

# Sample Path Properties of Bifractional Brownian Motion

Ciprian A. Tudor

SAMOS-MATISSE, Centre d'Economie de La Sorbonne,  
Université de Panthéon-Sorbonne Paris 1,  
90, rue de Tolbiac, 75634 Paris Cedex 13, France.  
tudor@univ-paris1.fr

URL: <http://pageperso.aol.fr/cipriantudor/>

Yimin Xiao \*

Department of Statistics and Probability,  
A-413 Wells Hall, Michigan State University,  
East Lansing, MI 48824, USA.

xiao@stt.msu.edu

URL: <http://www.stt.msu.edu/~xiaoyimi>

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## Abstract

Let  $B^{H,K} = \{B^{H,K}(t), t \in \mathbb{R}_+\}$  be a bifractional Brownian motion in  $\mathbb{R}^d$ . We prove that  $B^{H,K}$  is strongly locally nondeterministic. Applying this property and a stochastic integral representation of  $B^{H,K}$ , we establish Chung's law of the iterated logarithm for  $B^{H,K}$ , as well as sharp Hölder conditions and tail probability estimates for the local times of  $B^{H,K}$ .

We also consider the existence and the regularity of the local times of multiparameter bifractional Brownian motion  $B^{\bar{H},\bar{K}} = \{B^{\bar{H},\bar{K}}(t), t \in \mathbb{R}_+^N\}$  in  $\mathbb{R}^d$  using Wiener-Itô chaos expansion.

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# 1 Introduction

In recent years, there has been of considerable interest in studying fractional Brownian motion due to its applications in various scientific areas including telecommunications, turbulence, image processing and finance. On the other hand, many authors have proposed to use more general self-similar Gaussian processes and random fields as stochastic models; see e.g. Addie et al. (1999), Anh et al. (1999), Benassi et al. (2000), Mannersalo and Norros (2002), Bonami and Estrade (2003), Cheridito (2004), Benson et al. (2006). Such applications have raised many interesting theoretical questions about self-similar Gaussian processes and fields in general. However, contrast to the extensive studies on fractional Brownian motion, there has been little systematic investigation on other self-similar Gaussian processes. The main reasons for this, in our opinion, are the complexity of dependence structures and the non-availability of convenient stochastic integral representations for self-similar Gaussian processes which do not have stationary increments.

The objective of this paper is to fill this gap by developing systematic ways to study sample path properties of self-similar Gaussian processes. Our main tools are the Lamperti transformation [which provides a powerful connection between self-similar processes and stationary processes; see Lamperti (1962)] and the strong local nondeterminism of Gaussian processes [see Xiao (2005)]. In particular, for any self-similar Gaussian process  $X = \{X(t), t \in \mathbb{R}\}$ , the Lamperti transformation leads to a stochastic integral representation for  $X$ . We will show the usefulness of such a representation in studying sample path properties of  $X$ .

For concreteness, we only consider a rather special class of self-similar Gaussian processes, namely, the bifractional Brownian motions introduced by Houdré and Villa (2003), to illustrate our methods. Given constants  $H \in (0, 1)$  and  $K \in (0, 1]$ , the bifractional Brownian motion (bi-fBm, in short) in  $\mathbb{R}$  is a centered Gaussian process  $B_0^{H,K} = \{B_0^{H,K}(t), t \in \mathbb{R}_+\}$  with covariance function

$$R^{H,K}(s, t) := R(s, t) = \frac{1}{2K} \left[ (t^{2H} + s^{2H})^K - |t - s|^{2HK} \right] \quad (1.1)$$

and  $B_0^{H,K}(0) = 0$ .

Let  $B_1^{H,K}, \dots, B_d^{H,K}$  be independent copies of  $B_0^{H,K}$ . We define the Gaussian process  $B^{H,K} = \{B^{H,K}(t), t \in \mathbb{R}_+\}$  with values in  $\mathbb{R}^d$  by

$$B^{H,K}(t) = \left( B_1^{H,K}(t), \dots, B_d^{H,K}(t) \right), \quad \forall t \in \mathbb{R}_+. \quad (1.2)$$

By (1.1) one can verify easily that  $B^{H,K}$  is a self-similar process with index  $HK$ , that is, for every constant  $a > 0$ ,

$$\left\{ B^{H,K}(at), t \in \mathbb{R}_+ \right\} \stackrel{d}{=} \left\{ a^{HK} B^{H,K}(t), t \in \mathbb{R}_+ \right\}, \quad (1.3)$$

where  $X \stackrel{d}{=} Y$  means the two processes have the same finite dimensional distributions. Note that, when  $K = 1$ ,  $B^{H,K}$  is the ordinary fractional Brownian motion in  $\mathbb{R}^d$ . However, if  $K \neq 1$ ,  $B^{H,K}$  does not have stationary increments. In fact, fractional Brownian motion is the

only Gaussian self-similar process with stationary increments [see Samorodnitsky and Taqqu (1994)].

Russo and Tudor (2006) have established some properties on the strong variations, local times and stochastic calculus of real-valued bifractional Brownian motion. An interesting property that deserves to be recalled is the fact that, when  $HK = \frac{1}{2}$ , the quadratic variation of this process on  $[0, t]$  is equal to a constant times  $t$ . This is really remarkable since as far as we know this is the only Gaussian self-similar process with this quadratic variation besides Brownian motion. Taking into account this property, it is natural to ask if the bifractional Brownian motion  $B^{H,K}$  with  $KH = \frac{1}{2}$  shares other properties with Brownian motion (from the sample path regularity point of view). As it can be seen from the rest of the paper, the answer is often positive: for example, the bi-fBm with  $HK = \frac{1}{2}$  and Brownian motion satisfy the same forms of Chung's laws of the iterated logarithm and the Hölder conditions for their local times.

The rest of this paper is organized as follows. In Section 2 we apply the Lamperti transformation to prove the strong local nondeterminism of  $B_0^{H,K}$ . This property will play essential roles in proving most of our results. In Section 3 we derive small ball probability estimates and a stochastic integral representation for  $B_0^{H,k}$ . Applying these results, we prove a Chung's law of the iterated logarithm for bifractional Brownian motion.

Section 4 is devoted to the study of local times of one-parameter bifractional Brownian motion and the corresponding  $N$ -parameter fields. In general, there are mainly two methods in studying local times of Gaussian processes: the Fourier analysis approach introduced by Berman and the Malliavin calculus approach. It is known that, the Fourier analysis approach combined with various properties of *local nondeterminism* yields strong regularity properties such as the joint continuity and sharp Hölder conditions for the local times [see Berman (1973), Pitt (1978), Geman and Horowitz (1980), Xiao (1997, 2005)]; while the Malliavin calculus approach requires less conditions on the process and establishes regularity of the local times in the sense of Sobolev-Watanabe spaces [see Watanabe (1984), Imkeller et al. (1995), Eddahbi et al. (2005)]. In this paper we make use of both approaches to obtain more comprehensive results on local times of bifractional Brownian motion and fields.

Throughout this paper, an unspecified positive and finite constant will be denoted by  $c$ , which may not be the same in each occurrence. More specific constants in Section  $i$  are numbered as  $c_{i,1}, c_{i,2}, \dots$

## 2 Strong local nondeterminism

The following proposition is essential in this paper. From its proof, we see that the same conclusion holds for quite general self-similar Gaussian processes.

**Proposition 2.1** *For all constants  $0 < a < b$ ,  $B_0^{H,K}$  is strongly locally  $\varphi$ -nondeterministic on  $I = [a, b]$  with  $\varphi(r) = r^{2HK}$ . That is, there exist positive constants  $c_{2,1}$  and  $r_0$  such that for all  $t \in I$  and all  $0 < r \leq \min\{t, r_0\}$ ,*

$$\text{Var}\left(B_0^{H,K}(t) \mid B_0^{H,K}(s) : s \in I, r \leq |s - t| \leq r_0\right) \geq c_{2,1} \varphi(r). \quad (2.1)$$

**Proof** We consider the centered stationary Gaussian process  $Y_0 = \{Y_0(t), t \in \mathbb{R}\}$  defined through the Lamperti's transformation [Lamperti (1962)]:

$$Y_0(t) = e^{-HKt} B_0^{H,K}(e^t), \quad \forall t \in \mathbb{R}. \quad (2.2)$$

The covariance function  $r(t) := \mathbb{E}(Y_0(0)Y_0(t))$  is given by

$$\begin{aligned} r(t) &= \frac{1}{2^K} e^{-HKt} \left[ (e^{2Ht} + 1)^K - |e^t - 1|^{2HK} \right] \\ &= \frac{1}{2^K} e^{HKt} \left[ (1 + e^{-2Ht})^K - |1 - e^{-t}|^{2HK} \right]. \end{aligned} \quad (2.3)$$

Hence  $r(t)$  is an even function and, by (2.3) and the Taylor expansion, we verify that  $r(t) = O(e^{-\beta t})$  as  $t \rightarrow \infty$ , where  $\beta = \min\{H(2-K), HK\}$ . It follows that  $r(\cdot) \in L^1(\mathbb{R})$ . On the other hand, by using (2.3) and the Taylor expansion again, we also have

$$r(t) \sim 1 - \frac{1}{2^K} |t|^{2HK} \quad \text{as } t \rightarrow 0. \quad (2.4)$$

The stationary Gaussian process  $Y_0$  is sometimes called the Ornstein-Uhlenbeck process associated with  $B_0^{H,K}$  [Note that it does not coincide with the solution of the fractional Langevin equation, see Cheridito et al. (2003) for a proof in the case  $K = 1$ ]. By Bochner's theorem,  $Y_0$  has the following stochastic integral representation

$$Y_0(t) = \int_{\mathbb{R}} e^{i\lambda t} W(d\lambda), \quad \forall t \in \mathbb{R}, \quad (2.5)$$

where  $W$  is a complex Gaussian measure with control measure  $\Delta$  whose Fourier transform is  $r(\cdot)$ . The measure  $\Delta$  is called the spectral measure of  $Y$ .

