THE STOCHASTIC HEAT EQUATION WITH A FRACTIONAL-COLORED NOISE: EXISTENCE OF THE SOLUTION

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Abstract. In this article we consider the stochastic heat equation $u_t - \Delta u = \dot{B}$ in $(0,T) \times \mathbb{R}^d$, with vanishing initial conditions, driven by a Gaussian noise $\dot{B}$ which is fractional in time, with Hurst index $H \in (1/2, 1)$, and colored in space, with spatial covariance given by a function f. Our main result gives the necessary and sufficient condition on H for the existence of the process solution. When f is the Riesz kernel of order $\alpha \in (0,d)$ this condition is $H > (d-\alpha)/4$, which is a relaxation of the condition $H > d/4$ encountered when the noise $\dot{B}$ is white in space. When f is the Bessel kernel or the heat kernel, the condition remains $H > d/4$.

1. Introduction and Preliminaries

Stochastic partial differential equations (s.p.d.e.'s) perturbed by noise terms which bear a "colored" spatial covariance structure (but remain white in time) have become increasingly popular in the recent years, after the fundamental work of [2]. Such an equation can be viewed as a more flexible alternative to a classical s.p.d.e. driven by a space-time white noise, and therefore it can be used to model a more complex physical phenomenon which is subject to random perturbations. The major drawback of this theory is that it is mathematically more challenging than the classical theory, usually relying on techniques from potential analysis. One advantage is that an s.p.d.e. perturbed by a colored noise possesses a process solution (under relatively mild conditions on the covariance structure), in contrast with its white-noise driven counterpart, for which the solution is well understood only in the sense of distributions. Another advantage is the fact that such an equation can lead to a better understanding of a complex physical situation.

This article continues the line of research initiated by [2], the focus being on a relatively simple s.p.d.e., the stochastic heat equation. The novelty comes from the fact that random noise perturbing the equation possesses a colored temporal structure given by the covariance of a fractional Brownian motion (fBm), along with the colored spatial structure of [2].

We recall that a fBm on the real line is a zero-mean Gaussian process with covariance function $R_H(t,s) = (t^{2H} + s^{2H} - |t-s|^{2H})/2$, where $H \in (0,1)$. There is a huge amount of literature dedicated to the fBm, due to its mathematical tractability, and its many applications. We refer the reader to [9] for a comprehensive review on this subject.

Recently, the fBM made its entrance in the area of s.p.d.e.'s. We refer, among others, to [8], [16], [5], [10], [13] or [14]. For example, in the case of the stochastic heat equation driven by a Gaussian noise which is fractional in time and with a rather general covariance in space, it has been proved in [16] that if the time variable belongs to $[0,T]$ and the space variable belongs to S^1 (the unit circle) then

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the solution exists if and only if $H > 1/4$. The case of the same equation driven by a fractional-white noise with space variable in $\mathbb{R}^d$ has been treated in [8], and it follows that a process solution exists if $H > d/4$.

We note in passing that very few results are available in the literature, in the case of non-linear equations driven by a fractional Gaussian noise, due to difficulties encountered in the stochastic calculus associated to this noise. To circumvent these difficulties, a new pathwise method has been developed recently in [13] and [14], treating the nonlinear stochastic heat equation, respectively the nonlinear stochastic wave equation. In both these articles, the noise term carries a colored spatial covariance structure as well, which is the situation that we investigate in the present paper too.

In the present article we consider the (linear) stochastic heat equation in the domain $[0,T] \times \mathbb{R}^d$ driven by a Gaussian noise B which has a fractional time component, of Hurst index $H \in (1/2, 1)$, and a colored spatial component. Therefore, our work lies at the intersection of the two different lines of research mentioned above, namely those developed in [2], respectively [10]. The solution of the equation will be given in the mild formulation, but can also be viewed as a distribution solution. Therefore, the first step we need to take is to develop a stochastic calculus with respect to the noise B. Since our equation is linear, only spaces of deterministic integrands are considered in the present paper. Our main result identifies the necessary and sufficient condition on the Hurst index H for the existence of the process solution. We should mention that in this case, the solution is a Gaussian process, and hence Reproducing Kernel Hilbert Space techniques can be used to investigate its properties. This will be the subject of future work.

In preparation for treating the existence problem in its full generality, we studied first the case of the heat equation perturbed by a Gaussian noise which is fractional in time, but white in space (Section 2). In this case, it is known from [8] that the process solution exists if $H > d/4$, which forces a spatial dimension $d \in \{1, 2, 3\}$; in the present article, we strengthen this result by proving that the condition $H > d/4$ is in fact necessary for the existence of the solution. In contrast, when a spatial covariance structure is embedded in the noise (Section 3) the condition for the existence of the solution can be relaxed so that it imposes no restrictions on the spatial dimension d. When the color in space is given by the Riesz kernel of order α, we prove that the necessary and sufficient condition for the existence of the solution is $H > (d - \alpha)/4$. This demonstrates that a suitable choice of the spatial covariance structure can compensate for the drawbacks of the fractional component. However, it turns out that if the spatial covariance is given by the Bessel or the heat kernel, the condition remains $H > d/4$, whereas for the Poisson kernel the condition becomes $H > (d + 1)/4$.

This article contains 3 appendices. Appendix A contains a lemma which is heavily used in the present paper. This lemma is the tool which allows us to import the Fourier transform techniques from $\mathbb{R}$ to the bounded domain $[0, T]$. (So far, this type of techniques have been exploited only on $\mathbb{R}$; see e.g. [11].) Appendix B contains the proof of a technical statement. Appendix C contains a result which is essentially due to [2]; we include it since we could not find a direct reference.

We begin now to introduce the notation that will be used throughout this paper.

If $U \subset \mathbb{R}^n$ is an open set, we denote by $\mathcal{D}(U)$ the space of all infinitely differentiable functions whose support is compact and contained in U. By $\mathcal{D}'(U)$ we denote the set of continuous linear functionals on $\mathcal{D}(U)$ which is known as the space of *distributions*. We let $\mathcal{S}(\mathbb{R}^n)$ be the Schwarz space of all decreasing functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ be the space of *tempered distributions*, i.e. continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$. For an arbitrary function g on $\mathbb{R}^d$ the translation by x is denoted by g_x, i.e. $g_x(y) = g(x + y)$. The reflection by zero is denoted by $\tilde{g}$, i.e. $\tilde{g}(x) = g(-x)$.

For any function $\phi \in \mathcal{S}(\mathbb{R}^n)$ we define its Fourier transform by

$$\mathcal{F}\phi(\xi) = \int_{\mathbb{R}^d} \exp(-i\xi \cdot x)\phi(x)dx.$$

The map $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is an isomorphism which extends uniquely to a unitary isomorphism of $L_2(\mathbb{R}^n)$; this map can also be extended to $\mathcal{S}'(\mathbb{R}^n)$. We define the convolution $(f * g)(x) = \int_{R^n} f(x-y)g(y)dy$.

For any interval $(a,b) \subset \mathbb{R}$ and $\varphi \in L_2(a,b)$, we define the *restricted Fourier transform* of φ with respect to (a,b) by:

$$\mathcal{F}_{a,b}\varphi(\tau) = \int_a^b e^{-i\tau t}\varphi(t)dt.$$

As in [12], if $f \in L_1(0,T)$ and $\alpha > 0$, we define the *fractional integral* of f of order α by:

$$(I^{\alpha}_{T-}f)(t) = \frac{1}{\Gamma(\alpha)}\int_t^T (u-t)^{\alpha-1}f(u)du.$$

Finally, we denote by $\mathcal{B}_b(\mathbb{R}^d)$ the class of all bounded Borel sets in $\mathbb{R}^d$.

2. The Fractional-White Noise

The purpose of this section is to identify the necessary and sufficient condition for the existence of a distribution-solution of the stochastic heat equation, driven by a Gaussian noise which is *fractional in time* and *white in space*.

Our main result is similar to Theorem 11, [2], which identifies the necessary and sufficient condition for the existence of a distribution-solution of an arbitrary s.p.d.e.'s driven by a Gaussian noise which is *white in time* and *colored in space*.

In the first subsection we examine some spaces of deterministic integrands, which are relevant for the stochastic calculus with respect to fractional processes, and we explore the connection with Dalang's theory of stochastic integration. In the second subsection, we describe the Gaussian noise and its stochastic integral. In the third subsection, we introduce the process-solution and the distribution-solution of the stochastic heat equation driven by this noise, and we identify the necessary and sufficient condition for the existence of these processes.

2.1. **Spaces of deterministic integrands.** We begin by introducing the usual spaces associated with the fractional temporal noise. Throughout this article we suppose that $H \in (1/2, 1)$ and we let $\alpha_H = H(2H-1)$.

Let $\mathcal{H}(0,T)$ be the completion of $\mathcal{D}(0,T)$ with respect to the inner product

$$\begin{aligned}\langle \varphi, \psi \rangle_{\mathcal{H}(0,T)} &= \alpha_H \int_0^T \int_0^T \varphi(u)|u-v|^{2H-2}\psi(v)dvdu \\ &= \alpha_H c_H \int_{\mathbb{R}} \mathcal{F}_{0,T}\varphi(\tau)\overline{\mathcal{F}_{0,T}\psi(\tau)}|\tau|^{-(2H-1)}d\tau,\end{aligned}$$

where the second equality follows by Lemma A.1.(b) with $c_H = [2^{2(1-H)}\pi^{1/2}]^{-1}\Gamma(H-1/2)/\Gamma(1-H)$. Note that $c_H = q_{2H-1}$, where the constant q_α is defined in Lemma A.1 (Appendix A).

Let $\mathcal{E}(0,T)$ be the class of all linear combinations of indicator functions $1_{[0,t]}, t \in [0,T]$. One can see that $\mathcal{H}(0,T)$ is also the completion of $\mathcal{E}(0,T)$ with respect to the inner product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}(0,T)} = R_H(t,s).$$

(To see this, note that every $1_{[0,a]} \in \mathcal{E}(0,T)$ there exists a sequence $(\varphi_n)_n \subset \mathcal{D}(0,T)$ such that $\varphi_n(t) \to 1_{[0,a]}(t), \forall t$ and supp $\varphi_n \subset K$ for all n, where $K \subset (0,T)$ is a compact. By the dominated convergence theorem, it follows that $\|\varphi_n - 1_{[0,t]}\|_{\mathcal{H}(0,T)} \to 0$.)

In is important to emphasize that the space $\mathcal{H}(0,T)$ may contain *distributions*. We will justify this statement, using the argument of [2]. First, note that $\mathcal{H}(0,T) \subset \mathcal{H}(\mathbb{R})$, where $\mathcal{H}(\mathbb{R})$ is the completion of $\mathcal{D}(\mathbb{R})$ with respect to the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{H}(\mathbb{R})} = \alpha_H \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(u)\psi(v)|u-v|^{2H-2} dv du.$$

The space $\mathcal{H}(\mathbb{R})$ appears in several papers treating colored noises. (In fact $\mathcal{H}(\mathbb{R})$ is a particular instance of the space $\mathcal{P}_{0,x}^{(d)}$ of [1], in the case $\mu(d\xi) = |\xi|^{-(2H-1)} d\xi$ and $d = 1$.) From p. 9 of [2], we know that

$$\mathcal{H}(\mathbb{R}) \subset \overline{\mathcal{H}}(\mathbb{R}) := \{S \in \mathcal{S}'(\mathbb{R}); \mathcal{F}S \text{ is a function, } \int_{\mathbb{R}} |\mathcal{F}S(\tau)|^2 |\tau|^{-(2H-1)} d\tau < \infty\}. \quad (1)$$

Since $|\tau|^2 < 1 + |\tau|^2$, one can easily see that $\overline{\mathcal{H}}(\mathbb{R}) \subset \mathcal{H}^{-(H-1/2)}(\mathbb{R})$, where

$$\mathcal{H}^{-(H-1/2)}(\mathbb{R}) := \{S \in \mathcal{S}'(\mathbb{R}); \mathcal{F}S \text{ is a function, } \int_{\mathbb{R}} |\mathcal{F}S(\tau)|^2 (1+|\tau|^2)^{-(H-1/2)} d\tau < \infty\}$$

is the fractional Sobolev space of index $-(H-1/2)$ (see p. 191, [6]). Therefore, the elements of $\mathcal{H}(\mathbb{R})$ are tempered distributions on $\mathbb{R}$ of negative order $-(H-1/2)$. (This was also noticed by several authors; see e.g. p. 9, [9], or [11], [12])

On the other hand, similarly to (1), one can show that

$$\mathcal{H}(0,T) \subset \overline{\mathcal{H}}(0,T) := \{S \in \mathcal{S}'(\mathbb{R}); \mathcal{F}_{0,T}S \text{ is a function, } \int_{\mathbb{R}} |\mathcal{F}_{0,T}S(\tau)|^2 |\tau|^{-(2H-1)} d\tau < \infty\}, \quad (2)$$

where *the restricted Fourier transform* $\mathcal{F}_{0,T}S$ of a distribution $S \in \mathcal{S}'(\mathbb{R})$ is defined by: $\langle \mathcal{F}_{0,T}S, \varphi \rangle = \langle S, \mathcal{F}_{0,T}\varphi \rangle, \ \forall \varphi \in \mathcal{S}(\mathbb{R})$.

Remark 2.1. One can show that (see p.10, [9])

$$L_2(0,T) \subset L_{1/H}(0,T) \subset |\mathcal{H}(0,T)| \subset \mathcal{H}(0,T), \quad (3)$$

where $|\mathcal{H}(0,T)| = \{f : [0,T] \times \mathbb{R}^d \to \mathbb{R} \text{ measurable; } \int_0^T \int_0^T |f(u)||f(v)|||u-v|^{2H-2} du dv < \infty\}$.

A different approach of characterizing the space $\mathcal{H}(0,T)$ is based on the *transfer operator*. We recall that the kernel $K_H(t,s), t > s$ of the fractional covariance function R_H is defined by:

$$K_H(t,s) = c_H^* s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} du,$$

where $c_H^* = \{\alpha_H \Gamma(3/2-H)/[\Gamma(2-2H)\Gamma(H-1/2)]\}^{1/2}$.

Note that $R_H(t,s) = \int_0^{t\wedge s} K_H(t,u)K(s,u)du$ (see p.7-8, [9]) and hence

$$\langle K_H^* 1_{[0,t]}, K_H^* 1_{[0,s]} \rangle_{L_2(0,T)} = \int_0^{t\wedge s} K_H(t,u)K_H(s,u)du = R_H(t,s) = \langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}(0,T)},$$

i.e. K_H^* is an isometry between $(\mathcal{E}(0,T), \langle \cdot, \cdot \rangle_{\mathcal{H}(0,T)})$ and $L_2(0,T)$. Since $\mathcal{H}(0,T)$ is the completion of $\mathcal{E}(0,T)$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}(0,T)}$, this isometry can be extended to $\mathcal{H}(0,T)$. We denote this extension by K_H^*. In fact, one can prove that the map $K_H^* : \mathcal{H}(0,T) \to L_2(0,T)$ is also surjective (see the proof of Lemma 2.3).

