Université Paris VII and I

Master 2 M0 2018 – 2019

Analyse des séries financières

Examen final, Février 2019

3h00, sans aucun document

1. Let $(\varepsilon_t)_{t\in\mathbf{Z}}$ a sequence of centered independent and identically distributed random variables such as $\mathbb{E}(\varepsilon_0^2)=1$. If it exists we consider a sequence $(X_t)_{t\in\mathbf{Z}}$ such as:

$$X_t=\varepsilon_t\,\sigma_t \quad \text{with} \quad \sigma_t=a_0+a_1\,|X_{t-1}|+b_1\,\sigma_{t-1} \quad \text{for any } t\in\mathbf{Z} \tag{1}$$

where $(a_0,a_1,b_1)\in[0,\infty)^3$ are unknown parameters with $b_1>0$.

(a) Prove that (ε_t) is a stationary time series.

(b) Denote $\mathbf{R}^\infty$ the space of sequences of real numbers with finite number of non-zero real numbers. Consider $F:\mathbf{R}^\infty\to\mathbf{R}$ be a measurable function on $\mathbf{R}^\infty$. Assume also that F is Lipchitzian on $\mathbf{R}^\infty$: there exists a sequence $(\ell_i)_{i\in\mathbf{N}^*}$ of real non-negative numbers such that for any $x=(x_i)_i,\ y=(y_i)_i\in\mathbf{R}^\infty$,

$$\big|F(x)-F(y)\big|\le\sum_{i=1}^\infty \ell_i\,|x_i-y_i|,\quad \text{where} \quad \sum_{i=1}^\infty \ell_i<\infty.$$

For $t\in\mathbf{Z}$, prove that $(Y_t^{(n)})_n$ where $Y_t^{(n)}=F\big(\varepsilon_t,\varepsilon_{t-1},\dots,\varepsilon_{t-n},0,0,\dots\big)$ is a Cauchy sequence on $\mathbb{L}^2$. Deduce that $(Y_t)_{t\in\mathbf{Z}}$, where $Y_t=F\big((\varepsilon_{t-k})_{k\in\mathbf{N}}\big)$ for $t\in\mathbf{Z}$, is a stationary second order process.

(c) Let $\alpha(\cdot)$ be the function such as $\alpha(x)=a_1|x|+b_1$. Show that $\mathbb{E}\big[\big|\log\big(\alpha(\varepsilon_0)\big)\big|\big]<\infty$ using Jensen Inequality. Deduce also that for any $t\in\mathbf{Z}$:

$$\Big(\prod_{i=1}^n \alpha(\varepsilon_{t-i})\Big)^{1/n} \xrightarrow[n\to+\infty]{a.s.} e^\gamma \quad \text{with} \quad \gamma=\mathbb{E}\big[\log\big(\alpha(\varepsilon_0)\big)\big]. \tag{2}$$

(d) Suppose $(\sigma_t)_{t\in\mathbf{Z}}$ exists in (1). Then, establish that $\sigma_t=a_0+\alpha(\varepsilon_{t-1})\,\sigma_{t-1}$ for any $t\in\mathbf{Z}$. Deduce by recurrence that for any $t\in\mathbf{Z}$ and $k\in\mathbf{N}^*$,

$$\sigma_t=\beta_t(k)+\alpha(\varepsilon_{t-1})\times\cdots\times\alpha(\varepsilon_{t-k})\sigma_{t-k} \quad \text{with} \quad \beta_t(k)=a_0\Big(1+\sum_{i=1}^{k-1}\prod_{j=1}^{i}\alpha(\varepsilon_{t-j})\Big).$$

(e) Using (2) prove that if $\gamma<0$ then for any $t\in\mathbf{Z}$, $\beta_t(n)\xrightarrow[n\to+\infty]{a.s.}\beta_t$ with β_t a positive random variable and satisfies $\beta_t=a_0+\alpha(\varepsilon_{t-1})\beta_{t-1}$. Consequently, write $X_t=F(\varepsilon_t,\varepsilon_{t-1},\dots)$ and conclude about the existence and stationarity of $(X_t)_{t\in\mathbf{Z}}$. What's happening if $a_0=0$?

