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Weak convergence to the fractional Brownian sheet in Besov spaces

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Abstract. In this paper we study the problem of the approximation in law of the fractional Brownian sheet in the topology of the anisotropic Besov spaces. We prove the convergence in law of two families of processes to the fractional Brownian sheet: the first family is constructed from a Poisson procces in the plane and the second family is defined by the partial sums of two sequences of real independent fractional brownian motions.

Keywords: Fractional Brownian motion, weak convergence, Besov spaces.

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1 Introduction

Let $T = [0, 1]$ the unit interval and $(W_t^\alpha)_{t\in T}$ a fractional Brownian motion of Hurst parameter $\alpha \in (0, 1)$ on some probability space $(\Omega, \mathcal{F}, P)$. That is, W^α is a centered Gaussian process, starting from zero and its covariance is given by

$$R(t, s) = E\left(B_t^\alpha B_s^\alpha\right) = \frac{1}{2}\left(t^{2\alpha} + s^{2\alpha} - |t - s|^{2\alpha}\right).$$

Recall that the fractional Brownian motion of Hurst parameter $\alpha \in (0, 1)$ admits a Wiener integral representation with respect to W of the form

$$W_t^\alpha = \int_0^t K_\alpha(t, s) dW_s \tag{1}$$

where K_α is the kernel defined on the set $\{0 < s < t\}$ and it is given by

$$K_\alpha(t, s) = d_\alpha (t - s)^{\alpha - \frac{1}{2}} + d_\alpha(\frac{1}{2} - \alpha) \int_s^t (u - s)^{\alpha - \frac{3}{2}} \left(1 - \left(\frac{s}{u}\right)^{\frac{1}{2} - \alpha}\right) du, \tag{2}$$

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with d_α the following normalizing constant

$$d_\alpha = \left(\frac{2\alpha\Gamma(\frac{3}{2}-\alpha)}{\Gamma(\alpha+\frac{1}{2})\Gamma(2-2\alpha)} \right)^{\frac{1}{2}}.$$

Let now $(W_{u,v})_{(u,v)\in T^2}$ a Brownian sheet. The fractional Brownian sheet can be also defined by a Wiener integral with respect to the Brownian sheet (see [4])

$$W^{\alpha,\beta}_{s,t} = \int_0^s \int_0^{st} K_\alpha(s,u) K_\beta(t,v) dW_{u,v}. \tag{3}$$

where $\alpha, \beta \in (0,1)$ and the kernels K_α, K_β are defined above. Note that this process is a two-parameters centered Gaussian process, starting from (0, 0), and its covariance is given by

$$E\left(W^{\alpha,\beta}_{s,t} W^{\alpha,\beta}_{s',t'} \right) = \frac{1}{2}\left(s'^{2\alpha} + s^{2\alpha} - |s'-s|^{2\alpha} \right) \frac{1}{2}\left(t'^{2\beta} + t^{2\beta} - |t'-t|^{2\beta} \right), \tag{4}$$

and it coincides in law with the process introduced in [10] or [1].

The aim of this work is to study the weak convergence of some continuous processes in the anisotropic Besov spaces to the fractional Brownian sheet. We will consider two types of approximations. First, we let

$$y_\varepsilon(s,t) = \int_0^t \int_0^s \frac{1}{\varepsilon^2} \sqrt{xy} (-1)^{N(\frac{x}{\varepsilon},\frac{y}{\varepsilon})} dx dy, \quad \varepsilon > 0,$$

where $\{N(x,y),\ (x,y) \in \mathbb{R}^2_+\}$ is a standard Poisson process in the plane. It was proved in [3] (using the result of Stroock [11] for the one-dimensional case) that this process converges in law in the space of continuous functions on $[0,1]^2$, denoted by $C([0,1]^2)$, as ε tends to 0, to the ordinary Brownian sheet. Using this result and the representation (3) a natural way to obtain an approximation in law of the fractional Brownian sheet is to put

$$X_\varepsilon(s,t) = \int_0^t \int_0^s K_\alpha(s,u) K_\beta(t,v) \frac{1}{\varepsilon^2} \sqrt{uv} (-1)^{N(\frac{u}{\varepsilon},\frac{v}{\varepsilon})} du dv.$$