We are now introducing the space of deterministic integrands associated with the a noise which is fractional in time and white in space. This space was also considered in [10].

More precisely, let $\mathcal{H}$ be the completion of $\mathcal{D}((0,T)\times\mathbb{R}^d)$ with respect to the inner product

$$\begin{aligned}\langle\varphi,\psi\rangle_{\mathcal{H}} &= \alpha_H\int_0^T\int_0^T\int_{\mathbb{R}^d}\varphi(u,x)|u-v|^{2H-2}\psi(v,x)dxdvdu\\ &= \alpha_H c_H\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathcal{F}_{0,T}\varphi(\tau,x)\overline{\mathcal{F}_{0,T}\psi(\tau,x)}|\tau|^{-(2H-1)}dxd\tau\\ &= \int_{\mathbb{R}^d}\langle\varphi(\cdot,x),\psi(\cdot,x)\rangle_{\mathcal{H}(0,T)}dx\end{aligned}$$

where the second equality above follows by Lemma A.1.(b), and the third is due to Fubini's theorem.

If we let $\mathcal{E}$ be the space of all linear combinations of indicator functions $1_{[0,t]\times A}, t\in[0,T], A\in\mathcal{B}_b(\mathbb{R}^d)$, then one can prove that $\mathcal{H}$ is also the completion of $\mathcal{E}$ with respect to the inner product

$$\langle 1_{[0,t]\times A},1_{[0,s]\times B}\rangle_{\mathcal{H}}=R_H(t,s)\lambda(A\cap B)$$

where λ is the Lebesgue measure on $\mathbb{R}^d$. (The argument is similar to the one used in the temporal case.) Similarly to (2), one can show that

$$\begin{aligned}\mathcal{H}\subset\overline{\mathcal{H}}:=\{S:\mathbb{R}^d\to\mathcal{S}'(\mathbb{R});\mathcal{F}_{0,T}S(\cdot,x)\text{ is a function }\forall x\in\mathbb{R}^d,\ (\tau,x)\mapsto\mathcal{F}_{0,T}S(\tau,x)\text{ is measurable,}\\ \text{and }\int_{\mathbb{R}}\int_{\mathbb{R}^d}|\mathcal{F}_{0,T}S(\tau,x)|^2|\tau|^{-(2H-1)}dxd\tau<\infty\}.\end{aligned}$$

Using (3) and the fact that

$$\|S\|_{\mathcal{H}}^2=\int_{\mathbb{R}^d}\|S(\cdot,x)\|_{\mathcal{H}(0,T)}^2dx,\quad\forall S\in\mathcal{H},$$

one can show that

$$L_2((0,T)\times\mathbb{R}^d)\subset|\mathcal{H}|\subset\mathcal{H}\subset L_2(\mathbb{R}^d;\mathcal{H}(0,T)),$$

where $|\mathcal{H}|=\{\varphi:[0,T]\times\mathbb{R}^d\text{ measurable; }\int_0^T\int_0^T\int_{\mathbb{R}^d}|\varphi(u,x)||\varphi(v,x)||u-v|^{2H-2}dxdvdu<\infty\}$.

The next result gives an alternative criterion for verifying that a function φ lies in $\mathcal{H}$. (It can be compared to Theorems 2 and 3 of [2].)

Theorem 2.2. *Let $\varphi:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ be a function which satisfies the following conditions:*
(i) $\varphi(\cdot,x)\in L_2(0,T)$ for every $x\in\mathbb{R}^d$;
(ii) $(\tau,x)\mapsto\mathcal{F}_{0,T}\varphi(\tau,x)$ is measurable;
(iii) $\int_{\mathbb{R}}\int_{\mathbb{R}^d}|\mathcal{F}_{0,T}\varphi(\tau,x)|^2|\tau|^{-(2H-1)}dxd\tau<\infty$.
Then $\varphi\in\mathcal{H}$.

Proof: Similarly to Proposition 3.3, [11], we let

$$\begin{aligned}\tilde\Lambda &= \{\varphi:[0,T]\times\mathbb{R}^d\to\mathbb{R};\varphi(\cdot,x)\in L_2(0,T)\ \forall x,(\tau,x)\mapsto\mathcal{F}_{0,T}\varphi(\tau,x)\text{ is measurable, and}\\ &\qquad\|\varphi\|_{\tilde\Lambda}^2:=c_1\int_{\mathbb{R}}\int_{\mathbb{R}^d}|\mathcal{F}_{0,T}\varphi(\tau,x)|^2|\tau|^{-(2H-1)}dxd\tau<\infty\}\\ \Lambda &= \{\varphi:[0,T]\times\mathbb{R}^d\to\mathbb{R};\ \|\varphi\|_{\Lambda}^2:=c_2\int_0^T\int_{\mathbb{R}^d}[I_{T-}^{H-1/2}(u^{H-1/2}\varphi(u,x))(s)]^2s^{-(2H-1)}dxds<\infty\}\end{aligned}$$

where $c_1=\alpha_H c_H$ and $c_2=\{c_H^*\Gamma(H-1/2)\}^2$. We now prove that

$$\tilde\Lambda\subset\Lambda\quad\text{and}\quad\|\varphi\|_{\tilde\Lambda}=\|\varphi\|_{\Lambda},\quad\forall\varphi\in\tilde\Lambda.\tag{4}$$

Let $\varphi \in \tilde{\Lambda}$ be arbitrary. Since $\varphi(\cdot, x) \in L_2(0,T) \subset \mathcal{H}(0,T)$ and K_H^* is an isometry from $\mathcal{H}(0,T)$ to $L_2(0,T)$, we have $\|\varphi(\cdot,x)\|^2_{\mathcal{H}(0,T)} = \|K_H^* \varphi(\cdot,x)\|^2_{L_2(0,T)}$ for all $x \in \mathbb{R}^d$, that is

$$c_1 \int_{\mathbb{R}} |\mathcal{F}_{0,T}\varphi(\tau,x)|^2 |\tau|^{-(2H-1)} d\tau = c_2 \int_0^T [I_{T-}^{H-1/2}(u^{H-1/2}\varphi(u,x))(s)]^2 s^{-(2H-1)} ds, \quad \forall x \in \mathbb{R}^d.$$

Integrating with respect to dx and using Fubini's theorem, we get $\|\varphi\|_{\tilde{\Lambda}} = \|\varphi\|_{\Lambda}$. The fact that $\|\varphi\|_{\tilde{\Lambda}} < \infty$ forces $\|\varphi\|_{\Lambda} < \infty$, i.e. $\varphi \in \Lambda$. This concludes the proof of (4).

Next we prove that

$$\mathcal{E} \text{ is dense in } \Lambda \text{ with respect to } \|\cdot\|_{\Lambda}. \tag{5}$$

Let $\varphi \in \Lambda$ and $\varepsilon > 0$ be arbitrary. Since the map $(s,x) \mapsto I_{T-}^{H-1/2}(u^{H-1/2}\varphi(u,x))(s)$ belongs to $L_2((0,T) \times \mathbb{R}^d, d\lambda_H \times dx)$ where $\lambda_H(s) = s^{-(2H-1)}ds$, there exists a simple function $g(s,x) = \sum_{k=1}^n b_k 1_{[c_k,d_k)}(s) 1_{A_k}(x)$ on $(0,T) \times \mathbb{R}^d$, with $b_k \in \mathbb{R}$, $0 < c_k < d_k < T$ and $A_k \subset \mathbb{R}^d$ Borel set, such that

$$\int_0^T \int_{\mathbb{R}^d} [I_{T-}^{H-1/2}(u^{H-1/2}\varphi(u,x))(s) - g(s,x)]^2 s^{-(2H-1)} dx ds < \varepsilon. \tag{6}$$

By relation (8.1) of [12], we know that there exists an elementary function $l_k \in \mathcal{E}(0,T)$ such that

$$\int_0^T [1_{[c_k,d_k)}(s) - I_{T-}^{H-1/2}(u^{H-1/2} l_k(u))(s)]^2 s^{-(2H-1)} ds < \varepsilon / C_g,$$

where we chose $C_g := n \sum_{k=1}^n b_k^2 \lambda(A_k)$. Define $l(s,x) = \sum_{k=1}^n b_k l_k(t) 1_{A_k}(x)$ and note that $l \in \mathcal{E}$. Then

$$\int_0^T \int_{\mathbb{R}^d} [g(s,x) - I_{T-}^{H-1/2}(u^{H-1/2} l(u,x))(s)]^2 s^{-(2H-1)} ds < \varepsilon. \tag{7}$$

From (6) and (7), we get

$$\int_0^T \int_{\mathbb{R}^d} [I_{T-}^{H-1/2}(u^{H-1/2}\varphi(u,x))(s) - I_{T-}^{H-1/2}(u^{H-1/2} l(u,x))(s)]^2 s^{-(2H-1)} dx ds < 4\varepsilon,$$

i.e. $\|\varphi - l\|^2_{\Lambda} < 4\varepsilon c_2$. This concludes the proof of (5).

From (4) and (5), we infer immediately that $\mathcal{E}$ is dense in $\tilde{\Lambda}$ with respect to $\|\cdot\|_{\tilde{\Lambda}}$. Since $\|\cdot\|_{\tilde{\Lambda}} = \|\cdot\|_{\mathcal{H}}$ and $\mathcal{H}$ is the completion of $\mathcal{E}$ with respect to $\|\cdot\|_{\mathcal{H}}$, it follows that $\tilde{\Lambda} \subset \mathcal{H}$. This concludes the proof of the theorem. ∎

As in [10], we define the transfer operator by:

$$(K_H^* 1_{[0,t]\times A})(s,x) := K_H(t,s) 1_{[0,t]\times A}(s,x). \tag{8}$$

Note that

$$\begin{aligned}
\langle K_H^* 1_{[0,t]\times A}, K_H^* 1_{[0,s]\times B}\rangle_{L_2((0,T)\times\mathbb{R}^d)} &= \left(\int_0^{t\wedge s} K_H(t,u) K_H(s,u) du\right) \langle 1_A, 1_B\rangle_{L_2(\mathbb{R}^d)} \\
&= R_H(t,s)\lambda(A\cap B) = \langle 1_{[0,t]\times A}, 1_{[0,s]\times B}\rangle_{\mathcal{H}},
\end{aligned}$$

i.e. K_H^* is an isometry between $(\mathcal{E}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ and $L_2((0,T)\times\mathbb{R}^d)$. Since $\mathcal{H}$ is the completion of $\mathcal{E}$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, this isometry can be extended to $\mathcal{H}$. We denote this extension by $K_{\mathcal{H}}^*$.

Lemma 2.3. *$K_{\mathcal{H}}^* : \mathcal{H} \to L_2((0,T)\times\mathbb{R}^d)$ is surjective.*

Proof: Note that $1_{[0,t]\times A}\in K^*_{\mathcal{H}}(\mathcal{H})$ for all $t\in[0,T], A\in\mathcal{B}_b(\mathbb{R}^d)$; hence $\varphi\in K^*_{\mathcal{H}}(\mathcal{H})$ for every $\varphi\in\mathcal{E}$. Let $f\in L_2((0,T)\times\mathbb{R}^d)$ be arbitrary. Since $\mathcal{E}$ is dense in $L_2((0,T)\times\mathbb{R}^d)$, there exists a sequence $(f_n)_n\subset\mathcal{E}$ such that $\|f_n-f\|_{L_2((0,T)\times\mathbb{R}^d)}\to 0$. Since $f_n\in K^*_{\mathcal{H}}(\mathcal{H})$, there exists $\varphi_n\in\mathcal{H}$ such that $f_n=K^*_{\mathcal{H}}\varphi_n$. The sequence $(\varphi_n)_n$ is Cauchy in $\mathcal{H}$:

$$\|\varphi_n-\varphi_m\|_{\mathcal{H}}=\|K^*_{\mathcal{H}}\varphi_n-K^*_{\mathcal{H}}\varphi_m\|_{L_2((0,T)\times\mathbb{R}^d)}=\|f_n-f_m\|_{L_2((0,T)\times\mathbb{R}^d)}\to 0$$

as $m,n\to\infty$. Since $\mathcal{H}$ is complete, there exists $\varphi\in\mathcal{H}$ such that $\|\varphi_n-\varphi\|_{\mathcal{H}}\to 0$. Hence $\|f_n-K^*_{\mathcal{H}}\varphi\|_{L_2((0,T)\times\mathbb{R}^d)}=\|K^*_{\mathcal{H}}\varphi_n-K^*_{\mathcal{H}}\varphi\|_{L_2((0,T)\times\mathbb{R}^d)}\to 0$. But $\|f_n-f\|_{L_2((0,T)\times\mathbb{R}^d)}\to 0$. We conclude that $K^*_{\mathcal{H}}\varphi=f$. ■

Remark 2.4. Note that for every $\varphi\in\mathcal{E}$

$$\begin{aligned}(K^*_H\varphi)(s,x) &= \int_s^T \varphi(r,x)\frac{\partial K_H}{\partial r}(r,s)dr = c^*_H\int_s^T\varphi(r,x)\left(\frac{r}{s}\right)^{H-1/2}(r-s)^{H-3/2}dr\\ &= c^*_H\Gamma\left(H-\frac{1}{2}\right)s^{-(H-1/2)}I^{H-1/2}_{T-}(u^{H-1/2}\varphi(u,x))(s).\end{aligned}\tag{9}$$

Using Lemma 2.3, we can *formally* say that

$$\mathcal{H}=\{\varphi \text{ such that } (s,x)\mapsto s^{-(H-1/2)}I^{H-1/2}_{T-}(u^{H-1/2}\varphi(u,x))(s) \text{ is in } L_2((0,T)\times\mathbb{R}^d)\}.$$

2.2. **The Noise and the Stochastic Integral.** In this paragraph we describe the Gaussian noise which is randomly perturbing the heat equation. This noise is assumed to be fractional in time and white in space and was also considered by other authors (see [10]).

Let $F=\{F(\varphi);\varphi\in\mathcal{D}((0,T)\times\mathbb{R}^d)\}$ be a zero-mean Gaussian process with covariance

$$E(F(\varphi)F(\psi))=\langle\varphi,\psi\rangle_{\mathcal{H}}.\tag{10}$$

Let H^F be the Gaussian space of F, i.e. the closed linear span of $\{F(\varphi);\varphi\in\mathcal{D}((0,T)\times\mathbb{R}^d)\}$ in $L^2(\Omega)$.

