_1^2) < \infty$ alors

$$\frac{\overline{X_n} - \mathbb{E}X_1}{\sqrt{\frac{Var(X_1)}{n}}} \Rightarrow \mathcal{N}(0,1).$$

C'est-à-dire $$\frac{\hat{\lambda}_{MV} - \lambda}{\sqrt{\frac{\lambda(\lambda-1)}{n}}} \Rightarrow \mathcal{N}(0,1).$$

On en déduit que pour n grand $\hat{\lambda}_{MV}$ suit la loi gaussienne d'espérance λ et de variance $\frac{\lambda(\lambda-1)}{n}$, i.e. $\mathcal{N}(\lambda, \frac{\lambda(\lambda-1)}{n})$.

#. Notons $Y \sim \mathcal{N}(0,1)$ et $\mathbb{P}(|Y| \leq q_\alpha) = \alpha$.

On a $$\mathbb{P}\left(\frac{|\hat{\lambda}_{MV} - \lambda|}{\sqrt{\frac{\lambda(a-1)}{n}}} \leq q_\alpha \right) = \alpha.$$

① On en déduit $\mathbb{P}\left(\hat{\lambda}_{MV} - q_\alpha \sqrt{\frac{\lambda(a-1)}{n}} \leq \lambda \leq \hat{\lambda}_{MV} + q_\alpha \sqrt{\frac{\lambda(a-1)}{n}} \right) = \alpha$,

donc $IC_\alpha = \left[\bar{X}_n - q_\alpha \sqrt{\frac{\lambda(a-1)}{n}} , \bar{X}_n + q_\alpha \sqrt{\frac{\lambda(a-1)}{n}} \right]$.

② $-q_\alpha \leq \frac{\sqrt{n}(\hat{\lambda} - \lambda)}{\lambda} \leq q_\alpha \Leftrightarrow \frac{\hat{\lambda}}{1 + \frac{q_\alpha}{\sqrt{n}}} \leq \lambda \leq \frac{\hat{\lambda}}{1 - \frac{q_\alpha}{\sqrt{n}}}$

4.3 D'après l'énoncé, $\mathbb{P}(|Y| \leq 1{,}96) = 0{,}95$, i.e. $q_{95\%} = 1.96$

① On a donc $IC_{95\%} = \left[\bar{X}_n - 1{,}96 \frac{\hat{\lambda}}{\sqrt{n}} , \bar{X}_n + 1{,}96 \frac{\hat{\lambda}}{\sqrt{n}} \right]$

$= [1000 - 196 , 1000 + 196]$.

② $-1{,}96 \leq \frac{\sqrt{n}(\bar{X} - \lambda)}{\lambda} \leq 1{,}96 \Leftrightarrow \frac{\bar{X}}{1 + \frac{1{,}96}{\sqrt{n}}} \leq \lambda \leq \frac{\bar{X}}{1 - \frac{1{,}96}{\sqrt{n}}}$

Exo 2.

1. Oui, parce que $f_\theta(x) \geq 0$ et $\int f_\theta(x) dx = \int_0^\theta \frac{2x}{\theta^2} dx$

$= \left[\frac{x^2}{\theta^2} \right]_0^\theta = 1$.

2. $F_X(x) = \int_0^x \frac{2t}{\theta^2} dt = \left[\frac{t^2}{\theta^2} \right]_0^x = \frac{x^2}{\theta^2}$, $x \in [0, \theta]$.

$F_{max}(x) = \mathbb{P}(\max(X_1, X_2) \leq x) = \mathbb{P}(X_1 \leq x , X_2 \leq x) = (\mathbb{P}(X_1 \leq x))^2$

$= \frac{x^4}{\theta^4}$, $x \in [0, \theta]$.

$g_\theta(x) = F'_{max}(x) = \frac{4x^3}{\theta^4}$, $x \in [0, \theta]$.