Since  $r(\cdot) \in L^1(\mathbb{R})$ , so the spectral measure  $\Delta$  of  $Y$  has a continuous density function  $f(\lambda)$  which can be represented as the inverse Fourier transform of  $r(\cdot)$ :

$$f(\lambda) = \frac{1}{\pi} \int_0^\infty r(t) \cos(t\lambda) dt. \quad (2.6)$$

It follows from (2.4), (2.6) and the Tauberian theorem due to Pitman (1968, Theorem 5) [cf. Bingham et al. (1987)] that

$$f(\lambda) \sim c_{2,2} |\lambda|^{-(1+2HK)} \quad \text{as } \lambda \rightarrow \infty, \quad (2.7)$$

where  $c_{2,2} > 0$  is an explicit constant depending only on  $HK$ . Hence, by a result of Cuzick and DuPreez (1982, Lemma 1) [see also Xiao (2005) for more general results],  $Y_0 = \{Y_0(t), t \in \mathbb{R}\}$  is strongly locally  $\varphi$ -nondeterministic on any interval  $J = [-T, T]$  with  $\varphi(r) = r^{2HK}$  in the sense that there exist positive constants  $\delta$  and  $c_{2,3}$  such that for all  $t \in [-T, T]$  and all  $r \in (0, |t| \wedge \delta)$ ,

$$\text{Var}(Y_0(t) | Y_0(s) : s \in J, r \leq |s - t| \leq \delta) \geq c_{2,3} \varphi(r). \quad (2.8)$$

Now we prove the strong local nondeterminism of  $B_0^{H,K}$  on  $I$ . To this end, note that  $B_0^{H,K}(t) = t^{HK} Y_0(\log t)$  for all  $t > 0$ . We choose  $r_0 = a\delta$ . Then for all  $s, t \in I$  with  $r \leq |s - t| \leq r_0$  we have

$$\frac{r}{b} \leq |\log s - \log t| \leq \delta. \quad (2.9)$$

Hence it follows from (2.8) and (2.9) that for all  $t \in [a, b]$  and  $r < r_0$ ,

$$\begin{aligned} & \text{Var}\left(B_0^{H,K}(t) | B_0^{H,K}(s) : s \in I, r \leq |s - t| \leq r_0\right) \\ &= \text{Var}\left(t^{HK} Y_0(\log t) | s^{HK} Y_0(\log s) : s \in I, r \leq |s - t| \leq r_0\right) \\ &\geq t^{2HK} \text{Var}\left(Y_0(\log t) | Y_0(\log s) : s \in I, r \leq |s - t| \leq r_0\right) \\ &\geq a^{2HK} \text{Var}\left(Y_0(\log t) | Y_0(\log s) : s \in I, r/b \leq |\log s - \log t| \leq \delta\right) \\ &\geq c_{2,4} \varphi(r). \end{aligned} \quad (2.10)$$

This proves Proposition 2.1. □

For use in next section, we list two properties of the spectral density  $f(\lambda)$  of  $Y$ . They follow from (2.7) or, more generally, from (2.4) and the truncation inequalities in Loève (1977, p.209); see also Monrad and Rootzén (1995).

**Lemma 2.2** *There exist positive constants  $c_{2,5}$  and  $c_{2,6}$  such that for  $u > 1$ ,*

$$\int_{|\lambda| < u} \lambda^2 f(\lambda) d\lambda \leq c_{2,5} u^{2(1-HK)} \quad (2.11)$$

and

$$\int_{|\lambda| \geq u} f(\lambda) d\lambda \leq c_{2,6} u^{-2HK}. \quad (2.12)$$

We will also need the following lemma from Houdré and Villa (2003).

**Lemma 2.3** *There exist positive constants  $c_{2,7}$  and  $c_{2,8}$  such that for all  $s, t \in \mathbb{R}_+$ , we have*

$$c_{2,7} |t - s|^{2HK} \leq \mathbb{E}\left[\left(B_0^{H,K}(t) - B_0^{H,K}(s)\right)^2\right] \leq c_{2,8} |t - s|^{2HK}. \quad (2.13)$$

### 3 Chung's law of the iterated logarithm

As applications of small ball probability estimates, Monrad and Rootzen (1995), Xiao (1997) and Li and Shao (2001) established Chung-type laws of the iterated logarithm for fractional Brownian motion and other strongly locally nondeterministic Gaussian processes with stationary increments. However, there have been no results on Chung's LIL for self-similar Gaussian processes that do not have stationary increments [Recall that the class of self-similar Gaussian processes is large and fBm is the only such process with stationary increments].

In this section, we prove the following Chung's law of the iterated logarithm for bifractional Brownian motion in  $\mathbb{R}$ . It will be clear that our argument is applicable to a large class of self-similar Gaussian processes.

**Theorem 3.1** *Let  $B_0^{H,K} = \{B_0^{H,K}(t), t \in \mathbb{R}_+\}$  be a bifractional Brownian motion in  $\mathbb{R}$ . Then there exists a positive and finite constant  $c_{3,1}$  such that*

$$\liminf_{r \rightarrow 0} \frac{\max_{t \in [0,r]} |B_0^{H,K}(t)|}{r^{HK}/(\log \log(1/r))^{HK}} = c_{3,1} \quad a.s. \quad (3.1)$$

In order to prove Theorem 3.1, we need several preliminary results. Lemma 3.2 gives estimates on the small ball probability of  $B_0^{H,K}$ .

**Lemma 3.2** *There exist positive constants  $c_{3,2}$  and  $c_{3,3}$  such that for all  $t_0 \in [0,1]$  and  $x \in (0,1)$ ,*

$$\exp\left(-\frac{c_{3,2}}{x^{1/(HK)}}\right) \leq \mathbb{P}\left\{\max_{t \in [0,1]} |B_0^{H,K}(t) - B_0^{H,K}(t_0)| \leq x\right\} \leq \exp\left(-\frac{c_{3,3}}{x^{1/(HK)}}\right). \quad (3.2)$$

**Proof** By Proposition 2.1 and Lemma 2.3, we see that  $B_0^{H,K}$  satisfies Conditions (C1) and (C2) in Xiao (2005). Hence this lemma follows from Theorem 3.1 in Xiao (2005).  $\square$

Proposition 3.3 provides a zero-one law for ergodic self-similar processes, which complements the results of Takashima (1989). In order to state it, we need to recall some definitions.

Let  $X = \{X(t), t \in \mathbb{R}\}$  be a separable, self-similar process with index  $\kappa$ . For any constant  $a > 0$ , the scaling transformation  $S_{\kappa,a}$  of  $X$  is defined by

$$(S_{\kappa,a}X)(t) = a^{-\kappa}X(at), \quad \forall t \in \mathbb{R}. \quad (3.3)$$

Note that  $X$  is  $\kappa$ -self-similar is equivalent to saying that for every  $a > 0$ , the process  $\{(S_{\kappa,a}X)(t), t \in \mathbb{R}\}$  has the same finite dimensional distributions as those of  $X$ . That is, for a  $\kappa$ -self-similar process  $X$ , a scaling transformation  $S_{\kappa,a}$  preserves the distribution of  $X$ , and so the notion of ergodicity and mixing of  $S_{\kappa,a}$  can be defined in the usual way, cf. Cornfeld et al. (1982). Following Takashima (1989), we say that a  $\kappa$ -self-similar process  $X = \{X(t), t \in \mathbb{R}\}$  is ergodic (or strong mixing) if for every  $a > 0, a \neq 1$ , the scaling transformation  $S_{\kappa,a}$  is ergodic (or strong mixing, respectively). This, in turn, is equivalent to saying that the shift transformations for the corresponding stationary process  $Y = \{Y(t), t \in \mathbb{R}\}$  defined by  $Y(t) = e^{-\kappa t}X(e^t)$  are ergodic (or strong mixing, respectively).

**Proposition 3.3** *Let  $X = \{X(t), t \in \mathbb{R}\}$  be a separable, self-similar process with index  $\kappa$ . We assume that  $X(0) = 0$  and  $X$  is ergodic. Then for any increasing function  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , we have  $\mathbb{P}(E_{\kappa,\psi}) = 0$  or 1, where*

$$E_{\kappa,\psi} = \left\{ \omega : \text{there exists } \delta > 0 \text{ such that } \sup_{0 \leq s \leq t} |X(s)| \geq t^\kappa \psi(t) \text{ for all } 0 < t \leq \delta \right\}. \quad (3.4)$$

**Proof** We will prove that for every  $a > 0$ , the event  $E_{\kappa,\psi}$  is invariant with respect to the transformation  $S_{\kappa,a}$ . Then the conclusion follows from the ergodicity of  $X$ .

Fix a constant  $a > 0$  and  $a \neq 1$ . We consider two cases: (i)  $a > 1$  and (ii)  $a < 1$ . In the first case, since  $\psi$  is increasing, we have  $\psi(au) \geq \psi(u)$  for all  $u > 0$ . Assume that a.s. there is a  $\delta > 0$  such that

$$\sup_{0 \leq s \leq t} |X(s)| \geq t^\kappa \psi(t) \quad \text{for all } 0 < t \leq \delta, \quad (3.5)$$

then

$$\sup_{0 \leq s \leq t} |a^{-\kappa} X(as)| = a^{-\kappa} \sup_{0 \leq s \leq at} |X(s)| \geq t^\kappa \psi(t) \quad \text{for all } 0 < t \leq \delta/a. \quad (3.6)$$

This implies that  $E_{\kappa,\psi} \subset S_{\kappa,a}^{-1}(E_{\kappa,\psi})$ . By the self-similarity of  $X$ , these two events have the same probability, it follows that  $\mathbb{P}\{E_{\kappa,\psi} \Delta S_{\kappa,a}^{-1}(E_{\kappa,\psi})\} = 0$ . This proves that  $E_{\kappa,\psi}$  is  $S_{\kappa,a}$ -invariant and, hence, has probability 0 or 1.

In case (ii), we have  $\psi(au) \leq \psi(u)$  for all  $u > 0$  and the proof is similar to the above. If  $S_{\kappa,a}X \in E_{\kappa,\psi}$ , then we have  $X \in E_{\kappa,\psi}$ . This implies  $S_{\kappa,a}^{-1}(E_{\kappa,\psi}) \subset E_{\kappa,\psi}$  and again  $E_{\kappa,\psi}$  is  $S_{\kappa,a}$ -invariant. This finishes the proof.  $\square$

By a result of Manuyama (1949) on ergodicity and mixing properties of stationary Gaussian processes, we see that  $B_0^{H,K}$  is mixing. Hence we have the following corollary of Proposition 3.3.