The weak convergence of the family of processes X_ε to the fractional Brownian sheet $W^{\alpha,\beta}$ in the space of continuous functions has been showed in [4]. In our first result we prove that the sequence X_ε converges also to $W^{\alpha,\beta}$ in the stronger topology of the anisotropic Besov space $lip^*_p((\alpha,\beta),b))$. See the next Section for the definition of Besov spaces. To simplify the notation, we put $n = \frac{1}{\varepsilon^2}$ and we will consider the family of processes

$$X^n(s,t) = n \int_0^s \int_0^t K_\alpha(s,x) K_\beta(t,y) \sqrt{xy} (-1)^{N_n(x,y)} dx dy. \tag{5}$$

In the last expression $N_n(x,y) := N(x\sqrt{n}, y\sqrt{n})$. Observe that N_n is a Poisson process with intensity n.

On the other hand, it has been proved in [5] that if $(B^n)_n$, $(C^n)_n$ are two families of independent one dimensional Brownian motions then the process

$$W^n(s,t) = \frac{1}{\sqrt{n}} \sum_{j=1}^n B_s^j C_t^j$$

converges weakly to the Brownian sheet in the two dimensional Besov space $lip_p^*((\frac{1}{2}, \frac{1}{2}), b)$ for any $p > \frac{2}{b}$. Our second approximation result is an extension of the result of [5]. That is, if we put

$$Z^n(s,t) = \frac{1}{\sqrt{n}} \sum_{j=1}^n B_s^{j,\alpha} C_t^{j,\beta} \tag{6}$$

where $(B^{n,\alpha})_n$, $(C^{n,\beta})_n$ are two families of independent one dimensional fractional Brownian motions, then the process Z^n converges in law, as $n \to \infty$, to the fractional Brownian sheet $W^{\alpha,\beta}$ in the space $lip_p^*((\alpha, \beta), b)$.

For the weak convergence in Besov spaces to the one dimensional fractional Brownian motion we refer to [9].

2 Anisotropic Besov spaces

We recall in this section some basic elements on the two-dimensional Besov spaces. We refer to [10] for a complete presentation on this subject.

Let $T^2 = [0,1]^2$ and $D = \{1,2\}$. For a function $f : T^2 \to R$, $i \in D$, $h \in R$ and $e_i = (\delta_{i1}, \delta_{i2})$ unit vectors, let

$$\Delta_{h,i} = \begin{cases} f(z + he_i) - f(z), \text{ if } z, z + he_i \in T^2 \\ 0, \text{otherwise} \end{cases}$$

denote the progressive difference of f in the direction e_i. Let

$$\Delta_{(h_1,h_2)} f = \Delta_{h_1,1} \circ \Delta_{h_2,2} f, \quad \text{for } (h_1,h_2) \in R^2.$$

If $A = \{i\}$ is a subset of D with one element, then we put $\Delta_{(h_1,h_2),A} f = \Delta_{h_i,i} f$ and $\Delta_{(h_1,h_2),A} f = f$ if $A = \Phi$.

Now, for $f \in L^p(T^2)$ if $1 \le p < \infty$ or $f \in C(T^2)$ (the space of continuous functions on T^2) if $p = \infty$, we define its L^p -modulus of continuity by

$$\omega_{p,A}(f, (t_1, t_2)) = \sup_{0\le h_1 \le t_1, 0 \le h_2 \le t_2} \|\Delta_{(h_1,h_2),A} f\|_p \text{ for } (t_1, t_2) \in R^2_+.$$

For b real and $\bar{a} = (a_1, a_2)$, $a_1, a_2 > 0$ consider the real valued application on T^2 given by

$$\omega_{\bar{a},b}((t_1, t_2), A) = \prod_{i\in A} t_1^{a_1} \left(1 + \sum_{i\in A} \log \frac{1}{t_i}\right)^b$$

for any $A \subset D$ with $\omega_{\bar{a},b}((t_1, t_2), \Phi) = 1$.