For every indicator function $1_{[0,t]\times A}\in\mathcal{E}$, there exists a sequence $(\varphi_n)_n\subset\mathcal{D}((0,T)\times\mathbb{R}^d)$ such that $\varphi_n\to 1_{[0,t]\times A}$ and supp $\varphi_n\subset K$, $\forall n$, where $K\subset(0,T)\times\mathbb{R}^d$ is a compact. Hence $\|\varphi_n-1_{[0,t]\times A}\|_{\mathcal{H}}\to 0$ and $E(F(\varphi_m)-F(\varphi_n))^2=\|\varphi_m-\varphi_n\|_{\mathcal{H}}\to 0$ as $m,n\to\infty$, i.e. the sequence $\{F(\varphi_n)\}_n$ is Cauchy in $L_2(\Omega)$. A standard argument shows that its limit does not depend on $\{\varphi_n\}_n$. We set $F_t(A)=F(1_{[0,t]\times A})=_{L_2(\Omega)}\lim_n F(\varphi_n)\in H^F$. We extend F by linearity to $\mathcal{E}$. A limiting argument and relation (10) shows that

$$E(F(\varphi)F(\psi))=\langle\varphi,\psi\rangle_{\mathcal{H}},\quad\forall\varphi,\psi\in\mathcal{E},$$

i.e. $\varphi\mapsto F(\varphi)$ is an isometry between $(\mathcal{E},\langle\cdot,\cdot\rangle_{\mathcal{H}})$ and H^F. Since $\mathcal{H}$ is the completion of $\mathcal{E}$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, this isometry can be extended to $\mathcal{H}$, giving us the stochastic integral with respect to F. We will use the notation

$$F(\varphi)=\int_0^T\int_{\mathbb{R}^d}\varphi(t,x)F(dt,dx).$$

Remark 2.5. One can use the transfer operator $K^*_{\mathcal{H}}$ to explore the relationship between $F(\varphi)$ and Walsh's stochastic integral (introduced in [17]). More precisely, using Lemma 2.3, we define

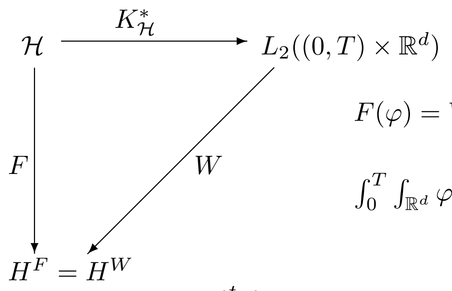
$$W(\phi):=F((K^*_{\mathcal{H}})^{-1}(\phi)),\quad \phi\in L_2((0,T)\times\mathbb{R}^d).\tag{11}$$

Note that

$$\mathbf{E}(W(\phi)W(\eta))=\langle(K^*_{\mathcal{H}})^{-1}(\phi),(K^*_{\mathcal{H}})^{-1}(\eta)\rangle_{\mathcal{H}}=\langle\phi,\eta\rangle_{L_2((0,T)\times\mathbb{R}^d)},$$

i.e. $W = \{W(\phi); \phi \in L_2((0,T) \times \mathbb{R}^d)\}$ is a space-time white noise. Using the stochastic integral notation, we write $W(\phi) = \int_0^T \int_{\mathbb{R}^d} \phi(t,x) W(dt,dx)$ for all $\phi \in L_2((0,T) \times \mathbb{R}^d)$. (Note that $W(\phi)$ is Walsh's stochastic integral with respect to the noise W.)

Let H^W be the Gaussian space of W, i.e. the closed linear span of $\{W(\phi); \phi \in L_2((0,T) \times \mathbb{R}^d)\}$ in $L_2(\Omega)$. By (11), we can see that $H^W = H^F$. The following diagram summarizes these facts:



$$F(\varphi) = W(K^*_{\mathcal{H}}\varphi), \quad \forall \varphi \in \mathcal{H}, \text{ i.e.}$$

$$\int_0^T \int_{\mathbb{R}^d} \varphi(t,x) F(dt,dx) = \int_0^T \int_{\mathbb{R}^d} (K^*_{\mathcal{H}}\varphi)(t,x) W(dt,dx), \quad \forall \varphi \in \mathcal{H}.$$

In particular, $F(t,A) = \int_0^t \int_A K_H(t,s) W(ds,dy)$. This relationship will not be used in the present paper.

2.3. **The Solution of the Stochastic Heat Equation.** We consider the stochastic heat equation driven by the noise F, written *formally* as:

$$v_t - \Delta v = \dot{F}, \quad \text{in } (0,T) \times \mathbb{R}^d, \quad v(0,\cdot) = 0, \tag{12}$$

where Δv denotes the Laplacian of v, and v_t is the partial derivative with respect to t.

Let G be the fundamental solution of the classical heat equation, i.e.

$$G(t,x) = \begin{cases} (4\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{4t}\right) & \text{if } t > 0, x \in R^d \\ 0 & \text{if } t \le 0, x \in R^d \end{cases} \tag{13}$$

Let $g_{tx}(s,y) := (G_{tx})\tilde{}\,(s,y) = G(t-s, x-y)$. The following result shows that the kernel G has the desired regularity, which is needed to apply Theorem 2.2.

Lemma 2.6. *If $\varphi = \eta * \tilde{G}$, where $\eta \in \mathcal{D}(0,T) \times \mathbb{R}^d)$, then*

$$\varphi(\cdot, x) \in L_2(0,T) \quad \forall x \in \mathbb{R}^d.$$

Proof: Without loss of generality, we suppose that $\eta(t,x) = \phi(t)\psi(x)$, where $\phi \in \mathcal{D}(0,T)$ and $\psi \in \mathcal{D}(\mathbb{R}^d)$. Using Minkowski's inequality for integrals (see p. 271, [15]), we have

$$\begin{aligned} \left(\int_0^T |\varphi(t,x)|^2 dt \right)^{1/2} &= \left(\int_0^T \left| \int_0^T \int_{\mathbb{R}^d} \phi(s)\psi(y) G(s-t, y-x) 1_{\{s>t\}} dy ds \right|^2 dt \right)^{1/2} \\ &\le \int_0^T \int_{\mathbb{R}^d} |\phi(s)\psi(y)| \left(\int_0^s |G(s-t, y-x)|^2 dt \right)^{1/2} dy ds. \end{aligned}$$

Using the change of variables $s - t = t'$ and $1/t' = u$, we get

$$\int_0^s |G(s-t, y-x)|^2 dt = \frac{1}{(4\pi)^d} \int_{1/s}^\infty u^{d-2} e^{-|y-x|^2 u/2} du \le \frac{C}{|y-x|^{2(d-1)}},$$

and

$$\left(\int_0^T |\varphi(t,x)|^2 dt\right)^{1/2} \leq C\int_0^T |\phi(s)| \int_{\mathbb{R}^d} \frac{|\psi(y)|}{|y-x|^{d-1}} dyds \leq C(|\psi| * R_1)(x) < \infty,$$

where $R_1(x) = \gamma_{1,d}|x|^{-(d-1)}$ is the Riesz kernel of order 1 in $\mathbb{R}^d$ and the convolution $|\psi| * R_1$ is well-defined by Theorem V.1, p. 119, [15]. ■

The next result gives the necessary and sufficient condition for the existence of the process solution.

Theorem 2.7. *If*

$$H > \frac{d}{4}\ , \tag{14}$$

then: (a) $g_{tx} \in |\mathcal{H}|$ *for every* $(t,x) \in [0,T] \times \mathbb{R}^d$*; (b)* $\eta * \tilde{G} \in \mathcal{H}$ *for every* $\eta \in \mathcal{D}((0,T)\times\mathbb{R}^d)$*. Moreover,*

$$\|g_{tx}\|_{\mathcal{H}} < \infty \ \forall (t,x) \in [0,T]\times\mathbb{R}^d \quad \textit{if and only if (14) holds.} \tag{15}$$

Remark 2.8. Note that condition (14) implies that $d \in \{1,2,3\}$, since $H < 1$.

Proof: (a) We will apply Theorem 2.2 to the function g_{tx}. Note that for every fixed $y \in \mathbb{R}^d$

$$\int_0^T |g_{tx}(s,y)|^2 ds = C\int_0^t \frac{1}{(t-s)^d} e^{-|y-x|^2/[2(t-s)]} ds = C\int_{1/t}^\infty u^{d-2} e^{-|y-x|^2 u/2} du < \infty,$$

i.e. $g_{tx}(\cdot,y) \in L_2(0,T)$. Clearly $(\tau,y) \mapsto \mathcal{F}_{0,T} g_{tx}(\tau,y)$ is measurable. We now calculate

$$\|g_{tx}\|^2_{\mathcal{H}} := \alpha_H c_H \int_{\mathbb{R}}\int_{\mathbb{R}^d} |\mathcal{F}_{0,T} g_{tx}(\tau,y)|^2 |\tau|^{-(2H-1)} dyd\tau.$$

For this, we write

$$\begin{aligned}
\|g_{tx}\|^2_{\mathcal{H}} &= \alpha_H c_H \int_{\mathbb{R}} |\tau|^{-(2H-1)} \int_{\mathbb{R}^d} \left(\int_0^T e^{-i\tau s} g_{tx}(s,y) ds\right)\left(\int_0^T e^{i\tau r} g_{tx}(r,y) dr\right) dyd\tau \\
&= \alpha_H c_H \int_{\mathbb{R}} |\tau|^{-(2H-1)} \int_0^t\int_0^t e^{-i\tau(s-r)} I(r,s) drdsd\tau, \\
&= \alpha_H \int_0^t\int_0^t |s-r|^{2H-2} I(r,s) drds
\end{aligned}$$

where we used Lemma A.1 (Appendix A) for the last equality and we denoted

$$I(r,s) := \int_{\mathbb{R}^d} g_{tx}(s,y) g_{tx}(r,y) dy = (2t-s-r)^{-d/2}$$

We obtain that

$$\|g_{tx}\|^2_{\mathcal{H}} = \alpha_H \int_0^t\int_0^t |s-r|^{2H-2}(2t-s-r)^{-d/2} drds = \alpha_H \int_0^t\int_0^t |u-v|^{2H-2}(u+v)^{-d/2} drds.$$

Relation (15) follows, since the last integral is finite if and only if $2H > d/2$. In this case, we have $\|g_{tx}\|_{|\mathcal{H}|} = \|g_{tx}\|_{\mathcal{H}} < \infty$ (since $g_{tx} \geq 0$), and hence $g_{tx} \in |\mathcal{H}|$.

(b) We will apply Theorem 2.2 to the function $\varphi = \eta * \tilde{G}$, since $\varphi(\cdot,x) \in L_2(0,T)$ for every $x \in \mathbb{R}^d$, by Lemma 2.6. By writing

$$\mathcal{F}_{0,T}\varphi(\tau,x) = \int_0^T e^{-i\tau s} \int_0^{T-s} \int_{\mathbb{R}^d} \eta(u+s,y) G(u,y-x) dyduds, \tag{16}$$

we see that $(\tau, x) \mapsto \mathcal{F}_{0,T}\varphi(\tau, x)$ is measurable. We now calculate

$$\|\varphi\|_{\mathcal{H}}^2 := \alpha_H c_H \int_{\mathbb{R}} \int_{\mathbb{R}^d} |\mathcal{F}_{0,T}\varphi(\tau, x)|^2 |\tau|^{-(2H-1)} dx d\tau.$$

Using (16), we get

$$\begin{aligned}\|\varphi\|_{\mathcal{H}}^2 &= \alpha_H c_H \int_{\mathbb{R}} |\tau|^{-(2H-1)} \int_0^T \int_0^T e^{-i\tau(s-r)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^{T-s} \int_0^{T-r} \eta(u+s, y)\eta(v+r, z) \\ &\quad J(u, v, y, z) dv\, du\, dz\, dy\, dr\, ds\, d\tau \\ &= \alpha_H \int_0^T \int_0^T |s-r|^{2H-2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^{T-s} \int_0^{T-r} \eta(u+s, y)\eta(v+r, z) \\ &\quad J(u, v, y, z) dv\, du\, dz\, dy\, dr\, ds, \end{aligned} \tag{17}$$

where we used Lemma A.1 (Appendix A) for the second equality and we denoted

$$J(u, v, y, z) := \int_{\mathbb{R}^d} G(u, y-x)G(v, z-x)dx = \exp\left\{-\frac{|y-z|^2}{4(u+v)}\right\}(u+v)^{-d/2}.$$

Clearly

$$J(u, v, y, z) \le (u+v)^{-d/2} \tag{18}$$

for every $u \in (0, T-s), v \in (0, T-r)$. Using (17), (18) and the fact that $\eta \in \mathcal{D}((0,T) \times \mathbb{R}^d)$, we get

$$\begin{aligned}\|\varphi\|_{\mathcal{H}}^2 &\le \alpha_H \int_0^T \int_0^T |s-r|^{2H-2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^{T-s} \int_0^{T-r} |\eta(u+s, y)\eta(v+r, z)| \\ &\quad (u+v)^{-d/2} dv\, du\, dz\, dy\, dr\, ds \\ &\le \alpha_H C_\eta \int_0^T \int_0^T |s-r|^{2H-2} \int_0^{T-s} \int_0^{T-r} (u+v)^{-d/2} dv\, du\, dr\, ds \\ &\le \alpha_H C_\eta \int_0^T \int_0^T (u+v)^{-d/2} (T-u)^{2H} dv\, du,\end{aligned}$$

where for the last inequality we used Fubini's theorem and the fact that $\int_0^{T-u} \int_0^{T-v} |s-r|^{2H-2} dr\, ds = R_H(T-u, T-v) = [(T-u)^{2H} + (T-v)^{2H} - (u-v)^{2H}]/2$. The last integral is clearly finite if $2H > d/2$, i.e. $H > d/4$. ■

Under the conditions of Theorem 2.7, $F(g_{tx})$ and $F(\eta * \tilde{G})$ are well-defined for every (t, x), respectively for every $\eta \in \mathcal{D}((0,T) \times \mathbb{R}^d)$ and we can introduce the following definition:

Definition 2.9. a) The process $\{v(t, x); t \in [0, T], x \in \mathbb{R}^d\}$ defined by

$$v(t, x) := F(g_{tx}) = \int_0^T \int_{\mathbb{R}^d} G(t-s, x-y)F(ds, dy) \tag{19}$$

is called the **process solution** of the stochastic heat equation (12), with vanishing initial conditions.

b) The process $\{v(\eta); \eta \in \mathcal{D}((0,T) \times \mathbb{R}^d)\}$ defined by

$$v(\eta) := F(\eta * \tilde{G}) = \int_0^T \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \eta(t+s, x+y)G(s, y) dy ds \right) F(dt, dx)$$

is called the **distribution-valued solution** of the stochastic heat equation (12), with vanishing initial conditions.