**Corollary 3.4** *There exists a constant  $c_{3,4} \in [0, \infty]$  such that*

$$\liminf_{t \rightarrow 0^+} \frac{(\log \log 1/t)^{HK}}{t^{HK}} \max_{0 \leq s \leq t} |B_0^{H,K}(s)| = c_{3,4}, \quad \text{a.s.} \quad (3.7)$$

**Proof** We take  $\psi_c(t) = c (\log \log 1/t)^{-HK}$  and define  $c_{3,4} = \sup \{c \geq 0 : \mathbb{P}\{E_{\kappa,\psi_c}\} = 1\}$ . It can be verified that (3.7) follows from Proposition 3.3.  $\square$

It follows from Corollary 3.4 that Theorem 3.1 will be established if we show  $c_{3,4} \in (0, \infty)$ . This is where Lemma 3.2 and the following lemma from Talagrand (1995) are needed.

**Lemma 3.5** *Let  $X = \{X(t), t \in \mathbb{R}\}$  be a centered Gaussian process in  $\mathbb{R}$  and let  $S \subset \mathbb{R}$  be a closed set equipped with the canonical metric defined by*

$$d(s, t) = \left[ \mathbb{E}(X(s) - X(t))^2 \right]^{1/2}.$$

*Then there exists a positive constants  $c_{3,5}$  such that for all  $u > 0$ ,*

$$\mathbb{P} \left\{ \sup_{s, t \in S} |X(s) - X(t)| \geq c_{3,5} \left( u + \int_0^D \sqrt{\log N_d(S, \varepsilon)} d\varepsilon \right) \right\} \leq \exp \left( -\frac{u^2}{D^2} \right), \quad (3.8)$$

*where  $N_d(S, \varepsilon)$  denotes the smallest number of open  $d$ -balls of radius  $\varepsilon$  needed to cover  $S$  and where  $D = \sup\{d(s, t) : s, t \in S\}$  is the diameter of  $S$ .*

Now we proceed to prove Theorem 3.1.

**Proof of Theorem 3.1** We prove the lower bound first. For any integer  $n \geq 1$ , let  $r_n = e^{-n}$ . Let  $0 < \gamma < c_{3,3}$  be a constant and consider the event

$$A_n = \left\{ \max_{0 \leq s \leq r_n} |B_0^{H,K}(s)| \leq \gamma^{HK} r_n^{HK} / (\log \log 1/r_n)^{HK} \right\}.$$

Then the self-similarity of  $B_0^{H,K}$  and Lemma 3.2 imply that

$$P\{A_n\} \leq \exp\left(-\frac{c_{3,3}}{\gamma} \log n\right) = n^{-c_{3,3}/\gamma}. \quad (3.9)$$

Since  $\sum_{n=1}^{\infty} P\{A_n\} < \infty$ , the Borel-Cantelli lemma implies

$$\liminf_{n \rightarrow \infty} \frac{\max_{s \in [0, r_n]} |B_0^{H,K}(s)|}{r_n^{HK} / (\log \log(1/r_n))^{HK}} \geq c_{3,3} \quad \text{a.s.} \quad (3.10)$$

It follows from (3.10) and a standard monotonicity argument that

$$\liminf_{r \rightarrow 0} \frac{\max_{t \in [0, r]} |B_0^{H,K}(t)|}{r^{HK} / (\log \log(1/r))^{HK}} \geq c_{3,6} \quad \text{a.s.} \quad (3.11)$$

The upper bound is a little more difficult to prove due to the dependence structure of  $B_0^{H,K}$ . In order to create independence, we will make use of the following stochastic integral representation of  $B_0^{H,K}$ : for every  $t > 0$ ,

$$B_0^{H,K}(t) = t^{HK} \int_{\mathbb{R}} e^{i\lambda \log t} W(d\lambda). \quad (3.12)$$

This follows from the spectral representation (2.5) of  $Y$  and its connection with  $B_0^{H,K}$ .

For every integer  $n \geq 1$ , we take

$$t_n = n^{-n} \quad \text{and} \quad d_n = n^\beta, \quad (3.13)$$

where  $\beta > 0$  is a constant whose value will be determined later. It is sufficient to prove that there exists a finite constant  $c_{3,7}$  such that

$$\liminf_{n \rightarrow \infty} \frac{\max_{s \in [0, t_n]} |B_0^{H,K}(s)|}{t_n^{HK} / (\log \log(1/t_n))^{HK}} \leq c_{3,7} \quad \text{a.s.} \quad (3.14)$$

Let us define two Gaussian processes  $X_n$  and  $\tilde{X}_n$  by

$$X_n(t) = t^{HK} \int_{|\lambda| \in (d_{n-1}, d_n]} e^{i\lambda \log t} W(d\lambda) \quad (3.15)$$

and

$$\tilde{X}_n(t) = t^{HK} \int_{|\lambda| \notin (d_{n-1}, d_n]} e^{i\lambda \log t} W(d\lambda), \quad (3.16)$$

respectively. Clearly  $B_0^{H,K}(t) = X_n(t) + \tilde{X}_n(t)$  for all  $t \geq 0$ . It is important to note that the Gaussian processes  $X_n$  ( $n = 1, 2, \dots$ ) are independent and, moreover, for every  $n \geq 1$ ,  $X_n$  and  $\tilde{X}_n$  are independent as well.

Denote  $h(r) = r^{HK} (\log \log 1/r)^{-HK}$ . We make the following two claims:



(i). There is a constant  $\gamma > 0$  such that

$$\sum_{n=1}^{\infty} \mathbb{P} \left\{ \max_{s \in [0, t_n]} |X_n(s)| \leq \gamma^{HK} h(t_n) \right\} = \infty. \quad (3.17)$$

(ii). For every  $\varepsilon > 0$ ,

$$\sum_{n=1}^{\infty} \mathbb{P} \left\{ \max_{s \in [0, t_n]} |\tilde{X}_n(s)| > \varepsilon h(t_n) \right\} < \infty. \quad (3.18)$$

Since the events in (3.17) are independent, we see that (3.14) follows from (3.17), (3.18) and a standard Borel-Cantelli argument.

It remains to verify the claims (i) and (ii) above. By Lemma 3.2 and Anderson's inequality [see Anderson (1955)], we have

$$\begin{aligned} \mathbb{P} \left\{ \max_{s \in [0, t_n]} |X_n(s)| \leq \gamma^{HK} h(t_n) \right\} &\geq \mathbb{P} \left\{ \max_{s \in [0, t_n]} |B_0^{H,K}(s)| \leq \gamma^{HK} h(t_n) \right\} \\ &\geq \exp \left( -\frac{c_{3,2}}{\gamma} \log(n \log n) \right) \\ &= (n \log n)^{-c_{3,2}/\gamma}. \end{aligned} \quad (3.19)$$

Hence (i) holds for  $\gamma \geq c_{3,2}$ .

In order to prove (ii), we divide  $[0, t_n]$  into  $p_n + 1$  non-overlapping subintervals  $J_{n,j} = [a_{n,j-1}, a_{n,j}]$ , ( $i = 0, 1, \dots, p_n$ ) and then apply Lemma 3.5 to  $\tilde{X}_n$  on each of  $J_{n,j}$ . Let  $\beta > 0$  be the constant in (3.13) and we take  $J_{n,0} = [0, t_n n^{-\beta}]$ . After  $J_{n,j}$  has been defined, we take  $a_{n,j+1} = a_{n,j}(1 + n^{-\beta})$ . It can be verified that the number of such subintervals of  $[0, t_n]$  satisfies the following bound:

$$p_n + 1 \leq c n^\beta \log n. \quad (3.20)$$

Moreover, for every  $j \geq 1$ , if  $s, t \in J_{n,j}$  and  $s < t$ , then we have  $t/s - 1 \leq n^{-\beta}$  and this yields

$$t - s \leq s n^{-\beta} \quad \text{and} \quad \log \left( \frac{t}{s} \right) \leq n^{-\beta}. \quad (3.21)$$

Lemma 2.3 implies that the canonical metric  $d$  for the process  $\tilde{X}_n$  satisfies

$$d(s, t) \leq c |s - t|^{HK} \quad \text{for all } s, t > 0 \quad (3.22)$$

and  $d(0, s) \leq c t_n^{HK} n^{-\beta HK}$  for every  $s \in J_{n,0}$ . It follows that  $D_0 := \sup\{d(s, t); s, t \in J_{n,0}\} \leq c t_n^{HK} n^{-\beta HK}$  and

$$N_d(J_{n,0}, \varepsilon) \leq c \frac{t_n n^{-\beta}}{\varepsilon^{1/(HK)}}. \quad (3.23)$$

Some simple calculation yields

$$\begin{aligned} \int_0^{D_0} \sqrt{\log N_d(J_{n,0}, \varepsilon)} \, d\varepsilon &\leq \int_0^{t_n^{HK} n^{-\beta HK}} \sqrt{\log \left( \frac{t_n n^{-\beta}}{\varepsilon^{1/(HK)}} \right)} \, d\varepsilon \\ &= t_n^{HK} n^{-\beta HK} \int_0^1 \sqrt{\log \left( \frac{1}{u} \right)} \, du \\ &= c_{3,8} t_n^{HK} n^{-\beta HK}. \end{aligned} \quad (3.24)$$

It follows from Lemma 3.5 and (3.24) that

$$\mathbb{P}\left\{\max_{s \in J_{n,0}} |\tilde{X}_n(s)| > \varepsilon h(t_n)\right\} \leq \exp\left(-c \frac{n^{2\beta HK}}{(\log(n \log n))^{2HK}}\right). \quad (3.25)$$

For every  $1 \leq j \leq p_n$ , we estimate the  $d$ -diameter of  $J_{n,j}$ . It follows from (3.16) that for any  $s, t \in J_{n,j}$  with  $s < t$ ,

$$\begin{aligned} \mathbb{E}\left(\tilde{X}_n(s) - \tilde{X}_n(t)\right)^2 &= \int_{|\lambda| \leq d_{n-1}} \left|t^{HK} e^{i\lambda \log t} - s^{HK} e^{i\lambda \log s}\right|^2 f(\lambda) d\lambda \\ &\quad + \int_{|\lambda| > d_n} \left|t^{HK} e^{i\lambda \log t} - s^{HK} e^{i\lambda \log s}\right|^2 f(\lambda) d\lambda \\ &:= \mathcal{J}_1 + \mathcal{J}_2. \end{aligned} \quad (3.26)$$

The second term is easy to estimate: for all  $s, t \in J_{n,j}$ ,

$$\mathcal{J}_2 \leq 4t_n^{2HK} \int_{|\lambda| > d_n} f(\lambda) d\lambda \leq c_{3,9} t_n^{2HK} n^{-2\beta HK}, \quad (3.27)$$

where the last inequality follows from (2.12).