3. $\mathbb{E}X_1 = \int_0^\theta \frac{2x^2}{\theta^2}\,dx = \left[\frac{2x^3}{3\theta^2}\right]_0^\theta = \frac{2}{3}\theta$

$\mathbb{E}\max(X_1, X_2) = \int_0^\theta \frac{4x^4}{\theta^4}\,dx = \left[\frac{4x^5}{5\theta^4}\right]_0^\theta = \frac{4}{5}\theta$

$\mathbb{E}\hat{\theta}_1 = \frac{3}{4}(\mathbb{E}X_1 + \mathbb{E}X_2) = \frac{3}{2}\mathbb{E}X_1 = \theta$

$\mathbb{E}\hat{\theta}_2 = \frac{5}{4}\mathbb{E}\max(X_1, X_2) = \theta$

4. $\mathbb{E}X_1^2 = \int_0^\theta \frac{2x^3}{\theta^2}\,dx = \left[\frac{x^4}{2\theta^2}\right]_0^\theta = \frac{1}{2}\theta^2$

$\mathbb{E}(\max(X_1, X_2))^2 = \int_0^\theta \frac{4x^5}{\theta^4}\,dx = \left[\frac{2x^6}{3\theta^4}\right]_0^\theta = \frac{2}{3}\theta^2$

$Var(X_1) = \frac{1}{2}\theta^2 - (\frac{2}{3}\theta)^2 = \frac{1}{18}\theta^2$

$Var(\max(X_1, X_2)) = \frac{2}{3}\theta^2 - (\frac{4}{5}\theta)^2 = (\frac{2}{3} - \frac{16}{25})\theta^2$

$RQM(\hat{\theta}_1) = Var(\hat{\theta}_1) = \frac{9}{16}(Var(X_1) + Var(X_2)) = \frac{9}{8}Var(X_1) = \frac{1}{16}\theta^2$

$RQM(\hat{\theta}_2) = Var(\hat{\theta}_2) = \frac{25}{16}Var(\max(X_1, X_2)) = \frac{25}{16}(\frac{2}{3} - \frac{16}{25})\theta^2 = \frac{1}{24}\theta^2$

5. Puisque $RQM(\hat{\theta}_1) > RQM(\hat{\theta}_2)$. $\hat{\theta}_2$ est meilleur.

Correction, Interrogation 2. Sujet B

Exo 1.

1. $L(\lambda) = \prod_{i=1}^{n} f_\lambda(x_i) = (\frac{1}{\lambda})^n e^{-\frac{1}{\lambda}\Sigma x_i} = \lambda^{-n} e^{-\frac{1}{\lambda}\Sigma x_i}$

$H(\lambda) = -n\ln\lambda - \frac{1}{\lambda}\Sigma x_i$

$$\frac{dH(\lambda)}{d\lambda} = -\frac{n}{\lambda} + \lambda^{-2}\Sigma x_i = 0 \Rightarrow \hat{\lambda}_{MV} = \frac{1}{n}\Sigma x_i = \overline{X_n}$$

$$\frac{d^2H(\lambda)}{d\lambda^2} = \lambda^{-2}n - 2\lambda^{-3}\Sigma x_i$$

Si $\lambda = \frac{1}{n}\Sigma x_i$, alors $\lambda^{-2}n - 2\lambda^{-3}\Sigma x_i = -\frac{n^3}{(\Sigma x_i)^2} < 0$.

Donc $\frac{d^2H(\lambda)}{d\lambda^2} < 0$ pour $\lambda = \overline{X_n}$.

2. D'après le théorème central limite, si $\mathbb{E}(X_1^2) < \infty$ alors

$$\frac{\overline{X_n} - \mathbb{E}X_1}{\sqrt{\frac{Var(X_1)}{n}}} \Rightarrow \mathcal{N}(0,1)$$

c'est-à-dire $$\frac{\hat{\lambda}_{MV} - \lambda}{\sqrt{\frac{\lambda^2}{n}}} \Rightarrow \mathcal{N}(0,1).$$

On en déduit que pour n grand $\hat{\lambda}_{MV}$ suit la loi gaussienne

d'espérance λ et de variance $\frac{\lambda^2}{n}$, i.e. $\mathcal{N}(\lambda, \frac{\lambda^2}{n})$.

~~5.~~ Notons $Y \sim \mathcal{N}(0,1)$ et $\mathbb{P}(|Y| \leq q_\alpha) = \alpha$.

On a $\mathbb{P}\left(\dfrac{|\hat{\lambda}_{MV} - \lambda|}{\frac{\lambda}{\sqrt{n}}} \leq q_\alpha \right) = \alpha$.

① On en déduit $\mathbb{P}\left(\hat{\lambda}_{MV} - q_\alpha \dfrac{\lambda}{\sqrt{n}} \leq \lambda \leq \hat{\lambda}_{MV} + q_\alpha \dfrac{\lambda}{\sqrt{n}} \right) = \alpha$,

donc $IC_\alpha = \left[\overline{X}_n - \dfrac{\lambda q_\alpha}{\sqrt{n}} , \overline{X}_n + \dfrac{\lambda q_\alpha}{\sqrt{n}} \right]$.