Definition 2.1. *Let $\bar{a} = (a_1, a_2)$, $a_1, a_2 > 0$, $b \in R$ and $1 \le p \le \infty$. The anisotropic Besov space $Lip_p(\bar{a}, b)$ is defined by*

$$Lip_p(\bar{a}, b) = \left\{ f \in L^p(T^2); \sum_{A\subset D} \sup_{t_1,t_2>0} \frac{\omega_{p,A}(f, (t_1, t_2))}{\omega_{\bar{a},b}((t_1, t_2), A)} < \infty \right\}$$

and this space is endowed with the norm

$$\|f\|_p^{\bar{a},b} = \sum_{A\subset D} \sup_{t_1,t_2>0} \frac{\omega_{p,A}(f, (t_1, t_2))}{\omega_{\bar{a},b}((t_1, t_2), A)}.$$

In this way $Lip_p(\bar{a}, b)$ becomes a non-separable Banach space.

We also introduce the subspace $lip_p^*(\bar{a}, b)$ of $Lip_p(\bar{a}, b)$ by

$$lip_p^*(\bar{a}, b) = \left\{ f \in L^p(T^2); \forall \Phi \ne A \subset D, \lim_{\delta_A(t_1,t_2)\to 0} \frac{\omega_{p,A}(f, (t_1, t_2))}{\omega_{\bar{a},b}((t_1, t_2), A)} = 0 \right\}$$

where $\delta_A(t_1, t_2) = \min\{t_i, i \in A\}$.

The following result on Besov spaces and fractional Brownian motion has been proved in [10].

Theorem 2.1. *For any $2 < p < \infty$, it holds*

$$P\left(W^{\alpha,\beta} \in Lip_p((\alpha, \beta), 0)\right) = 1 \text{ and } P\left(W^{\alpha,\beta} \in lip_p^*((\alpha, \beta), 0)\right) = 0.$$

For similar results in the one parameter case we refer to [7].

3 Weak convergence to the fractional Brownian sheet in Besov spaces

We will need the following tightness criterion in Besov spaces given by [5].

Lemma 3.1. *Let $(U^n(s,t))_{(s,t)\in[0,1]^2}$ be a sequence of continuous adapted processes such that there exists $a=(a_1,a_2)$, $a_1,a_2>0$ and for every $p\geq 1$ it exists a constant $C_p>0$ such that, for $s,s',t,t'\in[0,1]$, $s\leq s'$, $t\leq t'$ it holds*

$$E\left|\Delta_{s'-s,t'-t}U^n(s,t)\right|^p\leq C_p|s'-s|^{a_1p}|t'-t|^{a_2p} \tag{7}$$

and

$$\begin{aligned}E\left|\Delta_{s'-s,1}U^n(s,t)\right|^p&\leq C_p|s'-s|^{a_1p}\ ,\\ E\left|\Delta_{t'-t,2}U^n(s,t)\right|^p&\leq C_p|t'-t|^{a_2p}.\end{aligned} \tag{8}$$

Then U^n is tight in the space $lip_p^*(a,b)$ for any $p>\max_i\frac{1}{a_i}\vee\frac{2}{b}$.

We prove now the tightness of the approximating families X^n and Z^n given by (5) and (6).

Lemma 3.2.

1) *Let $(X^n(s,t))_{s,t\in[0,1]^2}$ be the family of processes given by (5). Then X^n is tight in the space $lip_p^*((\alpha,\beta),b))$ for any $p>\frac{2}{b}\vee\frac{1}{\alpha}\vee\frac{1}{\beta}$.*
2) *Let $(Z^n(s,t))_{s,t\in[0,1]^2}$ be the family of processes given by (6). Then Z^n is tight in the space $lip_p^*((\alpha,\beta),b))$ for any $p>\frac{2}{b}\vee\frac{1}{\alpha}\vee\frac{1}{\beta}$.*

Proof. Tightness of X^n: Note first that, since the Besov norms are increasing in p it suffices to prove the result for p even. We will show that

$$\sup_n E\Big[\Delta_{s,t}X^n(s',t')\Big]^p\leq C_p(s'-s)^{p\alpha}(t'-t)^{p\beta}$$

for any p even. By Lemma 4.1 of [4], it suffices to check this inequality for $s'\geq s>0$, $t'\geq t>0$, $s'-s<s$ and $t'-t<t$.