(f) Assume now $\gamma<0$ and $b_1<1$. Using an iterating decomposition, show for any $t\in\mathbf{Z}$:

$$\sigma_t=\frac{a_0}{1-b_1}+a_1\sum_{j=0}^\infty b_1^j|X_{t-j-1}|.$$

Deduce $\mathbb{E}(X_t\mid X_{t-1},X_{t-2},\dots)$ and $\mathrm{var}(X_t\mid X_{t-1},X_{t-2},\dots)$. Is (X_t) a conditionaly heteroskedastic process?

(g) Assume now that $(X_1, \cdots, X_N)$ is observed and let $\theta = {}^t(a_0, a_1, b_1)$. Prove that the quasi-maximum likelihood estimator $\widehat{\theta}$ of θ is:

$$\widehat{\theta} = \operatorname*{argmin}_{\theta \in \Theta} \Big\{ \log\Big(\frac{a_0}{1-b_1}\Big) + \frac{1}{2}\Big(\frac{(1-b_1)X_1}{a_0}\Big)^2 + \sum_{i=2}^N \log\Big(\frac{a_0}{1-b_1} + a_1 \sum_{j=0}^{i-2} b_1^j |X_{i-j-1}|\Big) + \frac{1}{2}\Big(\frac{X_i}{\frac{a_0}{1-b_1} + a_1 \sum_{j=0}^{i-2} b_1^j |X_{i-j-1}|}\Big)^2 \Big\}$$

where Θ is a set that should be specified.

(h) Is $\widehat{\theta}$ a consistent estimator? What is its convergence rate?

(i) Provide forecasting of X_{N+1} and X_{N+1}^2.

Proof. (a) Consider $k \in N^*$, $t_1 < \cdots < t_k \in \mathbf{Z}^k$ and $c \in \mathbf{Z}$. Then the characteristic function of $(\varepsilon_{t_1}, \ldots, \varepsilon_{t_k})$ is $\prod_{i=1}^k \phi_\varepsilon(u_i)$ for any $(u_1, \ldots, u_k) \in \mathbf{R}^k$ since (ε_t) is a sequence of i.i.d.r.v. Thus this is the same than the one of $(\varepsilon_{t_1+c}, \ldots, \varepsilon_{t_k+c})$, implying the stationarity of (ε_t).

(b) Set $\eta > 0$. For $n_1 < n_2 \in \mathbf{N}$, we have $\big|Y_t^{(n_1)} - Y_t^{(n_2)}\big| \le \sum_{k=n_1+1}^{n_2} \ell_k |\varepsilon_{t-k}|$ using the Lipchitz property. As a consequence $\operatorname{var}\big[\big|Y_t^{(n_1)} - Y_t^{(n_2)}\big|\big] \le \operatorname{var}[|\varepsilon_0|] \sum_{k=n_1+1}^{n_2} \ell_k^2 + \mathbb{E}[|\varepsilon_0|]^2 \big(\sum_{k=n_1+1}^{n_2} \ell_k\big)^2$ and since $\sum_{k=1}^\infty \ell_k < \infty$, we deduce that there exists n_0 such as for any $n_0 \le n_1 < n_2$, $\operatorname{var}\big[\big|Y_t^{(n_1)} - Y_t^{(n_2)}\big|\big] \le \eta$. As a consequence $(Y_t^{(n)})_n$ is a Cauchy sequence on $\mathbb{L}^2$.
Since $\mathbb{L}^2$ is a Banach space, we deduce that $(Y_t^{(n)})_n$ is also a consistent sequence on $\mathbb{L}^1$ and its limit is Y_t. We also have for any $n \in \mathbf{N}$, $(\varepsilon_t)_{t_1-n \le t \le t_k}$ has the same characteristic function than $(\varepsilon_t)_{t_1+c-n \le t \le t_k+c}$. As a consequence, if we consider $Y_t^{(n)} = F(\varepsilon_t, \varepsilon_{t-1}, \ldots, \varepsilon_{t-n}, 0, 0, \ldots)$, then it is clear than $(Y_{t_1}^{(n)}, \ldots, Y_{t_k}^{(n)})$ has the same distribution than $(Y_{t_1+c}^{(n)}, \ldots, Y_{t_k+c}^{(n)})$ and therefore $(Y_t^{(n)})_t$ is a stationary sequence. As this is true for any $n \in \mathbf{N}$, this is also true when $n \to \infty$, and this implies the stationarity of (Y_t), which is also a $\mathbb{L}^2$ sequence.