For the first term, we use the elementary inequality  $1 - \cos x \leq x^2$  to derive that for all  $s, t \in J_{n,j}$  with  $s < t$ ,

$$\begin{aligned} \mathcal{J}_1 &= \int_{|\lambda| \leq d_{n-1}} \left[ (t^{HK} - s^{HK})^2 + 2t^{HK} s^{HK} \left(1 - \cos\left(\lambda \log \frac{t}{s}\right)\right) \right] f(\lambda) d\lambda \\ &\leq s^{2HK} \left(\frac{t}{s} - 1\right)^{2HK} \int_{\mathbb{R}} f(\lambda) d\lambda + 2t^{2HK} \log^2\left(\frac{t}{s}\right) \int_{|\lambda| \leq d_{n-1}} \lambda^2 f(\lambda) d\lambda \\ &\leq c_{3,10} t_n^{2HK} n^{-2\beta HK}, \end{aligned} \quad (3.28)$$

where, in deriving the last inequality, we have used (3.21) and (2.11), respectively.

It follows from (3.26), (3.27) and (3.28) that the  $d$ -diameter of  $J_{n,j}$  satisfies

$$D_j \leq c_{3,11} t_n^{HK} n^{-\beta HK}. \quad (3.29)$$

Hence, similar to (3.25), we use Lemma 3.5 and (3.29) to derive

$$\mathbb{P}\left\{\max_{s \in J_{n,j}} |\tilde{X}_n(s)| > \varepsilon h(t_n)\right\} \leq \exp\left(-c \frac{n^{2\beta HK}}{(\log(n \log n))^{2HK}}\right). \quad (3.30)$$

By combining (3.20), (3.25) and (3.30) we derive that for every  $\varepsilon > 0$ ,

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{P}\left\{\max_{s \in [0, t_n]} |\tilde{X}_n(s)| > \varepsilon h(t_n)\right\} &\leq \sum_{n=1}^{\infty} \sum_{j=0}^{p_n} \mathbb{P}\left\{\max_{s \in J_{n,j}} |\tilde{X}_n(s)| > \varepsilon h(t_n)\right\} \\ &\leq c \sum_{n=1}^{\infty} n^\beta \log n \exp\left(-c \frac{n^{2\beta HK}}{(\log(n \log n))^{2HK}}\right) \\ &< \infty. \end{aligned} \quad (3.31)$$

This proves (3.18) and hence the theorem.  $\square$

**Remark 3.6** Let  $t_0 \in [0, 1]$  be fixed and we consider the process  $X = \{X(t), t \in \mathbb{R}_+\}$  defined by  $X(t) = B_0^{H,K}(t + t_0) - B_0^{H,K}(t_0)$ . By applying Lemma 3.2 and modifying the proof of Theorem 3.1, one can show that

$$c_{3,12}^{-1} \leq \liminf_{r \rightarrow 0} \frac{\max_{t \in [0, r]} |B_0^{H,K}(t + t_0) - B_0^{H,K}(t_0)|}{r^{HK} / (\log \log(1/r))^{HK}} \leq c_{3,12} \quad \text{a.s.}, \quad (3.32)$$

where  $c_{3,12} > 1$  is a constant depending on  $HK$  only.

Corresponding to Lemma 3.2, we can also consider the small ball probability of  $B_0^{H,K}$  under the Hölder-type norm. For  $\alpha \in (0, 1)$  and any function  $y \in C_0([0, 1])$ , we consider the  $\alpha$ -Hölder norm of  $y$  defined by,

$$\|y\|_\alpha = \sup_{s, t \in [0, 1], s \neq t} \frac{|y(s) - y(t)|}{|s - t|^\alpha}. \quad (3.33)$$

The following proposition extends the results of Stolz (1996) and Theorem 2.1 of Kuelbs, Li and Shao (1995) to bifractional Brownian motion.

**Proposition 3.7** *Let  $B_0^{H,K}$  be a bifractional Brownian motion in  $\mathbb{R}$  and  $\alpha \in (0, HK)$ . There exist positive constants  $c_{3,13}$  and  $c_{3,14}$  such that for all  $\varepsilon \in (0, 1)$ ,*

$$\exp\left(-c_{3,13} \varepsilon^{-1/(HK-\alpha)}\right) \leq \mathbb{P}\left\{\|B_0^{H,K}\|_\alpha \leq \varepsilon\right\} \leq \exp\left(-c_{3,14} \varepsilon^{-1/(HK-\alpha)}\right). \quad (3.34)$$

**Proof** It follows from Theorem 3.4 in Xiao (2005). □

## 4 Local times of bifractional Brownian motion

This section is devoted to the study of the local times of the bi-fBm both in the one-parameter and multi-parameter cases. As we pointed out in the Introduction there are essentially two ways to prove the existence and regularity properties of local times for Gaussian processes: the first is related to the Fourier analysis and the local nondeterminism property; the second is based on the Malliavin calculus and Wiener-Itô chaos expansion. We will apply the Fourier analysis approach for the one-parameter case and the Malliavin calculus approach for the multiparameter case.

### 4.1 The one-parameter case

Let  $B^{H,K} = \{B^{H,K}(t), t \in \mathbb{R}_+\}$  be a bifractional Brownian motion with indices  $H$  and  $K$  in  $\mathbb{R}^d$ . For any closed interval  $I \subset \mathbb{R}_+$  and for any  $x \in \mathbb{R}^d$ , the local time  $L(x, I)$  of  $B^{H,K}$  is defined as the density of the occupation measure  $\mu_I$  defined by

$$\mu_I(A) = \int_I \mathbb{1}_A(B^{H,K}(s)) ds, \quad A \in \mathcal{B}(\mathbb{R}^d).$$

It can be shown [cf. Geman and Horowitz (1980) Theorem 6.4] that the following *occupation density formula* holds: for every Borel function  $g(t, x) \geq 0$  on  $I \times \mathbb{R}^d$ ,

$$\int_I g(t, B^{H,K}(t)) dt = \int_{\mathbb{R}^d} \int_I g(t, x) L(x, dt) dx. \quad (4.1)$$

Lemma 2.3 and Theorem 21.9 in Geman and Horowitz (1980) imply that if  $1/(HK) > d$  then  $B^{H,K}$  has a local time  $L(x, t) := L(x, [0, t])$ , where  $(x, t) \in \mathbb{R}^d \times [0, \infty)$ . In fact, more regularity properties of  $L(x, t)$  can be derived from Theorem 3.14 in Xiao (2005) which we summarize in the following theorem. Besides interest in their own right, such results are also useful in studying the fractal properties of the sample paths of  $B^{H,K}$ .

**Theorem 4.1** *Let  $B^{H,K} = \{B^{H,K}(t), t \in \mathbb{R}\}$  be a bifractional Brownian motion with indices  $H$  and  $K$  in  $\mathbb{R}^d$ . If  $1/(HK) > d$ , then the following properties hold:*

- (i)  $B^{H,K}$  has a local time  $L(x, t)$  that is jointly continuous in  $(x, t)$  almost surely.
- (ii) [Local Hölder condition] For every  $B \in \mathcal{B}(\mathbb{R})$ , let  $L^*(B) = \sup_{x \in \mathbb{R}^d} L(x, B)$  be the maximum local time. Then there exists a positive constant  $c_{4,1}$  such that for all  $t_0 \in \mathbb{R}_+$ ,

$$\limsup_{r \rightarrow 0} \frac{L^*(B(t_0, r))}{\varphi_1(r)} \leq c_{4,1} \quad a.s. \quad (4.2)$$

Here and in the sequel,  $B(t, r) = (t - r, t + r)$  and  $\varphi_1(r) = r^{1-HKd}(\log \log 1/r)^{HKd}$ .

- (iii) [Uniform Hölder condition] For every finite interval  $I \subseteq \mathbb{R}$ , there exists a positive finite constant  $c_{4,2}$  such that

$$\limsup_{r \rightarrow 0} \sup_{t_0 \in I} \frac{L^*(B(t_0, r))}{\varphi_2(r)} \leq c_{4,2} \quad a.s., \quad (4.3)$$

where  $\varphi_2(r) = r^{1-HKd}(\log 1/r)^{HKd}$ .

**Proof** By Proposition 2.1 and Lemma 2.3, we see that the conditions of Theorem 3.14 in Xiao (2005) are satisfied. Hence the results follow.  $\square$

The following states that the local Hölder condition for the maximum local time is sharp.

**Remark 4.2** By the definition of local times, we have that for every interval  $Q \subseteq \mathbb{R}_+$ ,

$$|Q| = \int_{B^{H,K}(Q)} L(x, Q) dx \leq L^*(Q) \cdot \left( \max_{s, t \in Q} |B^{H,K}(s) - B^{H,K}(t)| \right)^d. \quad (4.4)$$

By taking  $Q = B(t_0, r)$  in (4.4) and using (3.32) in Remark 3.6, we derive the lower bound in the following

$$c_{4,3} \leq \limsup_{r \rightarrow 0} \frac{L^*(B(t_0, r))}{\varphi_1(r)} \leq c_{4,4} \quad a.s., \quad (4.5)$$

where  $c_{4,3} > 0$  is a constant independent of  $t_0$  and the upper bound is given by (4.2). A similar lower bound for (4.3) could also be established by using (4.4), if one proves that for every interval  $I \subseteq \mathbb{R}_+$ ,

$$\liminf_{r \rightarrow 0} \inf_{t \in I} \max_{s \in B(t,r)} \frac{|B^{H,K}(s) - B^{H,K}(t)|}{r^{HK}/(\log 1/r)^{HK}} \leq c_{4,5} \quad a.s. \quad (4.6)$$

Theorem 4.1 can be applied to determine the Hausdorff dimension and Hausdorff measure of the level set  $Z_x = \{t \in \mathbb{R}_+ : B^{H,K}(t) = x\}$ , where  $x \in \mathbb{R}^d$ . See Berman (1972), Monrad and Pitt (1987) and Xiao (1997, 2005). In the following theorem we prove a uniform Hausdorff dimension result for the level sets of  $B^{H,K}$ .

**Theorem 4.3** *If  $1/(HK) > d$ , then with probability one,*

$$\dim_{\mathbb{H}} Z_x = 1 - HKd \quad \text{for all } x \in \mathbb{R}^d, \quad (4.7)$$

where  $\dim_{\mathbb{H}}$  denotes Hausdorff dimension.