We will extend the kernels K_α and K_β over all $(0,1]$ by putting

$$\tilde{K}_\alpha(s,x)=\begin{cases}K_\alpha(s,x) & \text{if } s>x\\ 0 & \text{if } s\leq x\end{cases}$$

and for the sake of simplicity we will denote also by K_α this extension. We introduce also the following notations

$$K_{\alpha,\beta}(s,t,x,y)=K_\alpha(s,x)K_\beta(t,y),$$

and

$$\Delta_{s,t} K_{\alpha,\beta}(s', t', x, y) = (K_\alpha(s', x) - K_\alpha(s, x))(K_\beta(t', y) - K_\beta(t, y)).$$

We have that, with the notations introduced above,

$$E\left[\Delta_{s,t} X_n(s', t')\right]^p = n^p E\left[\int_{[0,1]^2} \Delta_{s,t} K_{\alpha,\beta}(s', t', x, y)\sqrt{xy}(-1)^{N_n(x,y)} dxdy\right]^p$$
$$= n^p E\left[\int_{[0,1]^{2p}} \prod_{i=1}^{p} \left(\Delta_{s,t} K_{\alpha,\beta}(s', t', x_i, y_i)\sqrt{x_i y_i}(-1)^{N_n(x_i,y_i)}\right) dx_1 \cdots dy_p\right].$$

We obtain now a bound for the expectation of the random part of the last expression. First of all, we have that

$$(-1)^{\sum_{i=1}^p N_n(x_i,y_i)} = (-1)^{\sum_{i=1}^p \Delta_{0,0} N_n(x_i,y_i)},$$

and that

$$\sum_{i=1}^{p} \Delta_{0,0} N_n(x_i, y_i) = \sum_{i=1}^{p} \Delta_{s,t} N_n(x_i, y_i) + \sum_{i=1}^{p} \Delta_{s,0} N_n(x_i, t)$$
$$+ \sum_{i=1}^{p} \Delta_{0,t} N_n(s, y_i) + \ p\, N_n(s, t),$$

and hence

$$(-1)^{\sum_{i=1}^p \Delta_{0,0} N_n(x_i,y_i)} =$$
$$(-1)^{\sum_{i=1}^p \Delta_{s,t} N_n(x_i,y_i)} (-1)^{\sum_{i=1}^p \Delta_{s,0} N_n(x_i,t)} (-1)^{\sum_{i=1}^p \Delta_{0,t} N_n(s,y_i)}.$$

Since for all x_i, y_i, x_j, y_k the three intervals $((s, t), (x_i, y_i)]$, $((s, 0), (x_j, t)]$ and $((0, t), (s, y_k)]$ are disjoint sets, the three factors of the last product are independent random variables. Moreover, if we majorize the expectation of the first factor by 1 we will obtain

$$E\left((-1)^{\sum_{i=1}^p \Delta_{0,0} N_n(x_i,y_i)}\right) \le \exp\{-2nt[(x_{(p)} - x_{(p-1)}) + \cdots + (x_{(2)} - x_{(1)})]\}$$
$$\times \exp\{-2ns[(y_{(p)} - y_{(p-1)}) + \cdots + (y_{(2)} - y_{(1)})]\},$$

where $x_{(1)}, \ldots, x_{(p)}$ are the variables $x_1 \ldots, x_p$ ordered in increasing order.