(c) If $b_1 > 0$ then $\log\big(\alpha(\varepsilon_0)\big) = \log(b_1) + \log(1 + a_1' |\varepsilon_0|)$ with $a_1' a_1 / b_1 \ge 0$. Since $x[0, \infty) \mapsto \log(1+x)$ is a concave function, the Jensen Inequality implies $\mathbb{E}\big|\big[\log\big(\alpha(\varepsilon_0)\big)\big|\big] \le |\log(b_1)| + \log\big(1 + a_1' \mathbb{E}(|\varepsilon_0|)\big) \le |\log(b_1)| + \log\big(1 + a_1'\big)\big) < \infty$ since $\mathbb{E}(|\varepsilon_0|) \le \sqrt{\operatorname{var}(\varepsilon_0)} \le 1$ from Jensen Inequality.
We have

$$\Big(\prod_{i=1}^n \alpha(\varepsilon_{t-i})\Big)^{1/n} = \exp\Big(\frac{1}{n} \sum_{i=1}^n \log(\alpha(\varepsilon_{t-i})\Big).$$

Using Strong Law of Large Numbers, since the sequence $(\log(\alpha(\varepsilon_k)))_k$ is a sequence of iidrv satisfying $\mathbb{E}(|\log(\alpha(\varepsilon_0))|) < \infty$, then $\Big(\frac{1}{n} \sum_{i=1}^n \log(\alpha(\varepsilon_{t-i})\Big) \xrightarrow[n \to +\infty]{a.s.} \gamma$.

(d) We have $\sigma_t = a_0 + a_1\,|X_{t-1}| + b_1\,\sigma_{t-1} = a_0 + \sigma_{t-1}(a_1|\varepsilon_{t-1}| + b_1)$.
We prove the relationship by recurrence. It is valid for $k = 1$. Now assume it is valid for k and replace σ_{t-k} by $a_0 + \sigma_{t-k-1}(a_1|\varepsilon_{t-k-1}| + b_1)$. Then this provides the relationship at rank $k+1$.

(e) Using the Cauchy Lemma for sum of positive real numbers, since $e^\gamma < 1$ for $\gamma < 0$, we deduce from (2) that $\beta_n(t) \xrightarrow[n \to +\infty]{a.s.} \beta(t)$ for any $t \in \mathbf{Z}$.
As we have $\alpha(\varepsilon_{t-1})\beta_{t-1}(n-1) = a_0\big(\alpha(\varepsilon_{t-1}) + \sum_{i=1}^{n-2} \prod_{j=1}^i \alpha(\varepsilon_{t-1-j})\big)\big) = a_0\big(\sum_{i=1}^{n-1} \prod_{j=1}^i \alpha(\varepsilon_{t-j})\big)\big) = \beta_t(n) - a_0$.
Therefore since each limit exists almost surely, we obtain $\alpha(\varepsilon_{t-1})\beta_{t-1} = \beta_t - a_0$.
We finally obtain that if $\gamma < 0$, $\prod_{j=1}^n \alpha(\varepsilon_{t-j}) \xrightarrow[n \to +\infty]{a.s.} 0$ and therefore

$$X_t = \beta_t\,\varepsilon_t = a_0\,\varepsilon_t\,\Big(1 + \sum_{i=1}^\infty \prod_{j=1}^i \alpha(\varepsilon_{t-j})\Big),$$

implying that (X_t) exists almost surely an is a stationary process.
If $a_0 = 0$ the only possibility is $X_t = 0$ a.s.