**Proof** It follows from Theorem 3.19 in Xiao (2005) that with probability one,

$$\dim_{\mathbb{H}} Z_x = 1 - HKd \quad \text{for all } x \in \mathcal{O}, \quad (4.8)$$

where  $\mathcal{O}$  is the random open set defined by

$$\mathcal{O} = \bigcup_{s,t \in \mathbb{Q}; s < t} \left\{ x \in \mathbb{R}^d : L(x, [s, t]) > 0 \right\}.$$

Hence it only remains to show  $\mathcal{O} = \mathbb{R}^d$  a.s. For this purpose, we consider the stationary Gaussian process  $Y = \{Y(t), t \in \mathbb{R}\}$  defined by  $Y(t) = e^{-HKt} B^{H,K}(e^t)$ , using the Lamperti transformation.

Note that the component processes of  $Y$  are independent and, as shown in the proof of Proposition 2.1, they are strongly locally  $\varphi$ -nondeterministic with  $\varphi(r) = r^{2HK}$ . It follows from Theorem 3.14 in Xiao (2005) that  $Y$  has a jointly continuous local time  $L_Y(x, t)$ , where  $(x, t) \in \mathbb{R}^d \times \mathbb{R}$ . From the proof of Proposition 2.1, it can be verified that  $Y$  satisfies the conditions of Theorem 2 in Monrad and Pitt (1987), it follows that almost surely for every  $y \in \mathbb{R}^d$ , there exists a finite interval  $J \subset \mathbb{R}$  such that  $L_Y(y, J) > 0$ .

On the other hand, by using the occupation density formula (4.1), we can verify that the local times of  $B^{H,K}$  and  $Y$  are related by the following equation: for all  $x \in \mathbb{R}^d$  and finite interval  $I = [a, b] \subset [0, \infty)$ ,

$$L(x, I) = \int_{[\log a, \log b]} e^{(1-HK)s} L_Y(e^{-HKs} x, ds). \quad (4.9)$$

Hence, there exists a.s. a finite interval  $I$  such that  $L(0, I) > 0$ . The continuity of  $L(x, I)$  implies the a.s. existence of  $\delta > 0$  such that  $L(y, I) > 0$  for all  $y \in \mathbb{R}^d$  with  $|y| \leq \delta$ . Observe that the scaling property of  $B^{H,K}$  implies that for all constants  $c > 0$ , the scaled local time  $c^{-(1-HKd)} L(x, ct)$  is a version of  $L(c^{-HK} x, t)$ . It follows that a.s. for every  $x \in \mathbb{R}^d$ ,  $L(x, J) > 0$  for some finite interval  $J \subset [0, \infty)$ .  $\square$

Since there is little knowledge on the explicit distribution of  $L(0, 1)$ , it is of interest in estimating the tail probability  $\mathbb{P}\{L(0, 1) > x\}$  as  $x \rightarrow \infty$ . This problem has been considered by Kasahara et al. (1999) for certain fractional Brownian motion and by Xiao (2005) for a large class of Gaussian processes. Our next result is a consequence of Theorem 3.20 in Xiao (2005).

**Theorem 4.4** *Let  $B^{H,K} = \{B^{H,K}(t), t \in \mathbb{R}\}$  be a bifractional Brownian motion in  $\mathbb{R}^d$  with indices  $H$  and  $K$ . If  $1/(HK) > d$ , then for  $x > 0$  large enough,*

$$-\log \mathbb{P}\{L(0, 1) > x\} \asymp x^{HK}, \quad (4.10)$$

where  $a(x) \asymp b(x)$  means  $a(x)/b(x)$  is bounded from below and above for  $x$  large enough.

**Proof** By Proposition 2.1 and Lemma 2.3, we see that the conditions of Theorem 3.20 in Xiao (2005) are satisfied. This proves (4.10).  $\square$

Let us also note that the existence of the jointly continuous version of the local time and the self-similarity allow us to prove the following renormalization result. The case  $d = 1$  has been proved in Russo and Tudor (2006).

**Proposition 4.5** *If  $1/(HK) > d$ , then for any integrable function  $F : \mathbb{R}^d \rightarrow \mathbb{R}$ ,*

$$t^{HK-1} \int_{[0,t]} F(B^{H,K}(u)) du \xrightarrow{(d)} \tilde{F} L(0, 1) \quad \text{as } t \rightarrow \infty, \quad (4.11)$$

where  $\tilde{F} = \int_{\mathbb{R}^d} F(x) dx$ .

**Proof** It holds that

$$\int_{[0,t]} F(B^{H,K}(u)) du = t \int_{[0,1]} F(B^{H,K}(tv)) dv \stackrel{d}{=} t \int_{[0,1]} F(t^{HK} B^{H,K}(v)) dv. \quad (4.12)$$

By using the occupation density formula, we derive

$$\int_{[0,t]} F(B^{H,K}(u)) du = t \int_{\mathbb{R}^d} F(t^{HK} x) L(x, 1) dx = t^{1-HK} \int_{\mathbb{R}^d} F(y) L(yt^{-HK}, 1) dy. \quad (4.13)$$

Since  $y \mapsto L(y, 1)$  is almost surely continuous and bounded, the dominated convergence theorem implies that, as  $t \rightarrow \infty$ , the last integral in (4.13) tends to  $\tilde{F} L(0, 1)$  almost surely. This and (4.12) yield (4.11).  $\square$

## 4.2 Oscillation of bifractional Brownian motion

The oscillations of certain classes of stochastic processes, especially Gaussian processes, in the measure space  $([0, 1], \lambda_1)$ , where  $\lambda_1$  is the Lebesgue measure in  $\mathbb{R}$ , have been studied, among others, by Wschebor (1992) and Azaïs and Wschebor (1996). The following is an analogous result for bifractional Brownian motion.

**Proposition 4.6** Let  $B^{H,K}$  be a bi-fBm in  $\mathbb{R}$  with indices  $H \in (0, 1)$  and  $K \in (0, 1]$ . For every  $t \in [0, 1]$ , let

$$Z_\varepsilon(t) = \frac{B^{H,K}(t + \varepsilon) - B^{H,K}(t)}{\varepsilon^{HK}}.$$

Then the following statements hold:

(i) For every integer  $k \geq 1$ , almost surely,

$$\int_0^1 (Z_\varepsilon(t))^k dt \rightarrow \mathbb{E}(\rho^k) \quad \text{as } \varepsilon \rightarrow 0,$$

where  $\rho$  is a centered normal random variable with variance  $\sigma^2 = 2^{1-K}$ .

(ii) For every interval  $J \subset [0, 1]$ , almost surely, for every  $x \in \mathbb{R}$

$$\lambda_1\{t \in J : Z_\varepsilon(t) \leq x\} \rightarrow \lambda_1(J) \mathbb{P}(\rho \leq x) \quad \text{as } \varepsilon \rightarrow 0.$$

**Proof** Let us denote

$$Y^{\varepsilon,k} = \int_0^1 (Z_\varepsilon(t))^k dt.$$

It is sufficient to prove that

$$\text{Var}(Y^{\varepsilon,k}) \leq c(k) \varepsilon^\beta \quad \text{for some } c(k) \text{ and } \beta > 0. \quad (4.14)$$

Then the conclusions (i) and (ii) will follow as in Azaïs and Wschebor (1996) by the means of a Borel-Cantelli argument.

Note that

$$\text{Var}(Y^{\varepsilon,k}) = \int_0^1 \int_0^1 \text{Cov}(Z_\varepsilon(u)^k, Z_\varepsilon(v)^k) dudv.$$

We will make use of the fact that for a centered Gaussian vector  $(U, V)$ ,

$$\text{Cov}(U^k, V^k) = \sum_{1 \leq p \leq k} c(p, k) [\text{Cov}(U, V)]^p [\text{Var}(U)\text{Var}(V)]^{k-p}.$$

Since the random variable  $Z_\varepsilon$  has clearly bounded variance [cf. Lemma 2.3], it suffices to show that for every  $1 \leq p \leq k$ ,

$$\int_0^1 \int_0^1 [\mathbb{E}(Z_\varepsilon(u)Z_\varepsilon(v))]^p dudv \leq c_{4,6} \varepsilon^\beta \quad (4.15)$$

We can write

$$\begin{aligned} \int_0^1 \int_0^1 [\mathbb{E}(Z_\varepsilon(u)Z_\varepsilon(v))]^p dvdu &= 2 \int_0^1 \int_0^u \mathbb{1}_{(u-v < \varepsilon)} [\mathbb{E}(Z_\varepsilon(u)Z_\varepsilon(v))]^p dvdu \\ &\quad + 2 \int_0^1 \int_0^u \mathbb{1}_{(u-v \geq \varepsilon)} [\mathbb{E}(Z_\varepsilon(u)Z_\varepsilon(v))]^p dvdu \\ &:= A + B. \end{aligned}$$

Clearly  $A \leq c\varepsilon$ , hence it suffices to bound the term  $B$ . Note that

$$\mathbb{E}(Z_\varepsilon(u)Z_\varepsilon(v)) = \frac{1}{\varepsilon^{2HK}} \int_{u-\varepsilon}^u \int_{v-\varepsilon}^v \frac{\partial^2 R}{\partial a \partial b} db da.$$

Since

$$\frac{\partial^2 R}{\partial a \partial b}(a, b) = \frac{2HK}{2^K} \left[ (a^{2H} + b^{2H})^{K-2} a^{2H-1} b^{2H-1} - (2HK - 1)|a - b|^{2HK-2} \right],$$

we have

$$\begin{aligned} B &\leq c(p, H, K) \int_0^1 \int_0^{u-\varepsilon} \left[ \frac{1}{\varepsilon^{2HK}} \int_{u-\varepsilon}^u \int_{v-\varepsilon}^v (a^{2H} + b^{2H})^{K-2} a^{2H-1} b^{2H-1} db da \right]^p dv du \\ &\quad + c(p, H, K) \int_0^1 \int_0^{u-\varepsilon} \left[ \frac{1}{\varepsilon^{2HK}} \int_{u-\varepsilon}^u \int_{v-\varepsilon}^v |a - b|^{2HK-2} db da \right]^p dv du \\ &:= B_1 + B_2. \end{aligned}$$