Using now the fact that $2t > t'$ and $2s > s'$, the last expression can be bounded by

$$\begin{aligned}&\exp\{-nt'[(x_{(p)} - x_{(p-1)}) + \cdots + (x_{(2)} - x_{(1)})]\}\\ &\times \exp\{-ns'[(y_{(p)} - y_{(p-1)}) + \cdots + (y_{(2)} - y_{(1)})]\}\\ &\quad \leq \exp\{-n[(x_{(p)} - x_{(p-1)})y_{(p-1)} + \cdots + (x_{(2)} - x_{(1)})y_{(1)}]\}\\ &\quad \times \exp\{-n[(y_{(p)} - y_{(p-1)})x_{(p-1)} + \cdots + (y_{(2)} - y_{(1)})x_{(1)}]\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&E\left[\Delta_{s,t} X_n(s',t')\right]^p\\ &\leq (p!)^2 n^p \int_{[0,1]^{2p}} \Big(\prod_{i=1}^{p} \left(\Delta_{s,t} K_{\alpha,\beta}(s',t',x_i,y_i)\sqrt{x_i y_i}\right)\\ &\quad \times \exp\{-n[(x_p - x_{p-1})y_{p-1} + \cdots + (x_2 - x_1)y_1]\}\\ &\quad \times \exp\{-n[(y_p - y_{p-1})x_{p-1} + \cdots + (y_2 - y_1)x_1]\}\\ &\quad \times 1_{\{x_1 \leq \cdots \leq x_p\}} 1_{\{y_1 \leq \cdots \leq y_p\}}\Big) dx_1 \cdots dy_p\\ &\leq C_p \Big(n^2 \int_{[0,1]^2} \Big(\Delta_{s,t} K_{\alpha,\beta}(s',t',x_1,y_1)\Delta_{s,t} K_{\alpha,\beta}(s',t',x_2,y_2)\sqrt{x_1 x_2 y_1 y_2}\\ &\quad \times \exp[-n(x_2 - x_1)y_1 - n(y_2 - y_1)x_1]\Big) 1_{\{x_1 \leq x_2\}} 1_{\{y_1 \leq y_2\}} dx_1 \cdots dy_2 \Big)^{\frac{p}{2}}.\end{aligned} \tag{9}$$

We now divide the region of integration in two parts: $A = \{x_1 \leq x_2 \leq 2x_1, \, y_1 \leq y_2 \leq 2y_1\}$ and A^c.

The integral of expression (9) over the region A can be majorized by

$$\begin{aligned}&C_p n^2 \int_{[0,1]^4} \Big(\Delta_{s,t} K_{\alpha,\beta}(s',t',x_1,y_1)\Delta_{s,t} K_{\alpha,\beta}(s',t',x_2,y_2) x_1 y_1\\ &\quad \times \exp\{-2n[(x_2 - x_1)y_1 + (y_2 - y_1)x_1]\} 1_{\{x_1 \leq x_2,\ y_1 \leq y_2\}}\Big) dx_1 \cdots dy_2\\ \leq\ &C_p n^2 \int_{[0,1]^4} \Big(K_\alpha(s',x_1) - K_\alpha(s,x_1)\Big)^2 \Big(K_\beta(t',y_2) - K_\beta(t,y_2)\Big)^2 x_1 y_1\\ &\quad \times \exp\{-2n[(x_2 - x_1)y_1 + (y_2 - y_1)x_1]\} 1_{\{x_1 \leq x_2,\ y_1 \leq y_2\}}\Big) dx_1 \cdots dy_2\\ +\ &C_p n^2 \int_{[0,1]^4} \Big(K_\alpha(s',x_2) - K_\alpha(s,x_2)\Big)^2 \Big(K_\beta(t',y_1) - K_\beta(t,y_1)\Big)^2 x_1 y_1\\ &\quad \times \exp\{-2n[(x_2 - x_1)y_1 + (y_2 - y_1)x_1]\} 1_{\{x_1 \leq x_2,\ y_1 \leq y_2\}}\Big) dx_1 \cdots dy_2.\end{aligned}$$

These last two summands can be treated in a similar way. In the first one we first integrate with respect to x_2 and then with respect to y_1, and in the second we integrate with respect x_1 and y_2. In both cases we obtain the bound

$$C_p\Big(\int_{[0,1]^2}(K_\alpha(s',x)-K_\alpha(s,x))^2(K_\beta(t',y)-K_\beta(t,y))^2dxdy\Big)^{\frac{p}{2}}$$
$$= C_p\left(E\left(\Delta_{s,t}W^{\alpha,\beta}_{s',t'}\right)^2\right)^{\frac{p}{2}} = C_p(s'-s)^{\alpha p}(t'-t)^{\beta p}.$$