(f) We have $\sigma_t = a_0 + a_1|X_{t-1}| + b_1(a_0 + a_1|X_{t-2}| + b_1\sigma_{t-2}) = a_0(1+b_1) + a_1(|X_{t-1}| + b_1|X_{t-2}|) + b_1^2\sigma_{t-2}$. By iteration, we obtain for any $k \in \mathbf{N}$:

$$\sigma_t = a_0\big(1 + b_1 + \cdots + b_1^k\big) + a_1\big(|X_{t-1}| + b_1|X_{t-2}| + \cdots + b_1^k|X_{t-k-1}|\big) + b_1^{k+1}\sigma_{t-k-1}.$$

As (σ_t) is stationary sequence, and $0 < b_1 < 1$ implies $b_1^{k+1} \xrightarrow[n \to \infty]{} 0$, $1 + b_1 + \cdots + b_1^k = (1-b_1)^{-1}$ and $|X_{t-1}| + b_1|X_{t-2}| + \cdots + b_1^k|X_{t-k-1}|$ converges almost surely as a linear combination of $(|X_t|)$ with $\sum |b_1^j| < \infty$, we finally obtain the relation.
We obtain $\mathbb{E}(X_t \mid X_{t-1}, \cdots) = \mathbb{E}(\varepsilon_t)\mathbb{E}(\sigma_t) = 0$ and $\operatorname{var}(X_t \mid X_{t-1}, \cdots) = \mathbb{E}(X_t^2 \mid X_{t-1}, \cdots) = \sigma_t^2$.
This implies that (X_t) is a conditionally heteroskedastic process (APARCH process).

(g) Since σ_t^2 is the volatility of (X_t), the conditional log-density of X_t with respect to $X_{t-1}, X_{t-2}, \ldots$ is $-\frac{1}{2}(\log(2\pi) + \log(\sigma_t^2) + X_t^2/\sigma_t^2)$. As a consequence, since $(X_1, \ldots, X_N)$ is observed, we replace in this formula X_t by 0 for $t \le 0$. Moreover as $\widehat{\theta}$ is supposed to be a maximizer of the quasi-log-likelihood, this is equivalent to minimize it multiplied by (-2). Finally $\Theta = (0, \infty) \times \big\{a_1 \ge 0,\ b_1 > 0,\ \mathbb{E}[|\log(a_1 + b_1|\varepsilon_0|)|] < \infty\big\}$.

(h) If we consider the relation $X_t = \varepsilon_t\Big(\frac{a_0}{1-b_1} + a_1 \sum_{j=0}^{\infty} b_1^j |X_{t-j-1}|\Big)$ it could be written as a causal affine model $X_t = M_\theta^t \varepsilon_t + f_\theta^t$ with $f_\theta^t = 0$ and $M_\theta^t = \sigma_t = G(X_{t-1}, X_{t-2}, \ldots)$. But the function G is Lipshitzian with coefficients $\alpha_j^{(0)}(M) = a_1 b_1^j$ and therefore $\sum_j \alpha_j^{(0)}(M) < \infty$. Moreover, $M_\theta^t = M_{\theta'}^t$ implies $\theta = \theta'$ almost surely. It is also possible to compute $\alpha_j^{(1)}(M)$ and $\alpha_j^{(2)}(M)$ and we conclude that $\widehat{\theta}$ is a consistent estimator of θ and there exists a definite positive matrix Σ_θ such as $\sqrt{n}\big(\widehat{\theta} - \theta\big) \underset{n\to\infty}{\overset{\mathcal{L}}{\longrightarrow}} \mathcal{N}\big(0\,,\,\Sigma_\theta\big)$.