Lemma 2.10. *The process $\{v(t, x); t \in [0, T], x \in \mathbb{R}^d\}$ is $L^2(\Omega)$-continuous.*

Proof: We first prove the continuity in t. We have

$$E|v(t+h,x)-v(t,x)|^2 = E\left|\int_0^{t+h}\int_{\mathbb{R}^d} g_{t+h,x}(s,y)F(ds,dy) - \int_0^t\int_{\mathbb{R}^d} g_{tx}(s,y)F(ds,dy)\right|^2$$

$$\begin{aligned}
&= E\left|\int_0^t\int_{\mathbb{R}^d}(g_{t+h,x}(s,y)-g_{tx}(s,y))F(ds,dy) + \int_t^{t+h}\int_{\mathbb{R}^d} g_{t+h,x}(s,y)F(ds,dy)\right|^2 \\
&\leq 2E\left|\int_0^t\int_{\mathbb{R}^d}(g_{t+h,x}(s,y)-g_{tx}(s,y))F(ds,dy)\right|^2 + 2E\left|\int_t^{t+h}\int_{\mathbb{R}^d} g_{t+h,x}(s,y)F(ds,dy)\right|^2 \\
&:= 2I_1(h)+2I_2(h).
\end{aligned}$$

We first treat the term $I_1(h)$. Note that

$$\begin{aligned}
I_1(h) &= \|F((g_{t+h,x}-g_{tx})1_{[0,t]})\|^2_{L_2(\Omega)} = \|(g_{t+h,x}-g_{tx})1_{[0,t]}\|^2_{\mathcal{H}} = \\
&= \alpha_H\int_0^t\int_0^t\int_{\mathbb{R}^d}(g_{t+h,x}-g_{tx})(u,y)|u-v|^{2H-2}(g_{t+h,x}-g_{tx})(v,y)dy\,dv\,du.
\end{aligned}$$

The continuity of $G(t,x)$ with respect to t shows that the integrand converges to zero as $h\to 0$. By applying the dominated convergence theorem (using the fact that $\|g_{tx}\|_{\mathcal{H}} < \infty$), we conclude that $I_1(h)\to 0$.

For the term $I_2(h)$, we have

$$\begin{aligned}
I_2(h) &= \|F(g_{t+h,x}1_{[t,t+h]})\|^2_{L_2(\Omega)} = \|g_{t+h,x}1_{[t,t+h]}\|^2_{\mathcal{H}} = \\
&= \alpha_H\int_t^{t+h}\int_t^{t+h}\int_{\mathbb{R}^d} g_{t+h,x}(u,y)g_{t+h,x}(v,y)|u-v|^{2H-2}dy\,dv\,du \\
&= \alpha_H\int_{t-h}^{t}\int_{t-h}^{t}\int_{\mathbb{R}^d} g_{tx}(u',y)g_{tx}(v',y)|u'-v'|^{2H-2}dy\,dv'\,du.
\end{aligned}$$

Since $1_{(t-h,t)}1_{(t-h,t)}\to 0$ as $h\to 0$ and $\|g_{tx}\|_{\mathcal{H}} < \infty$, we conclude that $I_2(h)\to 0$, by the dominated convergence theorem.

We now prove the continuity in x. We have

$$\begin{aligned}
E|v(t,x+h)-v(t,x)|^2 &= E\left|\int_0^t\int_{\mathbb{R}^d}(g_{t,x+h}(s,y)-g_{tx}(s,y))F(ds,dy)\right|^2 = \|(g_{t,x+h}-g_{tx})1_{[0,t]}\|^2_{\mathcal{H}} \\
&= \alpha_H\int_0^t\int_0^t(g_{t,x+h}-g_{tx})(u,y)||u-v|^{2H-2}(g_{t,x+h}-g_{tx})(v,y)dy\,dv\,du.
\end{aligned}$$

By the continuity in x of the function $G(t,x)$, the integrand converges to 0 as $h\to 0$. By the dominated convergence theorem, we conclude that $E|v(t,x+h)-v(t,x)|^2\to 0$ as $h\to 0$. ■

Theorem 2.11. *Let $\{v(\eta);\eta\in\mathcal{D}((0,T)\times\mathbb{R}^d)\}$ be the distribution-valued solution of the stochastic heat equation (12).*

In order that there exists a jointly measurable and locally mean-square bounded process $Y=\{Y(t,x);t\in[0,T],x\in\mathbb{R}^d\}$ such that

$$v(\eta)=\int_0^T\int_{\mathbb{R}^d} Y(t,x)\eta(t,x)dxdt \quad \forall\eta\in\mathcal{D}((0,T)\times\mathbb{R}^d) \quad a.s. \tag{20}$$

it is necessary and sufficient that (14) holds. In this case, Y is a modification of the process $v=\{v(t,x);t\in[0,T],x\in\mathbb{R}^d\}$ defined by (19).

Proof: The necessity part is similar to the proof of Theorem 11, [2]. Suppose that there exists a jointly measurable and locally mean-square bounded process $Y = \{Y(t,x); t \in [0,T], x \in \mathbb{R}^d\}$ such that (20) holds.

Let $(t_0, x_0) \in [0,T] \times \mathbb{R}^d$ be fixed. Let λ, ψ be nonnegative functions such that $\lambda \in \mathcal{D}(0,T), \psi \in \mathcal{D}(\mathbb{R}^d)$ and $\int_0^T \lambda(t)dt = \int_{\mathbb{R}^d} \psi(x)dx = 1$. Set $\lambda_n(t) = n\lambda(nt)$ and $\psi_n(x) = n^d \psi(nx)$. Define $\eta_n(t,x) = \lambda_n(t-t_0)\psi_n(x-x_0)$.

We calculate $E|v(\eta_n)|^2$ in two ways. First, using (20), we get

$$E|v(\eta_n)|^2 = \int_{[0,T]\times\mathbb{R}^d} E|Y(t,x)Y(s,y)|\ \eta_n(t,x)\eta_n(s,y)dydxdsdt.$$

Using Lebesgue differentiation theorem (see Exercise 2, Chapter 7, [18]), we get

$$\lim_n E|v(\eta_n)|^2 = E|Y(t_0,x_0)|^2. \tag{21}$$

Secondly,

$$E|v(\eta_n)|^2 = E|F(\eta_n * \tilde{G})|^2 = \|\eta_n * \tilde{G}\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^d}\int_{\mathbb{R}} |\mathcal{F}_{0,T}(\eta_n * \tilde{G})(\tau,x)|^2 |\tau|^{-(2H-1)} d\tau dx.$$

We claim that, for every $\tau \in \mathbb{R}, x \in \mathbb{R}^d$, we have: (see Appendix B for the proof)

$$\lim_n \mathcal{F}_{0,T}(\eta_n * \tilde{G})(\tau,x) = \mathcal{F}_{0,T} g_{t_0x_0}(\tau,x). \tag{22}$$

Using Fatou's lemma, (21) and (22), we get

$$\begin{aligned}
\|g_{t_0x_0}\|_{\mathcal{H}}^2 &= \int_{\mathbb{R}^d}\int_{\mathbb{R}} |\mathcal{F}_{0,T} g_{t_0x_0}(\tau,x)|^2 |\tau|^{-(2H-1)} d\tau dx \\
&\leq \varliminf_n \int_{\mathbb{R}^d}\int_{\mathbb{R}} |\mathcal{F}_{0,T}(\eta_n * \tilde{G})(\tau,x)|^2 |\tau|^{-(2H-1)} d\tau dx \\
&= \varliminf_n E|v(\eta_n)|^2 = E|Y(t_0,x_0)|^2 < \infty,
\end{aligned}$$

which forces $H > d/4$, by virtue of (15).

We now prove the sufficiency part. By Lemma 2.10, we know that the process v defined by (19) is continuous in $L^2(\Omega)$. Hence, it is continuous in probability and by Theorem IV.30, [4], there exists a jointly measurable modification Y of v. We have

$$E(Y(t,x)Y(s,y)) = E(v(t,x)v(s,y)) = E(F(g_{tx})F(g_{sy})) = \langle g_{tx}, g_{sy}\rangle_{\mathcal{H}} \tag{23}$$

and

$$E(v(\eta)Y(t,x)) = E(v(\eta)v(t,x)) = E(F(\eta * \tilde{G})F(g_{tx})) = \langle \eta * \tilde{G}, g_{tx}\rangle_{\mathcal{H}}. \tag{24}$$

To prove that (20) holds, we will show that

$$E\left|\int_0^T\int_{\mathbb{R}^d} Y(t,x)\eta(t,x)dxdt\right|^2 = E\left(v(\eta)\int_0^T\int_{\mathbb{R}^d} Y(t,x)\eta(t,x)dxdt\right) = E|v(\eta)|^2.$$

By Fubini's theorem and (23), we get

$$E\left|\int_0^T\int_{\mathbb{R}^d} Y(t,x)\eta(t,x)dxdt\right|^2 = \int_{([0,T]\times\mathbb{R}^d)^2}\eta(t,x)\eta(s,y)\langle g_{tx},g_{sy}\rangle_{\mathcal{H}}dy\,dx\,ds\,dt =$$
$$\int_{([0,T]\times\mathbb{R}^d)^2}\eta(t,x)\eta(s,y)\left(\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathcal{F}_{0,T}g_{tx}(\tau,z)\overline{\mathcal{F}_{0,T}g_{sy}(\tau,z)}\ |\tau|^{-(2H-1)}dzd\tau\right)dydxdsdt =$$
$$\int_{\mathbb{R}}\int_{\mathbb{R}^d}|\mathcal{F}_{0,T}(\eta*\tilde{G})(\tau,z)|^2|\tau|^{-(2H-1)}dz\,d\tau = \|\eta*\tilde{G}\|^2_{\mathcal{H}} = E|F(\eta*\tilde{G})|^2 = E|v(\eta)|^2.$$

On the other hand, using Fubini's theorem and (24), we get

$$E\left(v(\eta)\int_0^T\int_{\mathbb{R}^d}Y(t,x)\eta(t,x)dxdt\right) = \int_{[0,T]\times\mathbb{R}^d}\eta(t,x)\langle\eta*\tilde{G},g_{tx}\rangle_{\mathcal{H}}dxdt =$$
$$\int_{[0,T]\times\mathbb{R}^d}\eta(t,x)\left(\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathcal{F}_{0,T}(\eta*\tilde{G})(\tau,z)\overline{\mathcal{F}_{0,T}g_{tx}(\tau,z)}\ |\tau|^{-(2H-1)}dzd\tau\right)dxdt =$$
$$\int_{\mathbb{R}}\int_{\mathbb{R}^d}|\mathcal{F}_{0,T}(\eta*\tilde{G})(\tau,z)|^2|\tau|^{-(2H-1)}dzd\tau = E|v(\eta)|^2.$$

■

3. The Fractional-Colored Noise

In this section we examine the process-solution and the distribution-solution of the stochastic heat equation driven by a Gaussian noise which is *fractional in time* and *colored in space*. Most of the results of this section are obtained by mixing some colored spatial techniques with the fractional temporal techniques of Section 2. The results of this section can therefore be viewed as generalizations of the results of Section 2. The details are highly non-trivial.

The structure of this section is similar to that of Section 2. We first describe the spaces of deterministic integrands, then we introduce the Gaussian noise and the associated stochastic integral, and finally we examine the solution of the stochastic heat equation driven by this type of noise.

3.1. **Spaces of Deterministic Integrands.** We begin by introducing the space of deterministic integrands on $\mathbb{R}^d$.

We say that a function $f:\mathbb{R}\to\mathbb{R}$ is a *spatial covariance function*, if it is the Fourier transform of a tempered measure μ on $\mathbb{R}^d$, i.e. $f(x)=\int_{\mathbb{R}^d}e^{-i\xi\cdot x}\mu(d\xi)$.

Let $\mathcal{P}(\mathbb{R}^d)$ be the completion of $\mathcal{D}(\mathbb{R}^d)$ with respect to the inner product

$$\langle\varphi,\psi\rangle_{\mathcal{P}(\mathbb{R}^d)} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(x)f(x-y)\psi(y)dydx = \int_{\mathbb{R}^d}\mathcal{F}_2\varphi(\xi)\overline{\mathcal{F}_2\psi(\xi)}\mu(d\xi),$$

where $\mathcal{F}_2\varphi(\xi):=\int_{\mathbb{R}^d}e^{-i\xi x}\varphi(x)dx$ denotes the Fourier transform with respect to the x-variable.

Equivalently, we can say that $\mathcal{P}(\mathbb{R}^d)$ is the completion of $\mathcal{E}(\mathbb{R}^d)$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{P}(\mathbb{R}^d)}$, where $\mathcal{E}(\mathbb{R}^d)$ is the space of all linear combinations of indicator functions $1_A(x), A\in\mathcal{B}_b(\mathbb{R}^d)$)

The basic example of spatial covariance function is $f(x)=\delta(x)$, which gives rise to the spatial white noise. More interesting covariance structures are provided by potential analysis. Here are some examples (see e.g. p.149-151, [6], or p.117-132, [15]; our constants are slightly different than those given in these references, since our definition of the Fourier transform does not have the 2π factor):

Example 3.1. The Riesz kernel of order α:

$$f(x) = R_\alpha(x) := \gamma_{\alpha,d}|x|^{-d+\alpha}, \quad 0 < \alpha < d,$$

where $\gamma_{\alpha,d} = 2^{d-\alpha}\pi^{d/2}\Gamma((d-\alpha)/2)/\Gamma(\alpha/2)$. In this case, $\mu(d\xi) = |\xi|^{-\alpha}d\xi$.

Example 3.2. The Bessel kernel of order α:

$$f(x) = B_\alpha(x) := \gamma'_\alpha \int_0^\infty w^{(\alpha-d)/2-1}e^{-w}e^{-|x|^2/(4w)}dw, \quad \alpha > 0,$$

where $\gamma'_\alpha = (4\pi)^{\alpha/2}\Gamma(\alpha/2)$. In this case, $\mu(d\xi) = (1+|\xi|^2)^{-\alpha/2}d\xi$ and $\mathcal{P}(\mathbb{R}^d)$ coincides with $\mathcal{H}^{-\alpha/2}(\mathbb{R}^d)$, the fractional Sobolev space of order $-\alpha/2$; see e.g. p.191, [6].

Example 3.3. The heat kernel

$$f(x) = G_\alpha(x) := \gamma''_{\alpha,d}e^{-|x|^2/(4\alpha)}, \quad \alpha > 0,$$

where $\gamma''_{\alpha,d} = (4\pi\alpha)^{-d/2}$. In this case, $\mu(d\xi) = e^{-\pi^2\alpha|\xi|^2}d\xi$.

Example 3.4. The Poisson kernel

$$f(x) = P_\alpha(x) := \gamma'''_{\alpha,d}(|x|^2 + \alpha^2)^{-(d+1)/2}, \quad \alpha > 0,$$

where $\gamma'''_{\alpha,d} = \pi^{-(d+1)/2}\Gamma((d+1)/2)\alpha$. In this case, $\mu(d\xi) = e^{-4\pi^2\alpha|\xi|}d\xi$.

Remark 3.5. The space $\mathcal{P}$ defined as the completion of $\mathcal{D}((0,T)\times\mathbb{R}^d)$ (or the completion of $\mathcal{E}$) with respect to the inner product

$$\langle\varphi,\psi\rangle_{\mathcal{P}} = \int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(t,x)f(x-y)\psi(t,y)dydxdt = \int_0^T\langle\varphi(t,\cdot),\psi(t,\cdot)\rangle_{\mathcal{P}(\mathbb{R}^d)}dt.$$

has been studied by several authors in connection with a Gausssian noise which is white in time and colored in space. One can prove that $\mathcal{P} \subset L_2((0,T);\mathcal{P}(\mathbb{R}^d))$; see e.g. [2], or [1].