We consider now the integral given in (9) over the region A^c. This region is the union of $B_1=\{x_1\le x_2,\ y_1\le y_2,\ x_2>2x_1\}$ and $B_2=\{x_1\le x_2,\ y_1\le y_2,\ y_2>2y_1\}$. We will deal with the integral over B_2, the other one is analogous. In this case we obtain the following inequalities:

$$2(y_2-y_1)x_1+2(x_2-x_1)y_1\ge (y_2-y_1)x_1+x_1y_1+2(x_2-x_1)y_1$$
$$\ge \frac{1}{2}(y_2-y_1)x_1+x_1y_1+\frac{1}{2}(x_2-x_1)y_1=\frac{1}{2}(y_2x_1+y_1x_2).$$

Thus, the integral given in expression (9) over the region B_2 can be bounded by

$$C_pn^2\int_{[0,1]^4}\prod_{i=1}^{2}\left(\Delta_{s,t}K_{\alpha,\beta}(s',t',x_i,y_i)\sqrt{x_iy_i}\right)$$
$$\times\exp[-\frac{n}{2}(y_2x_1+x_2y_1)]\,1_{\{x_1\le x_2,\ y_1\le y_2\}}dx_1\cdots dy_2$$
$$\le C_pn^2\Big[\int_{[0,1]^4}\Big(K_\alpha(s',x_1)-K_\alpha(s,x_1)\Big)^2\Big(K_\beta(t',y_1)-K_\beta(t,y_1)\Big)^2x_1y_1$$
$$\times\exp[-\frac{n}{2}(y_2x_1+x_2y_1)]\,1_{\{x_1\le x_2,\ y_1\le y_2\}}dx_1\cdots dy_2$$
$$+\int_{[0,1]^4}\Big(K_\alpha(s',x_2)-K_\alpha(s,x_2)\Big)^2\Big(K_\beta(t',y_2)-K_\beta(t,y_2)\Big)^2x_2y_2$$
$$\times\exp[-\frac{n}{2}(y_2x_1+x_2y_1)]\,1_{\{x_1\le x_2,\ y_1\le y_2\}}dx_1\cdots dy_2\Big].$$

By integrating in the first summand of the last expression with respect to x_2 and y_2 and in the second summand with respect to x_1 and y_1, we obtain that the last expression is bounded by

$$C_p\Big(\int_{[0,1]^2}(K_\alpha(s',x)-K_\alpha(s,x))^2(K_\beta(t',y)-K_\beta(t,y))^2dxdy\Big)^{\frac{p}{2}}$$
$$= C_p\left(E\left(\Delta_{s,t}W^{\alpha,\beta}_{s',t'}\right)^2\right)^{\frac{p}{2}} = C_p(s'-s)^{\alpha p}(t'-t)^{\beta p}.$$

Using the same calculus we can prove that conditions (8) of Lemma 1 are also verified. This finishes the proof of tightness of X^n.

Tightness of Z^n: Concerning 2), observe that the rectangular increments of the process Z^n can be written as

$$\Delta_{s,t} Z^n_{s',t'} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left(B^{j,\alpha}_{s'} - B^{j,\alpha}_{s} \right) \left(C^{j,\beta}_{t'} - C^{j,\beta}_{t} \right)$$

and by decomposing $B^{j,\alpha}_{s'} - B^{j,\alpha}_{s} = \frac{B^{j,\alpha}_{s'} - B^{j,\alpha}_{s}}{\sqrt{E\left(B^{j,\alpha}_{s'} - B^{j,\alpha}_{s}\right)^2}} \left| s' - s \right|^{\alpha}$ and similarly for $C^{j,\beta}_{t'} - C^{j,\beta}_{t}$, we get, with $(\zeta_i, \eta_i)_i$ a double sequence of i.i.d. random variables with standard normal distribution,

$$\begin{aligned}
& E \left| \Delta_{s'-s,t'-t} Z^n_{s,t} \right|^p \\
\leq \;& (s'-s)^{p\alpha} (t'-t)^{p\beta} E \left[\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \zeta_i \eta_i \right]^p \\
\leq \;& C_p (s'-s)^{p\alpha} (t'-t)^{p\beta} E \left[\frac{1}{n} \sum_{i=1}^{n} \zeta_i^2 \right]^{\frac{p}{2}} \\
\leq \;& C_p (s'-s)^{p\alpha} (t'-t)^{p\beta} \frac{1}{n} \sum_{i=1}^{n} E \left| \zeta_i \right|^p \leq C_p (s'-s)^{p\alpha} (t'-t)^{p\beta}
\end{aligned}$$

and similarly (8) can be checked. □

We can state now our main result.