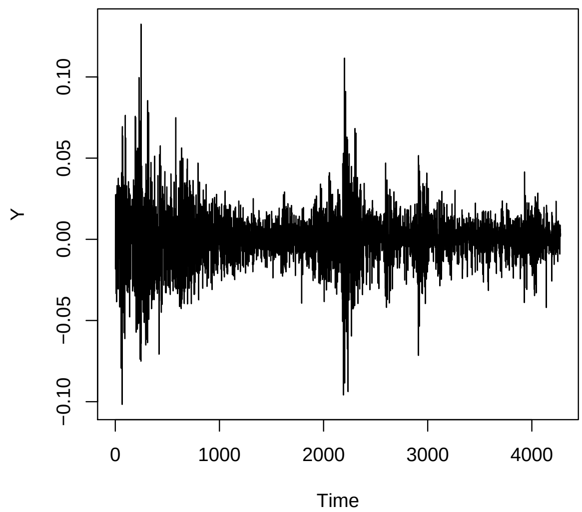
(i) From prÃ©vious computations, $\widehat{X}_{N+1} = 0$ and $\widehat{X}_{N+1}^2 = \Big(\frac{\widehat{a}_0}{1-\widehat{b}_1} + \widehat{a}_1 \sum_{j=0}^{\infty} \widehat{b}_1^j |X_{t-j-1}|\Big)^2$.

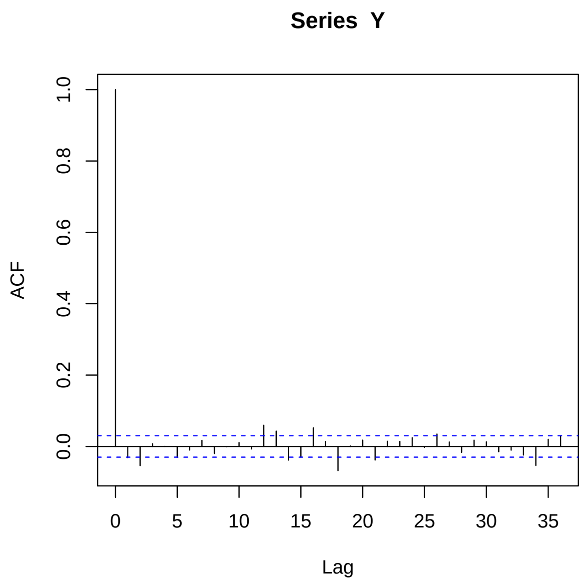
□

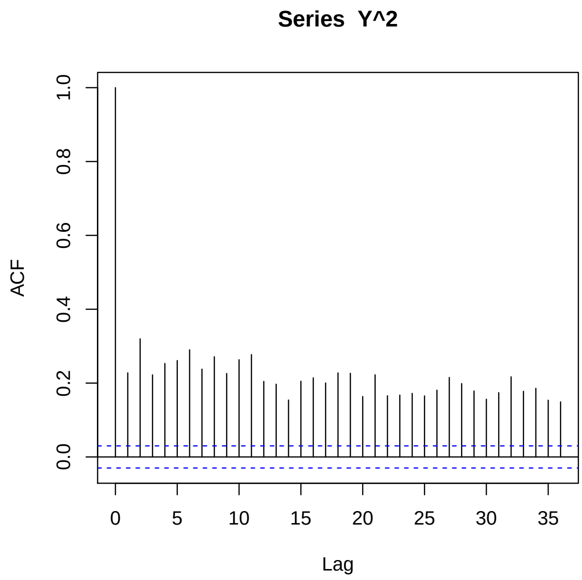
2. We study with R software the Nasdaq index from January 11st 2000 to January 11 2016.