In what follows, we need to extend the definition of $\mathcal{P}(\mathbb{R}^d)$ to allow for complex-valued functions. More precisely, let $\mathcal{D}_{\mathbb{C}}(\mathbb{R}^d)$ be the space of all infinitely differentiable functions $\varphi:\mathbb{R}^d\to\mathbb{C}$ with compact support, and $\mathcal{P}_{\mathbb{C}}(\mathbb{R}^d)$ be the completion of $\mathcal{D}_{\mathbb{C}}(\mathbb{R}^d)$ with respect to

$$\langle\varphi,\psi\rangle_{\mathcal{P}_{\mathbb{C}}(\mathbb{R}^d)} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(x)f(x-y)\psi(y)dydx.$$

Since $\mathcal{D}(\mathbb{R}^d)\subset\mathcal{D}_{\mathbb{C}}(\mathbb{R}^d)$ and $\langle\varphi,\psi\rangle_{\mathcal{P}(\mathbb{R}^d)} = \langle\varphi,\psi\rangle_{\mathcal{P}_{\mathbb{C}}(\mathbb{R}^d)}$ for every $\varphi,\psi\in\mathcal{D}(\mathbb{R}^d)$, we conclude that

$$\mathcal{P}(\mathbb{R}^d)\subset\mathcal{P}_{\mathbb{C}}(\mathbb{R}^d).$$

We are now introducing the space of deterministic integrands associated with a Gaussian noise which is fractional in time and colored in space. This space seems to be new in the literature. More precisely, let $\mathcal{HP}$ be the completion of $\mathcal{D}((0,T)\times\mathbb{R}^d)$ with respect to the inner product

$$\begin{aligned}\langle\varphi,\psi\rangle_{\mathcal{HP}} &= \alpha_H\int_0^T\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(u,x)|u-v|^{2H-2}f(x-y)\psi(u,y)dy\,dx\,dv\,du\\ &= \alpha_H c_H\int_{\mathbb{R}}|\tau|^{-(2H-1)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}f(x-y)\mathcal{F}_{0,T}\varphi(\tau,x)\overline{\mathcal{F}_{0,T}\psi(\tau,y)}dy\,dx\,d\tau\\ &= \alpha_H c_H\int_{\mathbb{R}}|\tau|^{-(2H-1)}\langle\mathcal{F}_{0,T}\varphi(\tau,\cdot),\overline{\mathcal{F}_{0,T}\psi(\tau,\cdot)}\rangle_{\mathcal{P}_{\mathbb{C}}(\mathbb{R}^d)}d\tau,\end{aligned}$$

where the second equality follows by Lemma A.1 (Appendix A). In particular,

$$\|\varphi\|^2_{\mathcal{HP}} = \alpha_H c_H \int_{\mathbb{R}} \|\mathcal{F}_{0,T}\varphi(\tau,\cdot)\|^2_{\mathcal{P}_{\mathbb{C}}(\mathbb{R}^d)} |\tau|^{-(2H-1)} d\tau,$$

or equivalently $\|\varphi\|^2_{\mathcal{HP}} = \int_{\mathbb{R}} \|\mathcal{F}_{0,T}\varphi(\tau,\cdot)\|^2_{\mathcal{P}_{\mathbb{C}}(\mathbb{R}^d)} \lambda_H(d\tau)$, where $\lambda_H(d\tau) = \alpha_H c_H |\tau|^{-(2H-1)} d\tau$.

One can prove that $\mathcal{HP}$ is also the completion of $\mathcal{E}$ with respect to the inner product

$$\langle 1_{[0,t]\times A}, 1_{[0,s]\times B}\rangle_{\mathcal{HP}} = R_H(t,s)\langle 1_A, 1_B\rangle_{\mathcal{P}(\mathbb{R}^d)}.$$

Clearly $|\mathcal{HP}| \subset \mathcal{HP}$, where $|\mathcal{HP}| = \{\varphi : [0,T]\times\mathbb{R}^d \text{ measurable};\ \|\varphi\|_{|\mathcal{HP}|} < \infty\}$ and

$$\|\varphi\|^2_{|\mathcal{HP}|} := \int_0^T \int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\varphi(u,x)||\varphi(v,y)||u-v|^{2H-2} f(x-y)\,dy\,dx\,dv\,du.$$

Remark 3.6. Using Fubini's theorem, we have the following alternative expression for calculating $\langle\varphi,\psi\rangle_{\mathcal{HP}}$: for every $\varphi,\psi \in \mathcal{D}((0,T)\times\mathbb{R}^d)$, we have

$$\begin{aligned}\langle\varphi,\psi\rangle_{\mathcal{HP}} &= \alpha_H \int_0^T\int_0^T |u-v|^{2H-2} \int_{\mathbb{R}^d} \mathcal{F}_2\varphi(u,\xi)\overline{\mathcal{F}_2\psi(u,\xi)}\mu(d\xi)dvdu \\ &= \alpha_H \int_{\mathbb{R}^d}\int_0^T\int_0^T \mathcal{F}_2\varphi(u,\xi)|u-v|^{2H-2}\overline{\mathcal{F}_2\psi(v,\xi)}dvdu\mu(d\xi) \\ &= \int_{\mathbb{R}^d} \langle \mathcal{F}_2\varphi(\cdot,\xi), \overline{\mathcal{F}_2\psi(\cdot,\xi)}\rangle_{\mathcal{H}_{\mathbb{C}}(0,T)}\mu(d\xi),\end{aligned}$$

where $\mathcal{H}_{\mathbb{C}}(0,T)$ denotes the completion of $\mathcal{D}_{\mathbb{C}}(0,T)$ with respect to the inner-product $\langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{C}}(0,T)}$ defined similarly to $\langle\cdot,\cdot\rangle_{\mathcal{H}(0,T)}$. In particular, $\|\varphi\|^2_{\mathcal{HP}} = \int_{\mathbb{R}^d} \|\mathcal{F}_2\varphi(\cdot,\xi)\|^2_{\mathcal{H}_{\mathbb{C}}(0,T)}\mu(d\xi)$. This expression will not be used in the present paper.

In this new context, the next theorem gives us a useful criterion for verifying that a function φ lies in $\mathcal{HP}$. To prove this theorem, we need the following lemma, generalizing Lemma 5.1, [11].

Lemma 3.7. *For every $A \in \mathcal{B}_b(\mathbb{R}^d)$, there exists a sequence $(g_n)_n \subset \mathcal{E}(\mathbb{R}^d)$ such that*

$$\int_{\mathbb{R}^d} |1_A(\xi) - \mathcal{F}_2 g_n(\xi)|^2 \mu(d\xi) \to 0.$$

Proof: Let $(\phi_n)_n \subset \mathcal{D}(\mathbb{R}^d)$ be such that $\phi_n(\xi) \to 1_A(\xi)$ uniformly and supp $\phi_n \subset K$, $\forall n$, where $K \subset \mathbb{R}^d$ is a compact (we may take $\phi_n = 1_A * \eta_n$, where $\eta_n(x) = n^d\eta(nx)$ and $\eta \in \mathcal{D}(\mathbb{R}^d)$, with $\int_{\mathbb{R}^d}|\eta(x)|dx = 1$). Using the dominated convergence theorem and the fact that μ is locally finite, we get $\int_{\mathbb{R}^d}|\phi_n(\xi) - 1_A(\xi)|^2\mu(d\xi) \to 0$.

Let $\psi_n \in \mathcal{S}(\mathbb{R}^d)$ be such that $\mathcal{F}_2\psi_n = \phi_n$. Then

$$\int_{\mathbb{R}^d} |\mathcal{F}_2\psi_n(\xi) - 1_A(\xi)|^2\mu(d\xi) \to 0. \tag{25}$$

Note that $\int_{\mathbb{R}^d}|\mathcal{F}_2\psi_n(\xi)|^2\mu(d\xi) = \int_{\mathbb{R}^d}|\phi_n(\xi)|^2\mu(d\xi) < \infty$. By Lemma C.1 (Appendix C), it follows that $\psi_n \in \mathcal{P}(\mathbb{R}^d)$. Since $\mathcal{P}(\mathbb{R}^d)$ is the completion of $\mathcal{E}(\mathbb{R}^d)$ with respect to $\|\cdot\|_{\mathcal{P}(\mathbb{R}^d)}$, there exists a sequence $(g_n)_n \subset \mathcal{E}(\mathbb{R}^d)$ such that

$$\|\psi_n - g_n\|^2_{\mathcal{P}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\mathcal{F}_2\psi_n(\xi) - \mathcal{F}_2 g_n(\xi)|^2\mu(d\xi) \to 0. \tag{26}$$

The conclusion follows from (25) and (26). ∎

Theorem 3.8. *Let $\varphi : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be a function which satisfies the following conditions:*
(i) $\varphi(\cdot, x) \in L_2(0,T)$ for every $x \in \mathbb{R}^d$;
(ii) $(\tau, x) \mapsto \mathcal{F}_{0,T}\varphi(\tau, \cdot)$ is measurable;
(iii) $\int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{F}_{0,T}\varphi(\tau,x) f(x-y) \overline{\mathcal{F}_{0,T}\varphi(\tau,y)} |\tau|^{-(2H-1)} dydxd\tau < \infty$.
Then $\varphi \in \mathcal{HP}$.

Proof: The proof follows the same lines as the proof of Theorem 2.2. The details are quite different though. Let

$$\begin{aligned}\tilde{\Lambda} &= \{\varphi : [0,T] \times \mathbb{R}^d \to \mathbb{R};\ \varphi(\cdot,x) \in L_2(0,T)\ \forall x, (\tau,x) \mapsto \mathcal{F}_{0,T}\varphi(\tau,x) \text{ is measurable, and} \\ &\quad \|\varphi\|^2_{\tilde{\Lambda}} := c_1 \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{F}_{0,T}\varphi(\tau,x) f(x-y) \overline{\mathcal{F}_{0,T}\varphi(\tau,y)} |\tau|^{-(2H-1)} dydxd\tau < \infty\} \\ \Lambda &= \{\varphi : [0,T] \times \mathbb{R}^d \to \mathbb{R};\ \|\varphi\|^2_{\Lambda} := c_2 \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} I^{H-1/2}_{T-}(u^{H-1/2}\varphi(u,x))(s) \cdot f(x-y) \\ &\quad \times I^{H-1/2}_{T-}(u^{H-1/2}\varphi(u,y))(s) \cdot s^{-(2H-1)} dydxds < \infty\}\end{aligned}$$

where $c_1 = \alpha_H c_H$ and $c_2 = \{c^*_H \Gamma(H-1/2)\}^2$. The fact that

$$\tilde{\Lambda} \subset \Lambda \quad \text{and} \quad \|\varphi\|_{\tilde{\Lambda}} = \|\varphi\|_{\Lambda}, \quad \forall \varphi \in \tilde{\Lambda}. \tag{27}$$

follows as in the proof of Theorem 2.2: let $\varphi \in \tilde{\Lambda}$ is arbitrary; using the fact K^*_H is an isometry from $\mathcal{H}(0,T)$ to $L_2(0,T)$, we get $\langle \varphi(\cdot,x), \varphi(\cdot,y)\rangle^2_{\mathcal{H}(0,T)} = \langle K^*_H\varphi(\cdot,x), K^*_H\varphi(\cdot,y)\rangle^2_{L_2(0,T)}$ for all $x, y \in \mathbb{R}^d$. Multiplying by $f(x-y)$, integrating with respect to $dx\,dy$ and using Fubini's theorem, we get $\|\varphi\|_{\tilde{\Lambda}} = \|\varphi\|_{\Lambda} < \infty$.

Next we prove that

$$\mathcal{E} \text{ is dense in } \Lambda \text{ with respect to } \|\cdot\|_{\Lambda}. \tag{28}$$

The proof of the theorem will follow from (27) and (28), as in the case of Theorem 2.2.

To prove (28), let $\varphi \in \Lambda$ and $\varepsilon > 0$ be arbitrary. Let $\lambda_H(s) = s^{-(2H-1)}ds$ and $a(s,x) = I^{H-1/2}_{T-}(u^{H-1/2}\varphi(u,x))(s)$. First, we claim that there exists $g \in \mathcal{E}$ such that

$$I_1 := \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} [a(s,x) - g(s,x)] f(x-y) [a(s,y) - g(s,y)] dydx\lambda_H(ds) < \varepsilon. \tag{29}$$

To see this, note that

$$\|\varphi\|^2_{\Lambda} = c_2 \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(s,x) f(x-y) a(s,y) dydx\lambda_H(ds) = c_2 \int_0^T \int_{\mathbb{R}^d} |\mathcal{F}_2 a(s,\xi)| \mu(d\xi) \lambda_H(ds) < \infty,$$

i.e. the map $(s,\xi) \mapsto \mathcal{F}_2 a(s,\xi)$ belongs to $L_2((0,T) \times \mathbb{R}^d, d\lambda_H \times d\mu)$. Hence, there exists a simple function $h(s,\xi) = \sum_{j=1}^m \beta_j 1_{[\gamma_j,\delta_j)}(s) 1_{B_j}(\xi)$ on $(0,T) \times \mathbb{R}^d$, with $\beta_j \in \mathbb{R}$, $0 < \gamma_j < \delta_j < T$ and $B_j \subset \mathbb{R}^d$ Borel sets, such that

$$\int_0^T \int_{\mathbb{R}^d} |\mathcal{F}_2 a(s,\xi) - h(s,\xi)|^2 \mu(d\xi) \lambda_H(ds) < \varepsilon/4. \tag{30}$$

By Lemma 3.7, for every $j = 1, \ldots, m$, there exists $g_j \in \mathcal{E}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} |1_{B_j}(\xi) - \mathcal{F}_2 g_j(\xi)| \mu(d\xi) < \varepsilon/(4D_h), \tag{31}$$

where we chose $D_h = m \sum_{j=1}^m \beta_j^2 \lambda_H([\gamma_j,\delta_j))$. Define $g(s,x) = \sum_{j=1}^m \beta_j 1_{[\gamma_j,\delta_j)}(s) g_j(x)$. Clearly $g \in \mathcal{E}$. Using (30) and (31), we get $I_1 = \int_0^T \int_{\mathbb{R}^d} |\mathcal{F}_2 a(s,\xi) - \mathcal{F}_2 g(s,\xi)|^2 \mu(d\xi) \lambda_H(ds) < \varepsilon$, which concludes the proof of (29).