Theorem 3.3. *Let X^n and Z^n be the families definied by (5) and(6). Then the following weak convergences hold*

$$X^n \to W^{\alpha,\beta} \textit{ in } lip^*_p\left((\alpha,\beta), b\right) \textit{ as } n \to \infty$$

and

$$Z^n \to W^{\alpha,\beta} \textit{ in } lip^*_p\left((\alpha,\beta), b\right) \textit{ as } n \to \infty\, .$$

Proof. We refer to [4] for the proof of the convergence of the finite dimensional distributions of X^n to those of $W^{\alpha,\beta}$. Concerning the family Z^n, let N integer, $a_1, \dots, a_N \in R$ and $(s_1, t_1), \dots, (s_N, t_N) \in [0,1]^2$. We must see that the random variables

$$\sum_{k=1}^{N} a_k Z_n(s_k, t_k) = \frac{1}{\sqrt{n}} \sum_{k=1}^{N} a_k \times \sum_{j=1}^{n} \left(\int_0^{s_k} K_\alpha(s_k, u_k) dW^j_{u_k} \right) \left(\int_0^{t_k} K_\beta(t_k, v_k) dW^j_{v_k} \right)$$

converge in law, as n tends to infinity to

$$\sum_{k=1}^{N} a_k W^{\alpha,\beta}_{s_k,t_k} = \sum_{k=1}^{N} \int_0^{s_k} \int_0^{t_k} K_\alpha(s_k, u_k) K_\beta(t_k, v_k) dW_{u_k, v_k}.$$

To prove this convergence, we will prove the convergence of the associated characteristic functions. In the sequel we will consider the extension of the kernel $K_\alpha(t, s)$ on $[0, 1]$ by putting zero if $s \geq t$. For the sake of simplicity we will denoted this extension also by $K_\alpha(t, s)$.

Let $(\gamma^{k,l})_l$ be a sequence of elementary functions converging, for every s_k, to $K_\alpha(s_k, \cdot)$ in $L^2(T)$ as l goes to ∞ and for every t_k, let $(\rho^{k,l})_l$ a sequence of elementary functions converging in $L^2(T)$, as $l \to \infty$, to $K_\beta(t_k, \cdot)$.

Let us denote by

$$Z^n_{k,l} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left(\int_0^{s_k} \gamma^{k,l}(u_k) dW^j_{u_k} \right) \left(\int_0^{t_k} \rho^{k,l}(v_k) dW^j_{v_k} \right)$$

and

$$Z_{k,l} = \int_0^{s_k} \int_0^{t_k} K_\alpha(s_k, u) K_\beta(t_k, v) dW_{u,v}.$$

For every λ real, we have the following bound

$$\begin{aligned} &\left| E\left(e^{i\lambda \sum_{k=1}^N a_k Z^n(s_k,t_k)}\right) - E\left(e^{i\lambda \sum_{k=1}^N a_k W^{\alpha,\beta}_{s_k,t_k}}\right) \right| \\ &\leq \left| E\left(e^{i\lambda \sum_{k=1}^N a_k Z^n(s_k,t_k)}\right) - E\left(e^{i\lambda \sum_{k=1}^N a_k Z^n_{k,l}}\right) \right| \\ &+ \left| E\left(e^{i\lambda \sum_{k=1}^N a_k Z^n_{k,l}}\right) - E\left(e^{i\lambda \sum_{k=1}^N a_k Z_{k,l}}\right) \right| \\ &+ \left| E\left(e^{i\lambda \sum_{k=1}^N a_k Z_{k,l}}\right) - E\left(e^{i\lambda \sum_{k=1}^N a_k W^{\alpha,\beta}_{s_k,t_k}}\right) \right| = I_1 + I_2 + I_3 \end{aligned}$$