(a) First the following commands have been executed with figures exhibited below:

```
Nas=read.csv("C:/Users/Admin/Dropbox/Enseignement/M2 MMMEF/TP/Nasdaq.csv")
X=Nas$Closing
n=length(X)
Y=log(X[2:n]/X[1:(n-1)])
ts.plot(Y)
acf(Y)
acf(Y^2)
```







Question II.1: Explain what is done. Which conclusions could you obtain from both the last commands? Are they compatible with a GARCH modelling?

(b) New commands are then executed:

```
library(fGarch)
QMLE=garchFit(~ garch(1,1), data = Y, trace = FALSE)
QMLE
```

Here there are the numerical results:

```
Call:
 garchFit(formula = ~garch(1, 1), data = Y, trace = FALSE)

Coefficient(s):
        mu       omega       alpha1       beta1
6.4040e-04  2.1146e-06  8.6732e-02  9.0314e-01

        Estimate  Std. Error  t value Pr(>|t|)
mu      6.404e-04   1.617e-04    3.961 7.46e-05 ***
omega   2.115e-06   4.052e-07    5.219 1.80e-07 ***
alpha1  8.673e-02   8.639e-03   10.039  < 2e-16 ***
beta1   9.031e-01   9.165e-03   98.544  < 2e-16 ***

                                Statistic p-Value
 Jarque-Bera Test   R    Chi^2  168.5587  0
 Shapiro-Wilk Test  R    W      0.9923722 2.300106e-14
 Ljung-Box Test     R    Q(10)  10.64455  0.3858725
 Ljung-Box Test     R    Q(15)  17.2698   0.3029932
 Ljung-Box Test     R    Q(20)  26.26701  0.1571734
 Ljung-Box Test     R^2  Q(10)  20.02718  0.02899662
 Ljung-Box Test     R^2  Q(15)  31.17497  0.008323147
 Ljung-Box Test     R^2  Q(20)  35.92566  0.01569342
 LM Arch Test       R    TR^2   20.11987  0.06485208

Information Criterion Statistics:
      AIC       BIC       SIC      HQIC
-5.886147 -5.880197 -5.886149 -5.884045
```

Question II.2: Explain what is done and explain which conclusions you deduce.

(c) Finally the following sequence of commands are executed:

```
M=matrix(0,3,4)
NAS10=garchFit(~ garch(1,0), data = Y, trace = FALSE)
M[1,1]=NAS10@fit$ics[2]
NAS11=garchFit(~ garch(1,1), data = Y, trace = FALSE)
M[1,2]=NAS11@fit$ics[2]
NAS12=garchFit(~ garch(1,2), data = Y, trace = FALSE)
M[1,3]=NAS12@fit$ics[2]
NAS13=garchFit(~ garch(1,3), data = Y, trace = FALSE)
M[1,4]=NAS13@fit$ics[2]
NAS20=garchFit(~ garch(2,0), data = Y, trace = FALSE)
M[2,1]=NAS20@fit$ics[2]
NAS21=garchFit(~ garch(2,1), data = Y, trace = FALSE)
M[2,2]=NAS21@fit$ics[2]
NAS22=garchFit(~ garch(2,2), data = Y, trace = FALSE)
..............................................
NAS33=garchFit(~ garch(3,3), data = Y, trace = FALSE)
M[3,4]=NAS33@fit$ics[2]
M
(Gop=which(M==min(M),2))
summary(NAS21)
```

The resuts are the following:

```
> M
          [,1]      [,2]      [,3]      [,4]
[1,] -5.501540 -5.880197 -5.878215 -5.876146
[2,] -5.665021 -5.881333 -5.879577 -5.877575
[3,] -5.717491 -5.879332 -5.877748 -5.875797
> (Gop=which(M==min(M),2))
     row col
[1,]   2   2
```

Question II.3: What is done here and what are your conclusions?