We claim now that there exists a function $l \in \mathcal{E}$ such that

$$I_2 := \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} [g(s,x) - b(s,x)] f(x-y) [g(s,y) - b(s,y)] dy dx \lambda_H(ds) < \varepsilon, \tag{32}$$

where $b(s,x) = I_{T-}^{H-1/2}(u^{H-1/2} l(u,x))(s)$. To see this, suppose that $g = \sum_{k=1}^n b_k 1_{[c_k,d_k)}(s) 1_{A_k}(x)$ for some $0 < c_k < d_k < T$ and $A_k \subset \mathbb{R}^d$ Borel sets. By relation (8.1) of [12], there exists an elementary function $l_k \in \mathcal{E}(0,T)$ such that

$$\int_0^T [1_{[c_k,d_k)}(s) - I_{T-}^{H-1/2}(u^{H-1/2} l_k(u))(s)]^2 \lambda_H(ds) < \varepsilon / C_g, \tag{33}$$

where we chose $C_g := \| \sum_{k=1}^n b_k 1_{A_k} \|^2_{\mathcal{P}(\mathbb{R}^d)}$. Let $l(s,x) = \sum_{k=1}^n b_k l_k(t) 1_{A_k}(x) \in \mathcal{E}$ and note that

$$\begin{aligned} I_2 &= \sum_{k,j=1}^n b_k b_j \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} [1_{[c_k,d_k)}(s) - I_{T-}^{H-1/2}(u^{H-1/2} l_k(u))(s)] 1_{A_k}(x) f(x-y) \\ &\quad [1_{[c_j,d_j)}(s) - I_{T-}^{H-1/2}(u^{H-1/2} l_j(u))(s)] 1_{A_j}(y) dy dx \lambda_H(ds) \\ &\leq \sum_{k,j=1}^n b_k b_j \langle 1_{A_k}, 1_{A_j} \rangle_{\mathcal{P}(\mathbb{R}^d)} \cdot 2 \left(\frac{\varepsilon}{C_g} + \frac{\varepsilon}{C_g} \right) = \varepsilon \end{aligned}$$

(we used (33) and the fact that $ab \leq 2(a^2 + b^2)$). The proof of (32) is complete.

Finally, we claim that from (29) and (32), we get

$$I := \int_0^T \int_{\mathbb{R}^d} [a(s,x) - b(s,x)] f(x-y) [a(s,y) - b(s,y)] dy dx \lambda_H(ds) < 4\varepsilon,$$

i.e. $\|\varphi - l\|^2_\Lambda < 4\varepsilon c_2$, which will conclude the proof of (28). To see this, note that $I = \sum_{k=1}^4 I_k$, where

$$\begin{aligned} I_3 &:= -\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} [a(s,x) - g(s,x)] f(x-y) [b(s,y) - g(s,y)] dy dx \lambda_H(ds) \\ I_4 &:= -\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} [a(s,y) - g(s,y)] f(x-y) [b(s,x) - g(s,x)] dy dx \lambda_H(ds). \end{aligned}$$

The fact that $|I_3| < \varepsilon$ and $|I_4| < \varepsilon$ follows from the Cauchy-Schwartz inequality in $\mathcal{P}(\mathbb{R}^d)$, (29) and (32). ■

It is again possible to describe the space $\mathcal{HP}$ using the transfer operator. Define the transfer operator on $\mathcal{E}$ by the same formula (8). Note that in this case we have

$$\begin{aligned} \langle K_H^* 1_{[0,t]\times A}, K_H^* 1_{[0,s]\times B} \rangle_{\mathcal{P}} &= \left(\int_0^{t\wedge s} K_H(t,u) K_H(s,u) du \right) \langle 1_A, 1_B \rangle_{\mathcal{P}(\mathbb{R}^d)} \\ &= R_H(t,s) \langle 1_A, 1_B \rangle_{\mathcal{P}(\mathbb{R}^d)} = \langle 1_{[0,t]\times A}, 1_{[0,s]\times B} \rangle_{\mathcal{HP}}, \end{aligned}$$

i.e. K_H^* is an isometry between $(\mathcal{E}, \langle \cdot, \cdot \rangle_{\mathcal{HP}})$ and $\mathcal{P}$. Since $\mathcal{HP}$ is the completion of $\mathcal{E}$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{HP}}$, this isometry can be extended to $\mathcal{HP}$. We denote this extension by $K^*_{\mathcal{HP}}$.

Lemma 3.9. *$K^*_{\mathcal{HP}} : \mathcal{HP} \to \mathcal{P}$ is surjective.*

Proof: The proof is similar to the proof of Lemma 2.3, using the fact that $1_{[0,t]\times A} \in K^*_{\mathcal{HP}}(\mathcal{HP})$ for all $t \in [0,T], A \in \mathcal{B}_b(\mathbb{R}^d)$, and $\mathcal{E}$ is dense in $\mathcal{P}$ with respect to $\| \cdot \|_{\mathcal{P}}$. ■

Remark 3.10. Using (9) and Lemma 3.9, we can *formally* say that

$$\mathcal{HP} = \{\varphi \text{ such that } (s,x) \mapsto s^{-(H-1/2)} I_{T-}^{H-1/2}(u^{H-1/2}\varphi(u,x))(s) \text{ is in } \mathcal{P}\}.$$

3.2. **The Noise and the Stochastic Integral.** In this subsection, we introduce the noise which is randomly perturbing the heat equation. This noise is assumed to be fractional in time and colored in space, with an arbitrary spatial covariance function f. It has been recently considered by other authors (see e.g. [14]) in the case when f is the Riesz kernel and the spatial dimension is $d=1$. The general definition that we consider in this subsection seems to be new in the literature.

Let $B = \{B(\varphi); \varphi \in \mathcal{D}((0,T)\times\mathbb{R}^d)\}$ be a zero-mean Gaussian process with covariance

$$E(B(\varphi)B(\psi)) = \langle \varphi, \psi \rangle_{\mathcal{HP}}.$$

Let H^B be the Gaussian space of B, i.e. the closed linear span of $\{B(\varphi); \varphi \in \mathcal{D}((0,T)\times\mathbb{R}^d)\}$ in $L^2(\Omega)$. As in subsection 2.2, we can define $B_t(A) = B(1_{[0,t]\times A})$ as the $L_2(\Omega)$-limit of the Cauchy sequence $\{B(\varphi)\}_n$, where $(\varphi_n)_n \subset \mathcal{D}((0,T)\times\mathbb{R}^d)$ converges pointwise to $1_{[0,t]\times A}$. We extend this definition by linearity to all elements in $\mathcal{E}$. A limiting argument shows that

$$E(B(\varphi)B(\psi)) = \langle \varphi, \psi \rangle_{\mathcal{HP}}, \quad \forall \varphi, \psi \in \mathcal{E},$$

i.e. $\varphi \mapsto B(\varphi)$ is an is isometry between $(\mathcal{E}, \langle\cdot,\cdot\rangle_{\mathcal{HP}})$ and H^B. Since $\mathcal{HP}$ is the completion of $\mathcal{E}$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, this isometry can be extended to $\mathcal{HP}$, giving us the stochastic integral with respect to B. We will use the notation

$$B(\varphi) = \int_0^T \int_{\mathbb{R}^d} \varphi(t,x) B(dt,dx).$$

Remark 3.11. Similarly to subsection 2.2, the transfer operator $K^*_{\mathcal{HP}}$ can be used to explore the relationship between $B(\varphi)$ and another stochastic integral. Using Lemma 3.9, we define

$$M(\phi) := B((K^*_{\mathcal{HP}})^{-1}(\phi)), \quad \phi \in \mathcal{P}. \tag{34}$$

Note that

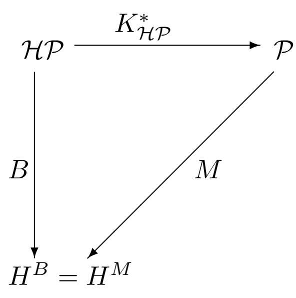
$$E(M(\phi)M(\eta)) = \langle (K^*_{\mathcal{HP}})^{-1}(\phi), (K^*_{\mathcal{HP}})^{-1}(\eta)\rangle_{\mathcal{HP}} = \langle \phi, \eta \rangle_{\mathcal{P}},$$

i.e. $M = \{M(\phi); \phi \in \mathcal{P}\}$ is a Gaussian noise which is white in time and has spatial covariance function f. This noise has been considered by Dalang in [2]. We use the following notation:

$$M(\phi) = \int_0^T \int_{\mathbb{R}^d} \phi(t,x) M(dt,dx), \quad \phi \in \mathcal{P}.$$

Note that $M(\phi)$ is in fact Dalang's stochastic integral with respect to the noise M.

Let H^M be the Gaussian space of M, i.e. the closed linear span of $\{M(\phi); \phi \in \mathcal{P}\}$ in $L_2(\Omega)$. By (34), it follows that $H^M = H^B$. The following diagram summarizes these facts:



$$B(\varphi) = M(K^*_{\mathcal{HP}}\varphi), \quad \forall \varphi \in \mathcal{HP}, \text{ i.e.}$$

$$\int_0^T \int_{\mathbb{R}^d} \varphi(t,x) B(dt,dx) = \int_0^T \int_{\mathbb{R}^d} (K^*_{\mathcal{HP}}\varphi)(t,x) M(dt,dx), \quad \forall \varphi \in \mathcal{HP}.$$

In particular, $B(t,A) = \int_0^t \int_A K_H(t,s)M(ds,dy)$. This relationship will not be used in the present article.

3.3. **The solution of the Stochastic Heat Equation.** We consider the stochastic heat equation driven by the noise B, written *formally* as:

$$u_t - \Delta u = \dot{B}, \quad \text{in } (0,T)\times\mathbb{R}^d, \quad u(0,\cdot)=0. \tag{35}$$

As in subsection 2.3, we let $G(t,x)$ be given by formula (13), and $g_{tx}(s,y) = G(t-s,x-y)$. The next theorem is the fundamental result leading to the necessary and sufficient condition for the existence of a process-solution and a distribution-solution of (35).

To state the theorem we need to introduce the following notations:

$$\begin{aligned} I_{f,tx}(r,s) &:= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_{tx}(s,y)f(y-z)g_{tx}(r,z)dy\,dz \\ J_f(u,v,y,z) &:= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(u,y-x)f(x-x')G(v,z-x')dx\,dx'. \end{aligned}$$

Theorem 3.12. *Suppose that the spatial covariance function f satisfies:*

$$\begin{aligned} A_f(2t-s-r)^{-(d-\alpha_f)/2} \le I_{f,tx}(r,s) &\le B_f(2t-r-s)^{-(d-\alpha_f)/2}, \qquad (36)\\ &\quad \forall r\in[0,t], \forall s\in[0,t], \forall t\in[0,T], \forall x\in\mathbb{R}^d \\ J_f(u,v,y,z) &\le C_f(u+v)^{-(d-\alpha_f)/2}, \qquad (37)\\ &\quad \forall u\in[0,T], \forall v\in[0,T], \forall y\in\mathbb{R}^d, \forall z\in\mathbb{R}^d \end{aligned}$$

for some constants $A_f, B_f, C_f > 0$ and $\alpha_f < d$. If

$$H > \frac{d-\alpha_f}{4}\ , \tag{38}$$

then: (a) $g_{tx} \in |\mathcal{HP}|$ for every $(t,x)\in[0,T]\times\mathbb{R}^d$; (b) $\eta * \tilde{G} \in \mathcal{HP}$ for every $\eta \in \mathcal{D}((0,T)\times\mathbb{R}^d)$. Moreover,

$$\|g_{tx}\|_{\mathcal{HP}} < \infty \ \forall (t,x)\in[0,T]\times\mathbb{R}^d \quad \textit{if and only if} \quad (38) \textit{ holds.} \tag{39}$$

Proof: (a) We will apply Theorem 3.8 to the function g_{tx}. As we noted in the proof of Theorem 2.7, $g_{tx}(\cdot,y)\in L_2(0,T)$ for every $y\in\mathbb{R}^d$, and the map $(\tau,y)\mapsto \mathcal{F}_{0,T}g_{tx}(\tau,y)$ is measurable. We calculate

$$\|g_{tx}\|^2_{\mathcal{HP}} := \alpha_H c_H \int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathcal{F}_{0,T}g_{tx}(\tau,x)f(x-y)\overline{\mathcal{F}_{0,T}g_{tx}(\tau,y)}|\tau|^{-(2H-1)}dydxd\tau.$$

For this, we write

$$\begin{aligned} \|g_{tx}\|^2_{\mathcal{HP}} &= \alpha_H c_H \int_{\mathbb{R}} |\tau|^{-(2H-1)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(\int_0^T e^{-i\tau s}g_{tx}(s,y)ds\right) f(y-z) \left(\int_0^T e^{i\tau r}g_{tx}(r,y)dr\right) dy\,dzd\tau \\ &= \alpha_H c_H \int_{\mathbb{R}} |\tau|^{-(2H-1)} \int_0^t\int_0^t e^{-i\tau(s-r)} I_{f,tx}(r,s)dr\,ds\,d\tau \\ &= \alpha_H \int_0^t\int_0^t |s-r|^{2H-2} I_{f,tx}(r,s)dr\,ds, \end{aligned} \tag{40}$$

where we used Lemma A.1 (Appendix) for the last equality and the definition of $I_{f,tx}(r,s)$. Using (36), we see that

$$\|g_{tx}\|_{\mathcal{HP}}^2 \leq \alpha_H B_f \int_0^t \int_0^t |s-r|^{2H-2}(2t-r-s)^{-(d-\alpha_f)/2} dr\, ds$$
$$\|g_{tx}\|_{\mathcal{HP}}^2 \geq \alpha_H A_f \int_0^t \int_0^t |s-r|^{2H-2}(2t-r-s)^{-(d-\alpha_f)/2} dr\, ds.$$

Relation (39) follows, since the integral above is finite if and only if $2H > (d-\alpha_f)/2$. In this case, we have $\|g_{tx}\|_{|\mathcal{HP}|} = \|g_{tx}\|_{\mathcal{HP}} < \infty$ (since $g_{tx} \geq 0$), and hence $g_{tx} \in |\mathcal{HP}|$.

(b) We will apply Theorem 3.8 to the function $\varphi = \eta * \tilde{G}$. By Lemma 2.6, $\varphi(\cdot, x) \in L_2(0,T)$ for every $x \in \mathbb{R}^d$. We now calculate

$$\|\varphi\|_{\mathcal{HP}}^2 := \alpha_H c_H \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{F}_{0,T}\varphi(\tau,x) f(x-y) \overline{\mathcal{F}_{0,T}\varphi(\tau,x')} |\tau|^{-(2H-1)} dx' dx d\tau.$$

By (16), we get

$$\begin{aligned} \|\varphi\|_{\mathcal{HP}}^2 &= \alpha_H c_H \int_{\mathbb{R}} |\tau|^{-(2H-1)} \int_0^T \int_0^T e^{-i\tau(s-r)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^{T-s} \int_0^{T-r} \eta(u+s,y)\eta(v+r,z) \\ &\quad J_f(u,v,y,z) dv\, du\, dz\, dy\, dr\, ds\, d\tau \\ &= \alpha_H \int_0^T \int_0^T |s-r|^{2H-2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^{T-s} \int_0^{T-r} \eta(u+s,y)\eta(v+r,z) \\ &\quad J_f(u,v,y,z) dv\, du\, dz\, dy\, dr\, ds, \end{aligned}$$

where we used Lemma A.1 (Appendix A) for the last equality and the definition of $J_f(u,v,y,z)$.