The term  $B_2$  can be treated as in the fBm case [see Azaïs and Wschebor (1996), Proposition 2.1] and we get  $B_2 \leq c\varepsilon^\beta$  for some constant  $\beta > 0$ . Finally, since  $a^{2HK} + b^{2HK} \geq a^{HK}b^{HK}$ , we can write

$$\begin{aligned} B_1 &\leq c(p, H, K) \int_0^1 \int_0^{u-\varepsilon} \left[ \frac{1}{\varepsilon^{2HK}} \int_{u-\varepsilon}^u \int_{v-\varepsilon}^v a^{HK-1} b^{HK-1} db da \right]^p dv du \\ &= c(p, H, K) \int_0^1 \int_0^{u-\varepsilon} \left( \frac{u^{HK} - (u-\varepsilon)^{HK}}{\varepsilon^{HK}} \right)^p \left( \frac{v^{HK} - (v-\varepsilon)^{HK}}{\varepsilon^{HK}} \right)^p dv du \\ &\leq c \left[ \int_0^1 \left( \frac{u^{HK} - (u-\varepsilon)^{HK}}{\varepsilon^{HK}} \right)^p dv du \right]^2. \end{aligned}$$

A change of variable shows that  $B_1 \leq c\varepsilon^{2(1-HK)}$ . Combining the above yields (4.15). Therefore, we have proved (4.14), and the proposition.  $\square$

The above result can be extended to obtain the almost sure weak approximation of the occupation measure of the bi-fBm  $B^{H,K}$  by means of normalized number of crossing of  $B_\varepsilon^{H,K}$ , where  $B_\varepsilon^{H,K}$  represents the convolution of  $B^{H,K}$  with an approximation of the identity  $\Phi_\varepsilon(t) = \frac{1}{\varepsilon} \Phi\left(\frac{t}{\varepsilon}\right)$  with  $\Phi = \mathbf{1}_{[-1,0]}$ . If  $g$  is a real function defined on an interval  $I$ , then the number of crossing of level  $u$  is

$$N_u(g, I) = \#\{t \in I, g(t) = u\},$$

where  $\#E$  denotes the cardinality of  $E$ .

**Proposition 4.7** *Almost surely for every continuous function  $f$  and for every bounded interval  $I \subset \mathbb{R}_+$ ,*

$$\left(\frac{\pi}{2}\right)^{1/2} \varepsilon^{1-HK} \int_{-\infty}^{\infty} f(u) N_u(B_\varepsilon^{H,K}, I) du \rightarrow \int_{-\infty}^{\infty} f(u) L(u, I) du \quad \text{as } \varepsilon \rightarrow 0.$$



**Proof** The arguments in Azaïs and Wschebor (1996), Section 5, apply. Details are left to the reader.  $\square$

### 4.3 The multi-parameter case

For any given vectors  $H = (H_1, \dots, H_N) \in (0, 1)^N$  and  $K = (K_1, \dots, K_N) \in (0, 1]^N$ , an  $(N, d)$ -bifractional Brownian sheet  $B^{H,K} = \{B^{H,K}(t), t \in \mathbb{R}_+^N\}$  is a centered Gaussian random field in  $\mathbb{R}^d$  with i.i.d. components whose covariance functions are given by

$$\mathbb{E}\left(B_1^{H,K}(s)B_1^{H,K}(t)\right) = \prod_{j=1}^N \frac{1}{2^{K_j}} \left[ \left(s_j^{2H_j} + t_j^{2H_j}\right)^{K_j} - |t_j - s_j|^{2H_j K_j} \right]. \quad (4.16)$$

It follows from (4.16) that, similar to an  $(N, d)$ -fractional Brownian sheet [cf. Xiao and Zhang (2002), Ayache and Xiao (2005)],  $B^{H,K}$  is operator-self-similar. However, it does not have convenient stochastic integral representations which have played essential rôles in the studies of fractional Brownian sheets. Nevertheless, we will prove that the sample path properties of  $B^{H,K}$  are very similar to those of fractional Brownian sheets, and we can describe the anisotropic properties of  $B^{H,K}$  in terms of the vectors  $H$  and  $K$ .

We start with the following useful lemma.

**Lemma 4.8** *For any  $\varepsilon > 0$ , there exist positive and finite constants  $c_{4,7}$  and  $c_{4,8}$  such that for all  $s, t \in [\varepsilon, 1]^N$ ,*

$$c_{4,7} \sum_{j=1}^N |s_j - t_j|^{2H_j K_j} \leq \mathbb{E} \left[ \left( B_1^{H,K}(s) - B_1^{H,K}(t) \right)^2 \right] \leq c_{4,8} \sum_{j=1}^N |s_j - t_j|^{2H_j K_j}, \quad (4.17)$$

and

$$c_{4,7} \sum_{j=1}^N |s_j - t_j|^{2H_j K_j} \leq \det \text{Cov} \left( B_1^{H,K}(s), B_1^{H,K}(t) \right) \leq c_{4,8} \sum_{j=1}^N |s_j - t_j|^{2H_j K_j}. \quad (4.18)$$

Here and in the sequel,  $\det \text{Cov}$  denotes determinant of the covariance matrix.

**Proof** We will make use of the following easily verifiable fact: For any Gaussian random vector  $(Z_1, Z_2)$ ,

$$\det \text{Cov}(Z_1, Z_2) = \text{Var}(Z_1) \text{Var}(Z_2|Z_1), \quad (4.19)$$

where  $\text{Var}(Z_1)$  and  $\text{Var}(Z_2|Z_1)$  denote the variance of  $Z_1$  and the conditional variance of  $Z_2$ , given  $Z_1$ , respectively.

By (4.19) we see that for all  $s, t \in [\varepsilon, 1]^N$ ,

$$\begin{aligned} \det \text{Cov} \left( B_1^{H,K}(s), B_1^{H,K}(t) \right) &= \mathbb{E} \left[ B_1^{H,K}(s)^2 \right] \text{Var} \left( B_1^{H,K}(t) | B_1^{H,K}(s) \right) \\ &\leq \mathbb{E} \left[ B_1^{H,K}(s)^2 \right] \mathbb{E} \left[ \left( B_1^{H,K}(s) - B_1^{H,K}(t) \right)^2 \right]. \end{aligned} \quad (4.20)$$

Since  $\text{Var}(B_1^{H,K}(s))$  is bounded from above and below by positive and finite constants, it is sufficient to prove the upper bound in (4.17) and the lower bound in (4.18).

When  $N = 1$ , Lemma 2.3, Proposition 2.1 and (4.19) imply that both (4.17) and (4.18) hold. Next we show that, if the lemma holds for any  $B^{H,K}$  with at most  $n$  parameters, then it holds for  $B^{H,K}$  with  $n + 1$  parameters.

We verify the upper bound in (4.17) first. For any  $s, t \in [\varepsilon, 1]^{n+1}$ , let  $s' = (s_1, \dots, s_n, t_{n+1})$ . Then we have

$$\begin{aligned} \mathbb{E} \left[ \left( B_1^{H,K}(s) - B_1^{H,K}(t) \right)^2 \right] &\leq 2\mathbb{E} \left[ \left( B_1^{H,K}(s) - B_1^{H,K}(s') \right)^2 \right] \\ &\quad + 2\mathbb{E} \left[ \left( B_1^{H,K}(s') - B_1^{H,K}(t) \right)^2 \right]. \end{aligned} \quad (4.21)$$

For the first term, we note that whenever  $s_1, \dots, s_n \in [\varepsilon, 1]$  are fixed,  $B^{H,K}$  is a (rescaled) bifractional Brownian motion in  $s_{n+1}$ . Hence Lemma 2.3 implies the first term in the right-hand side of (4.21) is bounded by  $c|t_n - s_n|^{2H_{n+1}K_{n+1}}$ , where the constant  $c$  is independent of  $s_1, \dots, s_n \in [\varepsilon, 1]$ . On the other hand, when  $t_{n+1} \in [\varepsilon, 1]$  is fixed,  $B^{H,K}$  is a (rescaled)  $(N, d)$ -bifractional Brownian sheet. Hence the induction hypothesis implies the second term in the right-hand side of (4.21) is bounded by  $c \sum_{j=1}^n |t_j - s_j|^{2H_j K_j}$ . This and (4.21) together prove the upper bound in (4.17).

Suppose the lower bound in (4.18) holds for any  $B^{H,K}$  with at most  $n$  parameters. For  $N = n + 1$ , we write  $\det\text{Cov}(B_1^{H,K}(s), B_1^{H,K}(t))$  as

$$\begin{aligned} &\prod_{j=1}^{n+1} t_j^{2H_j K_j} s_j^{2H_j K_j} - \prod_{j=1}^{n+1} \frac{1}{2^{2K_j}} \left[ (t_j^{2H_j} + s_j^{2H_j})^{K_j} - |t_j - s_j|^{2H_j K_j} \right]^2 \\ &= \prod_{j=2}^{n+1} t_j^{2H_j K_j} s_j^{2H_j K_j} \left\{ s_1^{2H_1 K_1} t_1^{2H_1 K_1} - \frac{1}{2^{2K_1}} \left[ (t_1^{2H_1} + s_1^{2H_1})^{K_1} - |t_1 - s_1|^{2H_1 K_1} \right]^2 \right\} \\ &\quad + \frac{1}{2^{2K_1}} \left[ (t_1^{2H_1} + s_1^{2H_1})^{K_1} - |t_1 - s_1|^{2H_1 K_1} \right]^2 \\ &\quad \times \left\{ \prod_{j=2}^{n+1} t_j^{2H_j K_j} s_j^{2H_j K_j} - \prod_{j=2}^{n+1} \frac{1}{2^{2K_j}} \left[ (t_j^{2H_j} + s_j^{2H_j})^{K_j} - |t_j - s_j|^{2H_j K_j} \right]^2 \right\} \\ &\geq c \sum_{j=1}^{n+1} |s_j - t_j|^{2H_j K_j} \end{aligned} \quad (4.22)$$

where the last inequality follows from the induction hypothesis. This proves the lower bound in (4.18).  $\square$

Applying Lemma 4.8, we can prove that many results in Xiao and Zhang (2002), Ayache and Xiao (2005) on sample path properties of fractional Brownian sheet also hold for  $B^{H,K}$  as well. Theorem 4.9 is concerned with the existence of local times of  $B^{H,K}$ .

**Theorem 4.9** Let  $B^{H,K} = \{B^{H,K}(t), t \in \mathbb{R}_+^N\}$  be an  $(N, d)$ -bifractional Brownian sheet with parameters  $H \in (0, 1)^N$  and  $K \in (0, 1]^N$ . If  $d < \sum_{j=1}^N \frac{1}{H_j K_j}$  then for any  $N$ -dimensional closed interval  $I \subset (0, \infty)^N$ ,  $B^{H,K}$  has a local time  $L(x, I)$ ,  $x \in \mathbb{R}^d$ . Moreover, the local time admits the following  $L^2$ -representation

$$L(x, I) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle y, x \rangle} \int_I e^{i\langle y, B^{H,K}(s) \rangle} ds dy, \quad x \in \mathbb{R}^d. \quad (4.23)$$

**Remark 4.10** Although the existence of local times can also be proved by using the Malliavin calculus [see Proposition 4.15 below], we prefer to provide a Fourier analytic proof because: 1) we can compare in this way the two methods and 2) the above theorem gives in addition the representation (4.23).