② $\dfrac{\hat{\lambda}}{1 + \frac{q_\alpha}{\sqrt{n}}} \leq \lambda \leq \dfrac{\hat{\lambda}}{1 - \frac{q_\alpha}{\sqrt{n}}}$

~~4.~~3 D'après l'énoncé, $\mathbb{P}(|Y| \leq 1{,}64) = 0{,}9$, i.e. $q_{90\%} = 1.64$

① On a donc $IC_{90\%} = \left[\overline{X}_n - \dfrac{\hat{\lambda}\, 1.64}{\sqrt{n}} , \overline{X}_n + \dfrac{\hat{\lambda}\, 1.64}{\sqrt{n}} \right]$

$= [1000 - 164 , 1000 + 164]$.

② $\dfrac{\overline{X}}{1 + \frac{1.64}{\sqrt{n}}} \leq \lambda \leq \dfrac{\overline{X}}{1 - \frac{1.64}{\sqrt{n}}}$

Exo 2.

1. Oui, parce que $f_\theta(x) \geq 0$ et $\int f_\theta(x)\,dx = \int_0^\theta \dfrac{3x^2}{\theta^3}\,dx$
$= \left[\dfrac{x^3}{\theta^3} \right]_0^\theta = 1$.

2. $F_X(x) = \int_0^x \dfrac{3t^2}{\theta^3}\,dt = \left[\dfrac{t^3}{\theta^3} \right]_0^x = \dfrac{x^3}{\theta^3}$, $x \in [0, \theta]$.

$F_{max}(x) = \mathbb{P}(\max(X_1, X_2) \leq x) = \mathbb{P}(X_1 \leq x, X_2 \leq x) = (\mathbb{P}(X_1 \leq x))^2$

$= \dfrac{x^6}{\theta^6}$, $x \in [0, \theta]$.

$g_\theta(x) = F'_{max}(x) = \dfrac{6x^5}{\theta^6}$, $x \in [0, \theta]$.

3. $\mathbb{E} X_1 = \int_0^\theta \frac{3x^3}{\theta^3}\,dx = \left[\frac{3x^4}{4\theta^3}\right]_0^\theta = \frac{3}{4}\theta$

$\mathbb{E} \max(X_1, X_2) = \int_0^\theta \frac{6x^6}{\theta^6}\,dx = \left[\frac{6x^7}{7\theta^6}\right]_0^\theta = \frac{6}{7}\theta$

$\mathbb{E}\,\hat{\theta}_1 = \frac{2}{3}(\mathbb{E}X_1 + \mathbb{E}X_2) = \frac{4}{3}\mathbb{E}X_1 = \theta$

$\mathbb{E}\,\hat{\theta}_2 = \frac{7}{6}\mathbb{E}\max(X_1, X_2) = \theta$

4. $\mathbb{E} X_1^2 = \int_0^\theta \frac{3x^4}{\theta^3}\,dx = \left[\frac{3x^5}{5\theta^3}\right]_0^\theta = \frac{3}{5}\theta^2$

$\mathbb{E}\left(\max(X_1, X_2)\right)^2 = \int_0^\theta \frac{6x^7}{\theta^6}\,dx = \left[\frac{6x^8}{8\theta^6}\right]_0^\theta = \frac{3}{4}\theta^2$

$Var(X_1) = \frac{3}{5}\theta^2 - \left(\frac{3}{4}\theta\right)^2 = \frac{3}{80}\theta^2$

$Var\left(\max(X_1, X_2)\right) = \frac{3}{4}\theta^2 - \left(\frac{6}{7}\theta\right)^2 = \left(\frac{3}{4} - \frac{36}{49}\right)\theta^2$

$RQM(\hat{\theta}_1) = Var(\hat{\theta}_1) = \frac{4}{9}\left(Var(X_1) + Var(X_2)\right) = \frac{8}{9}Var(X_1) = \frac{1}{30}\theta^2$

$RQM(\hat{\theta}_2) = Var(\hat{\theta}_2) = \frac{49}{36}Var(\max(X_1, X_2)) = \frac{49}{36}\left(\frac{3}{4} - \frac{36}{49}\right)\theta^2 = \frac{1}{48}\theta^2$

5. Puisque $RQM(\hat{\theta}_1) > RQM(\hat{\theta}_2)$, $\hat{\theta}_2$ est meilleur.