Using (37) and the fact that $\eta \in \mathcal{D}((0,T) \times \mathbb{R}^d)$ (and thus is bounded by a constant and its support is compact), we conclude that

$$\begin{aligned} \|\varphi\|_{\mathcal{HP}}^2 &\leq \alpha_H C_f \int_0^T \int_0^T |s-r|^{2H-2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^{T-s} \int_0^{T-r} |\eta(u+s,y)\eta(v+r,z)| \\ &\quad (u+v)^{-(d-\alpha_f)/2} dy\, dz\, dv\, du\, dr\, ds \\ &\leq \alpha_H C_f D_\eta \int_0^T \int_0^T |s-r|^{2H-2} \int_0^{T-s} \int_0^{T-r} (u+v)^{-(d-\alpha_f)/2} dv\, du\, dr\, ds. \end{aligned}$$

As in the proof of Theorem 2.7.(b), the last integral is finite if $2H > (d-\alpha_f)/2$.

■

The next theorem identifies identifies the constant α_f in the case of some particular covariance functions.

Theorem 3.13. *(i) If $f = R_\alpha$ with $0 < \alpha < d$, then (36) and (37) hold with $\alpha_f = \alpha$.*
(ii) If $f = B_\alpha$ with $\alpha > 0$, then (36) and (37) hold with $\alpha_f = 0$.
(iii) If $f = G_\alpha$ with $\alpha > 0$, then (36) and (37) hold with $\alpha_f = 0$.
(iv) If $f = P_\alpha$ with $\alpha > 0$, then (36) and (37) hold with $\alpha_f = -1$

Remark 3.14. a) If $f = R_\alpha$, condition (38) becomes $H > \max\{(d-\alpha)/4, 1/2\}$, which does not impose any restrictions on d. For any $d \geq 1$ arbitrary, it suffices to choose α such that $\max\{d-4H, 0\} < \alpha < d$.
b) If $f = B_\alpha$ or $f = G_\alpha$, condition (38) becomes $H > \max\{d/4, 1/2\}$, which forces $d < 4$.
c) If $f = P_\alpha$, condition (38) becomes $H > \max\{(d+1)/4, 1/2\}$, which forces $d < 3$.

Proof: We begin by examining condition (36). To simplify the notation, we will omit the index tx in the writing of $I_{f,tx}$. Using the definitions of I_f and G, we obtain

$$\begin{aligned} I_f(r,s) &= \frac{1}{(4\pi)^d[(t-s)(t-r)]^{d/2}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y-z) e^{-\frac{|x-y|^2}{4(t-s)} - \frac{|x-z|^2}{4(t-r)}} dydz \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\sqrt{2(t-s)}y' - \sqrt{2(t-r)}z') e^{-\frac{|y'|^2}{2} - \frac{|z'|^2}{2}} dy'dz' \\ &= E[f(\sqrt{2(t-s)}Y - \sqrt{2(t-r)}Z)] = E[f(U)]. \end{aligned} \tag{41}$$

Here we used the change of variables $x-y = \sqrt{2(t-s)}y', x-z = \sqrt{2(t-r)}z'$ and we denoted by $(Y,Z) = (Y_1,\dots Y_d, Z_1,\dots,Z_d)$ a random vector with independent $N(0,1)$ components, and

$$U = \sqrt{2(t-s)}Y - \sqrt{2(t-r)}Z.$$

Note that $U_i = \sqrt{2(t-s)}Y_i - \sqrt{2(t-r)}Z_i, i = 1,\dots,d$ are i.i.d. $N(0, 2(2t-s-r))$ random variables. Then $V_i = U_i/\sqrt{2(2t-s-r)}, i = 1,\dots,d$ are i.i.d. $N(0,1)$ random variables and

$$|U|^2 = \sum_{i=1}^d U_i^2 = 2(2t-s-r)\sum_{i=1}^d V_i^2 = 2(2t-s-r)W_d, \tag{42}$$

where W_d is a χ^2_d random variable.

We are now treating separately the four cases:

(i) In the case of the Riesz kernel, $f(x) = R_\alpha(x) = \gamma_{\alpha,d}|x|^{-(d-\alpha)}$ and $0<\alpha<d$. Using (41) and (42), the integral $I_f(r,s)$ becomes

$$\begin{aligned} I_{R_\alpha}(r,s) &= \gamma_{\alpha,d} E|U|^{-(d-\alpha)} = \gamma_{\alpha,d}[2(2t-s-r)]^{-(d-\alpha)/2} E|W_d|^{-(d-\alpha/2)} \\ &:= C_{\alpha,d}(2t-s-r)^{-(d-\alpha)/2}, \end{aligned} \tag{43}$$

where $C_{\alpha,d} = \gamma_{\alpha,d}2^{-(d-\alpha)/2}E|W_d|^{-(d-\alpha)/2}$. This proves that condition (36) is satisfied with $\alpha_f = \alpha$.

(ii) In the case of the Bessel kernel, $f(x) = B_\alpha(x) = \gamma'_\alpha \int_0^\infty w^{(\alpha-d)/2-1}e^{-w}e^{-|x|^2/4w}dw$ and $\alpha>0$. Using (41) and (42), the integral $I_f(r,s)$ becomes

$$\begin{aligned} I_{B_\alpha}(r,s) &= \gamma'_\alpha \int_0^\infty w^{-(\alpha-d)/2-1}e^{-w}E[e^{-|U|^2/(4w)}]dw \\ &= \gamma'_\alpha \int_0^\infty w^{(\alpha-d)/2-1}e^{-w}E\left[\exp\left(-\frac{2t-r-s}{2w}W_d\right)\right]dw \\ &= \gamma'_\alpha \int_0^\infty w^{(\alpha-d)/2-1}e^{-w}\left(1+\frac{2t-r-s}{w}\right)^{-d/2}dw \end{aligned}$$

where we used the fact that $E[e^{-cW_d}] = (1+2c)^{-d/2}$ for any $c>0$. Note that

$$\left(\frac{2t-r-s}{w}\right)^{d/2} \le \left(1+\frac{2t-r-s}{w}\right)^{d/2} \le C_d\left[1+\left(\frac{2t-r-s}{w}\right)^{d/2}\right]$$

where $C_d = 2^{d/2-1}$. Hence

$$\frac{1}{2C_d}(2t-r-s)^{-d/2} \le \frac{1}{C_d}\frac{w^{d/2}}{w^{d/2}+(2t-s-r)^{d/2}} \le \left(1+\frac{2t-r-s}{w}\right)^{-d/2} \le w^{d/2}(2t-s-r)^{-d/2} \tag{44}$$

where for the first inequality we used the fact that $a/(a+x) > 1/(2x)$ if x is small enough and $a>0$.

We conclude that

$$\begin{aligned} I_{B_\alpha}(r,s) &\leq \gamma'_\alpha \int_0^\infty w^{(\alpha-d)/2-1} e^{-w} w^{d/2} (2t-r-s)^{-d/2} dw \leq \gamma'_\alpha \Gamma\left(\frac{\alpha}{2}\right) (2t-r-s)^{-d/2} \\ I_{B_\alpha}(r,s) &\geq \frac{\gamma'_\alpha}{2C_d} \int_0^1 w^{(\alpha-d)/2-1} e^{-w} (2t-r-s)^{-d/2} dw \geq \frac{\gamma'_\alpha}{2C_d} \left(\int_0^1 w^{\alpha/2-1} e^{-w} dw \right) (2t-r-s)^{-d/2}, \end{aligned}$$

i.e. condition (36) is satisfied with $\alpha_f = 0$.

(iii) In the case of the heat kernel, $f(x) = G_\alpha(x) = \gamma''_{\alpha,d} e^{-|x|^2/(4\alpha)}$ and $\alpha > 0$. Using (41) and (42), the integral $I_f(r,s)$ becomes:

$$I_{G_\alpha}(r,s) = \gamma''_{\alpha,d} E[e^{-|U|^2/(4\alpha)}] = \gamma''_{\alpha,d} E\left[\exp\left(-\frac{2t-s-r}{2\alpha} W_d\right)\right] = \gamma''_{\alpha,d}\left(1 + \frac{2t-r-s}{\alpha}\right)^{-d/2}.$$

Using (44), we obtain that

$$\frac{\gamma''_{\alpha,d}}{2C_d}(2t-r-s)^{-d/2} \leq I_{G_\alpha}(r,s) \leq \gamma''_{\alpha,d}\alpha^{d/2}(2t-r-s)^{-d/2},$$

i.e. condition (36) is satisfied with $\alpha_f = 0$.

(iv) In the case of the Poisson kernel, $f(x) = P_\alpha(x) = \gamma'''_{\alpha,d}(|x|^2+\alpha^2)^{-(d+1)/2}$ and $\alpha > 0$. Using (41) and (42), the integral $I_f(r,s)$ becomes:

$$I_{P_\alpha}(r,s) = \gamma'''_{\alpha,d} E\left[(|U|^2+\alpha^2)^{-(d+1)/2}\right] = \gamma'''_{\alpha,d} E\left|2(2t-r-s)W_d + \alpha^2\right|^{-(d+1)/2}.$$

Using the fact that

$$A_d[(2t-r-s)W_d]^{-(d+1)/2} \leq [2(2t-r-s)W_d+\alpha^2]^{-(d+1)/2} \leq B_d[(2t-r-s)W_d]^{-(d+1)/2}$$

for some constants $A_d, B_d > 0$, we conclude that

$$A_d E|W_d|^{-(d+1)/2}(2t-r-s)^{-(d+1)/2} \leq I_{P_\alpha}(r,s) \leq B_d E|W_d|^{-(d+1)/2}(2t-r-s)^{-(d+1)/2},$$

i.e. condition (36) is satisfied with $\alpha_f = -1$.

We continue by examining condition (37). Using the definitions of J_f and G, we obtain that

$$\begin{aligned} J_f(u,v,y,z) &= \frac{1}{(4\pi)^d (uv)^{d/2}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-x') e^{-\frac{|x-y|^2}{4u} - \frac{|x'-z|^2}{4v}} dx dx' \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y-z+\sqrt{2}(\sqrt{u}a+\sqrt{v}a')) e^{-\frac{|a|^2}{2} - \frac{|a'|^2}{2}} da da' \\ &= E[f(y-z+\sqrt{2}(\sqrt{u}Y+\sqrt{v}Z))] = E[f(y-z+U)]. \qquad (45) \end{aligned}$$

Here we used the change of variables $x-y = \sqrt{2u}a$ and $x'-z = \sqrt{2v}a'$, we denoted by $(Y,Z) = (Y_1, \ldots Y_d, Z_1, \ldots, Z_d)$ a random vector with independent $N(0,1)$ components, and

$$U = \sqrt{2u}Y - \sqrt{2v}Z.$$

Note that $U_i = \sqrt{2u}Y_i - \sqrt{2v}Z_i, i = 1, \ldots, d$ are i.i.d. $N(0, 2(u+v))$ random variables. Then $V_i = U_i/\sqrt{2(u+v)}, i = 1, \ldots, d$ are i.i.d $N(0,1)$ random variables and

$$|y-z+U|^2 = \sum_{i=1}^d (y_i - z_i + U_i)^2 = 2(u+v)\sum_{i=1}^d (\mu_i + V_i)^2 = 2(u+v)\sum_{i=1}^d T_i^2, \qquad (46)$$

where $\mu_i = (y_i - z_i)/\sqrt{2(u+v)}$ and $T_i = \mu_i + V_i$ is $N(\mu_i, 1)$-distributed. It is known that (see e.g. p. 132, [7])

$$\sum_{i=1}^{d} T_i^2 \stackrel{d}{=} W_{d-1} + S^2, \tag{47}$$

where W_{d-1} and S are independent random variables with distributions χ^2_{d-1}, respectively $N(\sqrt{\sum_{i=1}^d \mu_i^2}, 1)$.

We are now treating separately the four cases:

(i) In the case of the Riesz kernel, $f(x) = \gamma_{\alpha,d}|x|^{-(d-\alpha)}$. Using (45), (46), and (47), the integral $J_f(u,v,y,z)$ becomes:

$$\begin{aligned} J_{R_\alpha}(u,v,y,z) &= \gamma_{\alpha,d} E|y-z+U|^{-(d-\alpha)} = \gamma_{\alpha,d}[2(u+v)]^{-(d-\alpha)/2} E\left|\sum_{i=1}^d T_i^2\right|^{-(d-\alpha)/2} \\ &= \gamma_{\alpha,d}[2(u+v)]^{-(d-\alpha)/2} E\left|W_{d-1}+S^2\right|^{-(d-\alpha)/2} \\ &\leq D_{\alpha,d}(u+v)^{-(d-\alpha)/2}, \end{aligned}$$

where $D_{\alpha,d} = \gamma_{\alpha,d} 2^{-(d-\alpha)/2} E|W_{d-1}|^{-(d-\alpha)/2}$, i.e. condition (37) is satisfied with $\alpha_f = \alpha$.

(ii) In the case of the Bessel kernel, $f(x) = B_\alpha(x) = \gamma'_\alpha \int_0^\infty w^{(\alpha-d)/2-1} e^{-w} e^{-|x|^2/4w} dw$ and $\alpha > 0$. Using (45), (46), and (47), the integral $I_f(r,s)$ becomes

$$\begin{aligned} J_{B_\alpha}(u,v,y,z) &= \gamma'_\alpha \int_0^\infty w^{(\alpha-d)/2-1} e^{-w} E[e^{-|y-z+U|^2/(4w)}] dw \\ &= \gamma'_\alpha \int_0^\infty w^{(\alpha-d)/2-1} e^{-w} E\left[\exp\left(-\frac{u+v}{2w}\sum_{i=1}^d T_i^2\right)\right] dw \\ &= \gamma'_\alpha \int_0^\infty w^{(\alpha-d)/2-1} e^{-w} E\left\{\exp\left[-\frac{u+v}{2w}(W_{d-1}+S^2)\right]\right\} dw \end{aligned}$$

Note that

$$E[e^{-c(W_{d-1}+S^2)}] \leq (1+2c)^{-d/2}, \quad \forall c > 0. \tag{48}$$

This follows by the independence of W_{d-1} and S, the fact that $E(e^{-cW_{d-1}}) = (1+2c)^{-(d-1)/2}$, and

$$E\left(e^{-cS^2}\right) = \frac{1}{\sqrt{1+2c}} \exp\left\{-\frac{\sum_{i=1}^d \mu_i^2}{2} \cdot \frac{2+2c}{1+2c}\right\} \leq (1+2c)^{-1/2}, \quad \forall c > 0$$

(recall that S has $N(\sqrt{\sum_{i=1}^d \mu_i^2}, 1)$ distribution). Therefore

$$\begin{aligned} J_{B_\alpha}(u,v,y,z) &\leq \gamma'_\alpha \int_0^\infty w^{(\alpha-d)/2-1} e^{-w} \left(1+\frac{u+v}{w}\right)^{-d/2} dvdu \\ &= \gamma'_\alpha \int_0^\infty w^{\alpha/2-1} e^{-w} (w+u+v)^{-d/2} dw \\ &\leq \gamma'_\alpha \Gamma(\alpha/2)(u+v)^{-d/2}, \end{aligned}$$

i.e. condition (37) is satisfied with $\alpha_f = 0$.