**Proof** Without loss of generality, we may assume that  $I = [\varepsilon, 1]^N$  where  $\varepsilon > 0$ . Let  $\lambda_N$  be the Lebesgue measure on  $I$ . We denote by  $\mu$  the image measure of  $\lambda_N$  under the mapping  $t \mapsto B^{H,K}(t)$ . Then the Fourier transform of  $\mu$  is

$$\widehat{\mu}(\xi) = \int_I e^{i\langle \xi, B^{H,K}(t) \rangle} dt. \quad (4.24)$$

It follows from Fubini's theorem and (4.17) that

$$\begin{aligned} \mathbb{E} \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 d\xi &= \int_I \int_I \int_{\mathbb{R}^d} \mathbb{E} \left( e^{i\langle \xi, B^{H,K}(s) - B^{H,K}(t) \rangle} \right) d\xi ds dt \\ &= c \int_I \int_I \frac{1}{[\mathbb{E}(B_1^{H,K}(s) - B_1^{H,K}(t))^2]^{d/2}} ds dt \\ &\leq c \int_I \int_I \frac{1}{[\sum_{j=1}^N |s_j - t_j|^{2H_j K_j}]^{d/2}} ds dt. \end{aligned} \quad (4.25)$$

The same argument in Xiao and Zhang (2002, p. 214) shows that the last integral is finite whenever  $d < \sum_{j=1}^N \frac{1}{H_j K_j}$ . Hence, in this case,  $\widehat{\mu} \in L^2(\mathbb{R}^d)$  a.s. and Theorem 4.9 follows from the Plancherel theorem.  $\square$

**Remark 4.11** Recently, Ayache, Wu and Xiao (2006) have shown that fractional Brownian sheets have jointly continuous local times based on the ‘‘sectorial local nondeterminism’’. It would be interesting to prove that  $B^{H,K}$  is sectorially locally nondeterministic and to establish joint continuity and sharp Hölder conditions for the local times of  $B^{H,K}$ .

Now we consider the Hausdorff and packing dimensions of the image, graph and level set of  $B^{H,K}$ . In order to state our theorems conveniently, we assume

$$0 < H_1 K_1 \leq \dots \leq H_N K_N < 1. \quad (4.26)$$

We denote packing dimension by  $\dim_P$ ; see Falconer (1990) for its definition and properties. The following theorems can be proved by using Lemma 4.8 and the same arguments as in Ayache and Xiao (2005, Section 3). We leave the details to the interested reader.

**Theorem 4.12** *With probability 1,*

$$\dim_{\mathbb{H}} B^{H,K}([0, 1]^N) = \dim_{\mathbb{P}} B^{H,K}([0, 1]^N) = \min \left\{ d; \sum_{j=1}^N \frac{1}{H_j K_j} \right\} \quad (4.27)$$

and

$$\begin{aligned} \dim_{\mathbb{H}} \text{Gr} B^{H,K}([0, 1]^N) &= \dim_{\mathbb{P}} \text{Gr} B^{H,K}([0, 1]^N) \\ &= \begin{cases} \sum_{j=1}^N \frac{1}{H_j K_j} & \text{if } \sum_{j=1}^N \frac{1}{H_j K_j} \leq d, \\ \sum_{j=1}^k \frac{H_k K_k}{H_j K_j} + N - k + (1 - H_k K_k)d & \text{if } \sum_{j=1}^{k-1} \frac{1}{H_j K_j} \leq d < \sum_{j=1}^k \frac{1}{H_j K_j}, \end{cases} \end{aligned} \quad (4.28)$$

where  $\sum_{j=1}^0 \frac{1}{H_j K_j} := 0$ .

**Theorem 4.13** *Let  $L_x = \{t \in (0, \infty)^N : B^{H,K}(t) = x\}$  be the level set of  $B^{H,K}$ . The following statements hold:*

- (i) *If  $\sum_{j=1}^N \frac{1}{H_j} < d$ , then for every  $x \in \mathbb{R}^d$  we have  $L_x = \emptyset$  a.s.*
- (ii) *If  $\sum_{j=1}^N \frac{1}{H_j} > d$ , then for any  $x \in \mathbb{R}^d$  and  $0 < \varepsilon < 1$ , with positive probability*

$$\begin{aligned} \dim_{\mathbb{H}} (L_x \cap [\varepsilon, 1]^N) &= \dim_{\mathbb{P}} (L_x \cap [\varepsilon, 1]^N) \\ &= \min \left\{ \sum_{j=1}^k \frac{H_k}{H_j} + N - k - H_k d, 1 \leq k \leq N \right\} \\ &= \sum_{j=1}^k \frac{H_k}{H_j} + N - k - H_k d, \quad \text{if } \sum_{j=1}^{k-1} \frac{1}{H_j} \leq d < \sum_{j=1}^k \frac{1}{H_j}. \end{aligned} \quad (4.29)$$

#### 4.4 A Malliavin calculus approach

Using the Malliavin calculus approach, we can study the local times of more general bifractional Brownian sheets. Consider the  $(N \times d)$ -matrices

$$\overline{H} = (\overline{H}_1, \dots, \overline{H}_d) \quad \text{and} \quad \overline{K} = (\overline{K}_1, \dots, \overline{K}_d),$$

where for any  $i = 1, \dots, d$

$$\overline{H}_i = (H_{i,1}, \dots, H_{i,N}) \quad \text{and} \quad \overline{K}_i = (K_{i,1}, \dots, K_{i,N})$$

with  $H_{i,j} \in (0, 1)$  and  $K_{i,j} \in (0, 1]$  for every  $i = 1, \dots, d$  and  $j = 1, \dots, N$ .

We will say that the Gaussian field  $B^{\overline{H}, \overline{K}}$  is an  $(N, d)$ -bifractional Brownian sheet with indices  $\overline{H}$  and  $\overline{K}$  if

$$B^{\overline{H}, \overline{K}}(t) = \left( B^{\overline{H}_1}(t), \dots, B^{\overline{H}_d}(t) \right), \quad t \in [0, \infty)^N$$

and for every  $i = 1, \dots, d$ , the process  $\{B^{\overline{H}_i}(t), t \in \mathbb{R}_+^N\}$  is centered and has covariance function

$$\mathbb{E} \left( B^{\overline{H}_i, \overline{K}_i}(t) B^{\overline{H}_i, \overline{K}_i}(s) \right) = R^{\overline{H}_i, \overline{K}_i}(s, t) = \prod_{j=1}^N R^{H_{i,j}, K_{i,j}}(s_j, t_j).$$

As in subsection 4.1, the local time  $L(x, t)$  ( $t \in \mathbb{R}_+^N$  and  $x \in \mathbb{R}^d$ ) of  $B^{\overline{H}, \overline{K}}$  is defined as the density of the occupation measure  $\mu_t$ , defined by

$$\mu_t(A) = \int_{[0, t]} \mathbb{1}_A(B^{\overline{H}, \overline{K}}(s)) ds, \quad A \in \mathcal{B}(\mathbb{R}^d).$$

Formally, we can write

$$L(x, t) = \int_{[0, t]} \delta_x(B^{\overline{H}, \overline{K}}(s)) ds,$$

where  $\delta_x$  denotes the Dirac function and  $\delta_x(B_s^{\overline{H}, \overline{K}})$  is therefore a distribution in the Watanabe sense (see Watanabe (1984)).

We need some notation. For  $x \in \mathbb{R}$ , let  $p_\sigma(x)$  be the centered Gaussian kernel with variance  $\sigma > 0$ . Consider also the Gaussian kernel on  $\mathbb{R}^d$  given by

$$p_\sigma^d(x) = \prod_{i=1}^d p_\sigma(x_i), \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

Denote by  $\mathbf{H}_n(x)$  the  $n$ -th Hermite polynomial defined by  $\mathbf{H}_0(x) = 1$  and for  $n \geq 1$ ,

$$\mathbf{H}_n(x) = \frac{(-1)^n}{n!} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \exp\left(-\frac{x^2}{2}\right), \quad x \in \mathbb{R}.$$

We will make use of the following technical lemma.

**Lemma 4.14** *For any  $H \in (0, 1)$  and  $K \in (0, 1]$ , let us define the function*

$$Q_{H,K}(z) = \frac{R^{H,K}(1, z)}{z^{HK}}, \quad z \in (0, 1]$$

and  $Q_{H,K}(0) = 0$ . Then the function  $Q_{H,K}$  takes values in  $[0, 1]$ ,  $Q_{H,K}(1) = 1$  and it is strictly increasing. Moreover, there exists  $\delta > 0$  such that for all  $z \in (1 - \delta, 1)$ ,

$$(Q_{H,K}(z))^n \leq \exp(-c(\delta, H, K)n(1 - z)^{2H}). \quad (4.30)$$

**Proof** Clearly, the Cauchy-Schwarz inequality implies  $0 \leq Q_{H,K}(z) \leq 1$ . Let us prove that the function  $Q_{H,K}$  is strictly increasing. By computing the derivative  $Q'_{H,K}(z)$  and multiplying this by  $z^{HK+1}$ , we observe that this is equivalent to show

$$(1-z)^{2HK-1}(1+z) - (1+z^{2H})^{K-1}(1-z^{2H}) > 0 \quad \text{for all } z \in (0,1). \quad (4.31)$$

If  $HK \leq \frac{1}{2}$ , since  $(1+z^{2H})^{K-1} \leq 1+z$ , the left side in (4.31) can be minorized by  $(1+z^{2H})^K((1-z)^{2HK-1} - 1 + z^{2H})$  and this is positive since  $(1-z)^{2HK-1} \geq 1$ .