By the mean value theorem we can bound the term I_1 by

$$\max_k E\left|Z^n(s_k,t_k) - Z^n_{k,l}\right|$$

and moreover

$$\begin{aligned}
&E\left|Z^n(s_k,t_k) - Z^n_{k,l}\right| \\
&\leq \frac{1}{\sqrt{n}} E\left|\sum_{j=1}^{n}\left(\int_0^1 K_\alpha(s_k,u_k)dW^j_{u_k}\right)\left(\int_0^1 K_\beta(t_k,v_k)dW^j_{v_k}\right)\right. \\
&\left. - \left(\int_0^1 \gamma^{k,l}(u_k)dW^j_{u_k}\right)\left(\int_0^1 \rho^{k,l}(v_k)dW^j_{v_k}\right)\right| \\
&\leq \frac{1}{\sqrt{n}} E\sum_{j=1}^{n}\left|\int_0^1 \left(K_\alpha(s_k,u) - \gamma^{k,l}(u)\right)dW^j(u)\right|\left|\int_0^1 K_\beta(t_k,v)dW^j_v\right| \\
&+ \frac{1}{\sqrt{n}} E\sum_{j=1}^{n}\left|\int_0^1 \gamma^{k,l}(u)dW^j_u\right|\left|\int_0^1 \left(K_\beta(t_k,v) - \rho^{k,l}(u)\right)dW^j(v)\right| \\
&\leq \frac{1}{\sqrt{n}}\left(E\left(\sum_{j=1}^{n}\left|\int_0^1 \left(K_\alpha(s_k,u) - \gamma^{k,l}(u)\right)dW^j(u)\right|\left|\int_0^1 K_\beta(t_k,v)dW^j_v\right|\right)\right)^{\frac{1}{2}} \\
&+ \frac{1}{\sqrt{n}}\left(E\left(\sum_{j=1}^{n}\left|\int_0^1 \gamma^{k,l}(u)dW^j(u)\right|\left|\int_0^1 \left(K_\beta(t_k,v) - \rho^{k,l}(v)\right)dW^j_v\right|\right)\right)^{\frac{1}{2}}.
\end{aligned}$$

Using the independence of increments, we can majorize the last expression by

$$\begin{aligned}
&\left(\int_0^1 \left(K_\alpha(s_k,u) - \gamma^{k,l}(u)\right)^2 du\right)\left(\int_0^1 K^2_\beta(t_k,v)dv\right) \\
&+\left(\int_0^1 (\gamma^{k,l}(u))^2 du\right)\left(\int_0^1 \left(K_\beta(t_k,v) - \rho^{k,l}(v)\right)^2 dv\right).
\end{aligned}$$

Now, since $\gamma^{k,l}$ and $\rho^{k,l}$ are elementary functions, the convergence of I_2 to 0 as $n \to \infty$ follows from the convergence of W^n to $W^{\alpha,\beta}$. Finally, concerning the lasr term I_3, we have

$$\begin{aligned}
&\left|E\left(e^{i\lambda\sum_{k=1}^N a_k Z_{k,l}}\right) - E\left(e^{i\lambda\sum_{k=1}^N a_k W^{\alpha,\beta}_{s_k,t_k}}\right)\right| \leq C\max_k E\left|Z_{k,l} - W^{\alpha,\beta}(s_k,t_k)\right| \\
&= C\max_j\{\bar{E}|\int_{[0,1]^2}[\gamma^{kl}(x)\rho^{kl}(y) - K_\alpha(s_k,x)K_\beta(t_k,y)]dW_{x,y}|\} \\
&\leq C\max_j \|\gamma^{kl}\otimes\rho^{kl} - K_\alpha(s_k,\cdot)\otimes K_\beta(t_k,\cdot)\|_{L^2([0,1]^2)}.
\end{aligned}$$

This last norm in $L^2([0, 1]^2)$ tends to zero, as $l \to \infty$, independently of n. This fact concludes the proof. □

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