(iii) In the case of the heat kernel, $f(x)=G_\alpha(x)=\gamma''_{\alpha,d}e^{-|x|^2/(4\alpha)}$ and $\alpha>0$. Using (45), (46), and (47), the integral $J_f(u,v,y,z)$ becomes

$$\begin{aligned} J_{G_\alpha}(u,v,y,z) &= \gamma''_{\alpha,d}E[e^{-|U|^2/(4\alpha)}]=\gamma''_{\alpha,d}E\left\{\exp\left[-\frac{u+v}{2\alpha}(W_{d-1}+S^2)\right]\right\} \\ &\leq \gamma''_{\alpha,d}\left(1+\frac{u+v}{\alpha}\right)^{-d/2}\leq \gamma''_{\alpha,d}\alpha^{d/2}(u+v)^{-d/2} \end{aligned}$$

where we used (48) for the first inequality. This proves that condition (37) is satisfied with $\alpha_f=0$.

(iv) In the case of the Poisson kernel, $f(x)=P_\alpha(x)=\gamma'''_{\alpha,d}(|x|^2+\alpha^2)^{-(d+1)/2}$ and $\alpha>0$. Using (45), (46), and (47), the integral $J_f(u,v,y,z)$ becomes

$$\begin{aligned} J_{P_\alpha}(u,v,y,z) &= \gamma'''_{\alpha,d}E\left[(|y-z+U|^2+\alpha^2)^{-(d+1)/2}\right]=\gamma'''_{\alpha,d}E\left|2(u+v)\sum_{i=1}^{d}T_i^2+\alpha^2\right|^{-(d+1)/2} \\ &= \gamma'''_{\alpha,d}E\left|2(u+v)(W_{d-1}+S^2)+\alpha^2\right|^{-(d+1)/2}\leq \gamma'''_{\alpha,d}[2(u+v)]^{-(d+1)/2}E|W_{d-1}|^{-(d+1)/2}, \end{aligned}$$

i.e. condition (37) is satisfied with $\alpha_f=-1$. ■

Under the conditions of Theorem 3.12), $B(g_{tx})$ and $B(\eta*\tilde{G})$ are well-defined for every (t,x), respectively for every $\eta\in\mathcal{D}((0,T)\times\mathbb{R}^d)$, and we can introduce the following definition:

Definition 3.15. a) The process $\{u(t,x);t\in[0,T],x\in\mathbb{R}^d\}$ defined by

$$u(t,x):=B(g_{tx})=\int_0^T\int_{\mathbb{R}^d}G(t-s,x-y)B(ds,dy) \tag{49}$$

is called the **process solution** of the stochastic heat equation (35), with vanishing initial conditions.

b) The process $\{u(\eta);\eta\in\mathcal{D}((0,T)\times\mathbb{R}^d)\}$ defined by

$$u(\eta):=B(\eta*\tilde{G})=\int_0^T\int_{\mathbb{R}^d}\left(\int_0^\infty\int_{\mathbb{R}^d}\eta(t+s,x+y)G(s,y)dyds\right)B(dt,dx)$$

is called the **distribution-valued solution** of the stochastic heat equation (35), with vanishing initial conditions.

Lemma 3.16. *The process $\{u(t,x);t\in[0,T],x\in\mathbb{R}^d\}$ is $L^2(\Omega)$-continuous.*

Proof: The proof is identical to the proof of Lemma 2.10, based on the continuity of the function $G(t,x)$ with respect to each of its arguments, and the fact that $\|g_{tx}\|_{\mathcal{HP}}<\infty$, which is a consequence of Theorem 3.12.(a). ■

The next theorem can be viewed as a counterpart of the result obtained by Maslowski and Nualart (see Example 3.5, [8] in the case $m=1$, $L_1=\Delta$, $f=0$, $\Phi=1$).

Theorem 3.17. *Suppose that the spatial covariance function f satisfies (36) and (37). Let $\{u(\eta);\eta\in\mathcal{D}((0,T)\times\mathbb{R}^d)\}$ be the distribution-valued solution of the stochastic heat equation (35).*

In order that there exists a jointly measurable and locally mean-square bounded process $X=\{X(t,x);t\in[0,T],x\in\mathbb{R}^d\}$ such that

$$u(\eta)=\int_0^T\int_{\mathbb{R}^d}X(t,x)\eta(t,x)dxdt\quad\forall\eta\in\mathcal{D}((0,T)\times\mathbb{R}^d)\quad a.s.$$

it is necessary and sufficient that (38) holds. In this case, X is a modification of the process $u = \{u(t,x); t \in [0,T], x \in \mathbb{R}^d\}$ defined by (49).

Proof: The proof is omitted since it is identical to the proof of Theorem 2.11, using $\|\cdot\|_{\mathcal{HP}}$ instead of $\|\cdot\|_{\mathcal{H}}$, and relation (39) instead of (15). ■

Appendix A. An Auxiliary Lemma

The following result is the analogue of Lemma 1, p.116, [15], for functions on bounded domains. It plays a crucial role in the present paper. For our purposes, it is stated only for $d = 1$, but it can be easily generalized to $d \geq 2$.

Lemma A.1. *Let $0 < \alpha < 1$ be arbitrary. (a) For every $\varphi \in L_2(a,b)$, we have*

$$\int_a^b |t|^{-(1-\alpha)}\varphi(t)dt = q_\alpha \int_{\mathbb{R}} |\tau|^{-\alpha}\mathcal{F}_{a,b}\varphi(\tau)d\tau$$

where $q_\alpha = (2^{1-\alpha}\pi^{1/2})^{-1}\Gamma(\alpha/2)/\Gamma((1-\alpha)/2)$. (b) For every $\varphi, \psi \in L_2(a,b)$, we have

$$\int_a^b \int_a^b \varphi(u)|u-v|^{-(1-\alpha)}\psi(v)dvdu = q_\alpha \int_{\mathbb{R}} |\tau|^{-\alpha}\mathcal{F}_{a,b}\varphi(\tau)\overline{\mathcal{F}_{a,b}\psi(\tau)}d\tau$$

Remark A.2. Note that $q_\alpha = 1/\gamma_{\alpha,1}$ where $\gamma_{\alpha,d}$ is the constant defined in Example 3.1.

Proof: (a) We use the fact that

$$\int_{\mathbb{R}} e^{-i\tau t}e^{-\pi\delta|\tau|^2}d\tau = \delta^{-1/2}e^{-|t|^2/(4\pi\delta)}, \quad \forall \delta > 0. \tag{50}$$

Using the definition of $\mathcal{F}_{a,b}\varphi$, Fubini's theorem and (50), we have

$$\int_{\mathbb{R}} e^{-\pi\delta|\tau|^2}\mathcal{F}_{a,b}\varphi(\tau)d\tau = \int_a^b \left(\int_{\mathbb{R}} e^{-i\tau t}e^{-\pi\delta|\tau|^2}d\tau\right)\varphi(t)dt = \delta^{-1/2}\int_a^b e^{-|t|^2/(4\pi\delta)}\varphi(t)dt.$$

Multiply by $\delta^{\alpha/2-1}$ and integrate with respect to $\delta > 0$. Using Fubini's theorem, we get

$$\int_{\mathbb{R}} \left(\int_0^\infty \delta^{\alpha/2-1}e^{-\pi\delta|\tau|^2}d\delta\right)\mathcal{F}_{a,b}\varphi(\tau)d\tau = \int_a^b \left(\int_0^\infty \delta^{-(1-\alpha)/2-1}e^{-\pi|t|^2/(4\pi\delta)}d\delta\right)\varphi(t)dt.$$

Using the change of variable $1/\delta = u$ for the inner integral on the right hand side, and the definition of the Gamma function for evaluating both inner integrals, we get the conclusion.

(b) Note that for every $u \in [a,b]$,

$$\int_a^b |u-v|^{-(1-\alpha)}\psi(v)dv = \int_{u-b}^{u-a} |w|^{-(1-\alpha)}\psi(u-w)dw = q_\alpha \int_{\mathbb{R}} |\tau|^{-\alpha}\mathcal{F}_{u-b,u-a}(\psi_u)\tilde{\ }(\tau)d\tau,$$

where we used the result in (a) for the last equality. Now,

$$\mathcal{F}_{u-b,u-a}(\psi_u)\tilde{\ }(\tau) = \int_{u-b}^{u-a} e^{-\tau t}\psi(u-v)dv = \int_a^b e^{-i\tau(u-w)}\psi(w)dw = e^{-i\tau u}\overline{\mathcal{F}_{a,b}\psi(\tau)}.$$

Using Fubini's theorem

$$\int_a^b \varphi(u) \int_a^b |u-v|^{-(1-\alpha)}\psi(v)dvdu = q_\alpha \int_a^b \varphi(u) \int_{\mathbb{R}} |\tau|^{-\alpha} e^{-i\tau u}\overline{\mathcal{F}_{a,b}\psi(\tau)}d\tau du =$$

$$q_\alpha \int_{\mathbb{R}} \left(\int_a^b \varphi(u) e^{-i\tau u} du \right) |\tau|^{-\alpha} \overline{\mathcal{F}_{a,b}\psi(\tau)} d\tau = q_\alpha \int_{\mathbb{R}} |\tau|^{-\alpha} \mathcal{F}_{a,b}\varphi(\tau)\overline{\mathcal{F}_{a,b}\psi(\tau)}d\tau.$$

■

Appendix B. Proof of (22)

Let $\eta_n(t,x) = \lambda_n(t-t_0)\psi_n(x-x_0) := \alpha_n(t)\beta_n(x)$. Then

$$\begin{aligned}
\mathcal{F}_{0,T}(\eta_n * \tilde{G})(\tau,x) &= \int_0^T e^{-i\tau t} \left(\int_{-\infty}^0 \int_{\mathbb{R}^d} \alpha_n(t-s)\beta_n(x-y)\tilde{G}(s,y)dy\,ds \right) dt \\
&= \int_{\mathbb{R}^d} \beta_n(x-y) \int_{-\infty}^0 e^{-i\tau s}\tilde{G}(s,y) \left(\int_0^T e^{-i\tau(t-s)}\alpha_n(t-s)dt \right) ds\,dy.
\end{aligned}$$

Since supp $\alpha_n \subset (t_0, t_0+T/n)$, we obtain that

$$\int_0^T e^{-i\tau(t-s)}\alpha_n(t-s)dt = \int_{-s}^{T-s} e^{-i\tau u}\alpha_n(u)du = \begin{cases} \mathcal{F}_{t_0,t_0+T/n}\alpha_n(\tau) & \text{if } -s < t_0 \\ \mathcal{F}_{-s,t_0+T/n}\alpha_n(\tau) & \text{if } t_0 < -s < t_0+T/n \\ 0 & \text{if } -s > t_0+T/n \end{cases}$$

and hence

$$\mathcal{F}_{0,T}(\eta_n * \tilde{G})(\tau,x) = \mathcal{F}_{t_0,t_0+T/n}\alpha_n(\tau) \int_{\mathbb{R}^d} \beta_n(x-y) \int_{-t_0}^0 e^{-i\tau s}\tilde{G}(s,y)dsdy +$$

$$\int_{\mathbb{R}^d} \beta_n(x-y) \int_{-t_0-T/n}^{-t_0} e^{-i\tau s}\tilde{G}(s,y)\mathcal{F}_{-s,t_0+T/n}\alpha_n(\tau)dsdy := A_n(\tau,x) + B_n(\tau,x).$$

Note that $\lim_n B_n(\tau,x) = 0$. Whereas for $A_n(\tau,x)$, we have

$$\mathcal{F}_{t_0,t_0+T/n}\alpha_n(\tau) = \int_{t_0}^{t_0+T/n} e^{-i\tau t}\lambda_n(t-t_0)dt = e^{-i\tau t_0} \int_0^{T/n} e^{-i\tau u}\lambda_n(u)du$$

$$= e^{-i\tau t_0}\mathcal{F}_{0,T}\lambda(\tau/n) \to e^{-i\tau t_0}, \quad \text{as } n \to \infty$$

and

$$\begin{aligned}
\int_{\mathbb{R}^d} \beta_n(x-y) \int_{-t_0}^0 e^{-i\tau s}\tilde{G}(s,y)dsdy &= \int_{\mathbb{R}^d} \psi_n(x-y-x_0)\mathcal{F}_{-t_0,0}\tilde{G}(\tau,y)dy \\
&= \int_{\mathbb{R}^d} \psi_n(z-x_0)\mathcal{F}_{-t_0,0}\tilde{G}(\tau,x-z)dz \\
&\to \mathcal{F}_{-t_0,0}\tilde{G}(\tau,x-x_0), \quad \text{as } n \to \infty
\end{aligned}$$

by Lebesgue differentiation theorem (see Exercise 2, Chapter 7, [18]). Therefore

$$\lim_n A_n(\tau,x) = e^{-i\tau t_0}\mathcal{F}_{-t_0,0}\tilde{G}(\tau,x-x_0) = \mathcal{F}_{0,T}g_{t_0x_0}(\tau,x).$$

Appendix C. A Result about the Space $\mathcal{P}(\mathbb{R}^d)$

The next result is embedded in Theorem 3, [2]. We have used this result in the proof of Lemma 3.7. We include its proof for the sake of completeness.

Lemma C.1. *If $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} |\mathcal{F}_2\varphi(\xi)|^2\mu(d\xi) < \infty$, then $\varphi \in \mathcal{P}(\mathbb{R}^d)$.*

Proof: Let $\eta \in \mathcal{D}(\mathbb{R}^d)$ be such that $\eta > 0$ and $\int_{\mathbb{R}^d} \eta(x)dx = 1$. Define $\eta_n(x) = n^d\eta(nx)$ and $\varphi_n = \varphi * \eta_n \in \mathcal{S}(\mathbb{R}^d)$. We have $\varphi_n \in |\mathcal{P}(\mathbb{R}^d)| \subset \mathcal{P}(\mathbb{R}^d)$, since

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\varphi_n(x)|f(x-y)|\varphi_n(y)|dydx = \int_{\mathbb{R}^d} f(z)(|\varphi_n| * |\tilde{\varphi}_n|)(z)dz < \infty,$$

by Leibnitz's formula (see p. 13, [2]). Note that

$$\|\varphi_n - \varphi\|^2_{\mathcal{P}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\mathcal{F}_2\varphi_n(\xi) - \mathcal{F}_2\varphi(\xi)|^2\mu(d\xi) = \int_{\mathbb{R}^d} |\mathcal{F}_2\eta_n(\xi) - 1|^2|\mathcal{F}_2\varphi(\xi)|^2\mu(d\xi) \to 0,$$

where we used the dominated convergence theorem, and the fact that $\mathcal{F}_2\eta(\xi) = \mathcal{F}_2\eta(\xi/n) \to 1$ and $|\mathcal{F}_2\eta(\xi)| \leq 1$ for all n. The conclusion follows.

■

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