If  $HK > \frac{1}{2}$ , we note that

$$\begin{aligned} (1-z)^{2HK-1}(1+z) + (1+z^{2H})^{K-1}z^{2H} &\geq (1-z)(1+z) + (1+z^{2H})^{K-1}z^2 \\ &\geq (1+z^{2H})^{K-1}(1-z^2) + (1+z^{2H})^{K-1}z^2 \geq (1+z^{2H})^{K-1}. \end{aligned}$$

and this implies (4.31). Concerning the inequality (4.30), we note that

$$Q_{H,K}(z)^n = \exp(n \log Q_{H,K}(z)) \geq \exp(-n(1 - Q_{H,K}(z))).$$

Now by Taylor's formula

$$(1+z^{2H})^K z^{-HK} \leq 2^K + c(H, K, \delta)(1-z)^2$$

and therefore

$$\begin{aligned} Q_{H,K}(z) &\leq 1 + c(H, K, \delta)(1-z)^2 - \frac{1}{2^K}(1-z)^{2HK} \\ &\leq 1 + c(H, K, \delta)(1-z)^{2HK} \delta^{2-2HK} - \frac{1}{2^K}(1-z)^{2HK}. \end{aligned}$$

The conclusion follows as in the proof of Lemma 2 in Eddahbi et al. (2005), since

$$1 - Q_{H,K}(z) \geq \frac{1}{2^K}(1-z)^{2HK}(1 - c(H, K, \delta))$$

for any  $z \in (1-\delta, 1)$  with  $\delta$  close to zero and with  $c(H, K, \delta)$  tending to zero as  $\delta \rightarrow 0$ .  $\square$

The following proposition gives a chaotic expansion of the local time of the  $(N, d)$ -bifractional Brownian sheet. The stochastic integral  $I_n(h)$  appeared below is the multiple Wiener-Itô integral of order  $n$  of the function  $h$  of  $nN$  variables with respect to an  $(N, 1)$  bifractional Brownian motion with parameters  $H = (H_1, \dots, H_N)$  and  $K = (K_1, \dots, K_N)$ . Recall that such integrals can be constructed in general on a Gaussian space [see, for example, Major (1981), or Nualart (1995)]. We will only need the following isometry formula:

$$\mathbb{E} \left( I_n(\mathbb{1}_{[0,t]}^{\otimes n}) I_m(\mathbb{1}_{[0,s]}^{\otimes m}) \right) = R^{H,K}(t, s)^n \mathbb{1}_{(n=m)} = \prod_{j=1}^N (R^{H_j, K_j}(t_j, s_j))^n \mathbb{1}_{(n=m)} \quad (4.32)$$

for all  $s, t \in \mathbb{R}_+^N$ .

**Proposition 4.15** For any  $x \in \mathbb{R}^d$  and  $t \in (0, \infty)^N$ , the local times  $L(x, t)$  admits the following chaotic expansion

$$L(x, t) = \sum_{n_1, \dots, n_d \geq 0} \int_{[0, t]} \prod_{i=1}^d \frac{p_{\underline{s}^{2\bar{H}_i \bar{K}_i}}(x_i)}{\underline{s}^{n_i \bar{H}_i \bar{K}_i}} \mathbf{H}_{n_i} \left( \frac{x_i}{\underline{s}^{\bar{H}_i}} \right) I_{n_i}^i (\mathbb{1}_{[0, s]}(\cdot)^{\otimes n_i}) ds, \quad (4.33)$$

where  $\underline{s} = s_1 \cdots s_N$  and  $\underline{s}^{\bar{H}_i \bar{K}_i} = \prod_{j=1}^N s_j^{H_{i,j} K_{i,j}}$ . The integrals  $I_{n_i}^i$  denotes the multiple Itô stochastic integrals with respect to the independent  $N$ -parameter bifractional Brownian motion  $B^{\bar{H}_i, \bar{K}_i}$ .

Moreover, if  $\sum_{j=1}^N \frac{1}{H_j^* K_j^*} > d$ , where  $H_j^* = \max\{H_{i,j} : i = 1, \dots, d\}$  and  $K_j^* = \max\{K_{i,j} : i = 1, \dots, d\}$ , then  $L(x, t)$  is a random variable in  $L^2(\Omega)$ .

**Proof** The chaotic expression (4.33) can be obtained similarly as in Eddahbi et al. (2005) or Russo and Tudor (2006). It is based on the approximation of the Dirac delta function by Gaussian kernels with variance converging to zero. Let us evaluate the  $L^2(\Omega)$  norm of  $L(x, t)$ . By the independence of components and the isometry of multiple stochastic integrals, we obtain

$$\|L(x, t)\|_2^2 = \sum_{m \geq 0} \sum_{n_1 + \dots + n_d = m} \int_{[0, t]} du \int_{[0, t]} dv \prod_{i=1}^d \beta_{n_i}(u) \beta_{n_i}(v) R^{\bar{H}_i, \bar{K}_i}(u, v)^{n_i}, \quad (4.34)$$

where

$$\beta_{n_i}(u) = \frac{p_{\underline{s}^{2\bar{H}_i \bar{K}_i}}(x_i)}{\underline{s}^{n_i \bar{H}_i \bar{K}_i}} \mathbf{H}_{n_i} \left( \frac{x_i}{\underline{s}^{\bar{H}_i}} \right)$$

By Propositions 3 and 6 in Imkeller et al. (1995) [see also Lemma 11 in Eddahbi et al. (1995)], we have the bound

$$\beta_{n_i}(u) \beta_{n_i}(v) \leq c_{4,9} \frac{1}{(n_i \vee 1)^{\frac{8\beta-1}{6}}} \frac{1}{\underline{u}^{n_i \bar{H}_i \bar{K}_i} \underline{v}^{n_i \bar{H}_i \bar{K}_i}} \quad (4.35)$$

for any  $\beta \in [\frac{1}{4}, \frac{1}{2})$ . Using the inequality (4.35), we derive from (4.34) that  $\|L(x, t)\|_2^2$  is at most

$$\begin{aligned} & c \sum_{m \geq 0} \sum_{n_1 + \dots + n_d = m} \left( \prod_{i=1}^d \frac{1}{(n_i \vee 1)^{\frac{8\beta-1}{6}}} \right) \int_{[0, t]} du \int_{[0, u]} dv \prod_{i=1}^d \prod_{j=1}^N \frac{R^{H_{i,j}, K_{i,j}}(u_j, v_j)^{n_i}}{(u_j v_j)^{n_i H_{i,j} K_{i,j}}} \\ & = c \sum_{m \geq 0} \sum_{n_1 + \dots + n_d = m} \left( \prod_{i=1}^d \frac{1}{(n_i \vee 1)^{\frac{8\beta-1}{6}}} \right) \prod_{j=1}^N \int_0^{t_j} u_j du_j \int_0^1 \left( \prod_{i=1}^d Q_{H_{i,j}, K_{i,j}}(z)^{n_i} \right) dz \quad (4.36) \\ & = c_{4,10} \underline{t}^2 \sum_{m \geq 0} \sum_{n_1 + \dots + n_d = m} \left( \prod_{i=1}^d \frac{1}{(n_i \vee 1)^{\frac{8\beta-1}{6}}} \right) \prod_{j=1}^N \int_0^1 \left( \prod_{i=1}^d Q_{H_{i,j}, K_{i,j}}(z)^{n_i} \right) dz, \end{aligned}$$

where we used the change of variables  $u_j = u_j$  and  $v_j = z_j u_j$ . Using the above lemma and as in the proof of Lemma 2 in Eddahbi et al. (2005), we can prove the bound

$$\int_0^1 \left( \prod_{i=1}^d Q_{H_{i,j}, K_{i,j}}(z)^{n_i} \right) dz \leq c_{4,11} m^{-\frac{1}{2H_j^* K_j^*}}. \quad (4.37)$$

Here  $c_{4,11} = c_{4,11}(\overline{H}, \overline{K})$  depends on  $\overline{H}, \overline{K}$ . Finally, (4.37) implies that

$$\begin{aligned} \|L(x, t)\|_2^2 &\leq c_{4,12} \sum_{m \geq 1} \left( \prod_{j=1}^N m^{-\frac{1}{2H_j^* K_j^*}} \right) \sum_{n_1 + \dots + n_d = m} \left( \prod_{i=1}^d \frac{1}{(n_i \vee 1)^{\frac{8\beta-1}{6}}} \right) \\ &\leq c_{4,13} \sum_{m \geq 1} m^{-\sum_{j=1}^N \frac{1}{2H_j^* K_j^*} + d(1 - \frac{8\beta-1}{6}) - 1}, \end{aligned} \quad (4.38)$$

where  $c_{4,12}$  and  $c_{4,13}$  depend on  $\overline{H}, \overline{K}$  and  $\underline{t}$  only. The last series in (4.38) converges if

$$\sum_{j=1}^N \frac{1}{2H_j^* K_j^*} > d \left( 1 - \frac{8\beta-1}{6} \right). \quad (4.39)$$

To conclude, observe that by choosing  $\beta$  close to  $\frac{1}{2}$ ,  $\sum_{j=1}^N \frac{1}{H_j^* K_j^*} > d$  implies the required condition (4.39).  $\square$

We recall that a random variable  $F = \sum_n I_n(f_n)$  belongs to the Watanabe space  $\mathbb{D}^{\alpha,2}$  if

$$\|F\|_{\alpha,2}^2 := \sum_{n \geq 0} (1+m)^\alpha \|I_n(f_n)\|_2^2 < \infty.$$

**Corollary 4.16** *For any  $t \in (0, \infty)^N$  and  $x \in \mathbb{R}^d$ , the local time  $L(x, t)$  of the  $(N, d)$ -bifractional Brownian sheet  $B^{\overline{H}, \overline{K}}$  belongs to the Watanabe space  $\mathbb{D}^{\alpha,2}$  for every  $0 < \alpha < \sum_{j=1}^N \frac{1}{2H_j^* K_j^*} - \frac{d}{2}$ .*

**Proof** This is a consequence of the proof of Proposition 4.15. Using the computation contained there, we obtain for any  $\beta \in [\frac{1}{4}, \frac{1}{2})$ ,

$$\|L(x, t)\|_{\alpha,2}^2 \leq c_{4,14}(\overline{H}, \overline{K}, d, t) \sum_{m \geq 1} (1+m)^\alpha m^{d(1 - \frac{8\beta-1}{6}) - 1 - \sum_{j=1}^N \frac{1}{2H_j^* K_j^*}}$$

which is convergent if  $\alpha < \sum_{j=1}^N \frac{1}{2H_j^* K_j^*} - d(1 - \frac{8\beta-1}{6}) - 1 - \sum_{j=1}^N \frac{1}{2H_j^* K_j^*}$ . Choosing  $\beta$  close to  $\frac{1}{2}$ , we get the conclusion.  $\square$

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