COMPLEMENTS on STOCHASTIC CALCULUS (1).

Conditional expectation:

1. Recall that two processes X and Y on $(\Omega, \mathcal{F}, P)$ ont the same finite dimensional distributions if

$$\forall\, 0 \leq t_1 < t_2 < \ldots < t_n < \infty, (X_{t_1}, \ldots, X_{t_n})\ et\ (Y_{t_1}, \ldots, Y_{t_n})$$

   have the same law.

   Show that if two processes are modifications one of the other one, then they have the same finite dimensional distributions.

2. If (X, Y) is a random vector with density $f_{(X,Y)}$ such that $\forall x, y \in R : f_{(X,Y)}(x, y) > 0$, and if $\mathcal{B} := \sigma(Y)$;
   1. Show that $E[X|\mathcal{B}] = g(Y)$, où

$$g : y \to g(y) := \frac{\int_{\mathbb{R}} x f_{(X,Y)}(x, y)dx}{\int_{\mathbb{R}} f_{(X,Y)}(x, y)dx}$$

   2. If (X, Y) is a Gaussian vector such that $E[X] = E[Y] = 0$, $var[X] = 1 = var[Y]$ et $cov[X, Y] = \rho \in [0, 1]$, calculate $E[X|\mathcal{B}]$.

3. If $\mathcal{B} \subset \mathcal{C} \subset \mathcal{F}$ and $X \in L^2(\Omega, \mathcal{F}, P)$
   1. Prove that $E[X|\mathcal{B}] = E[(E[X|\mathcal{C}])|\mathcal{B}]$.
   2. Prove that the above relation is not true when $\mathcal{B}$ is not included in $\mathcal{C}$ by using the point 2 in the previous exercise with $\mathcal{B} := \sigma(X)$ and $\mathcal{C} := \sigma(Y)$.

4. The law of the random vector $X = (X_1, \ldots, X_n)$ is called exchangeable if for any permutation π of $\{1, \ldots, n\}$ the vectors X et X_π ont the same law, where $X_\pi := (X_{\pi(1)}, \ldots, X_{\pi(n)})$.

   If the law of X is exchangeable , and if $S := X_1 + \ldots + X_n$, calculate $E[X_1|S]$ ($E[X_1|S]$ is a notation for $E[X_1|\sigma(S)]$).

5. If $Z_1, Z_2, \ldots$ is a i.i.d. sequence of random variables in $\mathcal{N}(0, 1)$, if $\mathcal{F}_n := \sigma(Z_1, \ldots, Z_n)$ and if $X_n := Z_1 + \ldots + Z_n$,
   1. Show that $\forall n \geq m : E[X_n|\mathcal{F}_m] = X_m$.
   2. Show that $\forall n \geq m : E[Y_n|\mathcal{F}_m] = Y_m$, where $Y_n := X_n^2 - n$.
   3. Show that $\forall n \geq m : E[M_n|\mathcal{F}_m] = M_m$, where $M_n := \exp(X_n - n/2)$.

   (that means that X, Y and M are martingales.)

Brownian motion and related topics:

1. Prove that if B is a Brownian motion, then $cov(B_s, B_t) = s \wedge t$, where $s \wedge t$ denotes $\min(s, t)$.

   Prove that if X is a continuous centered Gaussian process and if $\forall s, t \geq 0 : cov(X_s, X_t) = s \wedge t$, then B is a Brownian motion for its natural filtration. If X is a centered Gaussian process with covariance $cov(X_s, X_t) = s \wedge t$ then it is a.s. a Brownian motion.

2. Let $(X_t)_{t\geq 0}$ be a continuous adapted process. Show that if

$$\lim_{|\Delta|\to 0} T_t^{\Delta,p}(X) = L_t$$

in probability, where L_t is a r.v. with values in $[0,\infty[$ then

$$\forall q > p, \lim_{|\Delta|\to 0} T_t^{\Delta,q}(X) = 0$$

in probability and

$$\forall 0 < q < p, \lim_{|\Delta|\to 0} T_t^{\Delta,q}(X) = \infty$$

in probability on the set $(L_t > 0)$.

Deduce (from the quadratic variation) that the trajectories of the Brownian motion are not with bounded variation.

3. The fractional Brownian motion is a centered Gaussian process B^H with covariance

$$E(B_t^H B_s^H) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

where $H \in (0,1)$ is called the Hurst index . Show that if $H = \frac{1}{2}$ we retrieve the Brownian motion. Prove that B^H is H -self-similar, i.e. for every $c > 0$, $c^H B_{ct}, t \geq 0$ is again a fractional Brownian motion. Prove that this process has a.s. continuous paths.

4. (Stopping times)

   a. Prove that the sum of two stopping times is a stopping time.

   b. If $(\tau_n)_{n\geq 1}$ is a sequence of stopping times for a right-continuous filtration, then

   $$\sup_{n\geq 1} \tau_n, \inf_{n\geq 1} \tau_n$$

   are stopping times.

5. Let T be a $\mathcal{F}_t$ stopping time. Consider the sequence $(T_n)_{n\geq 1}$ defined by $T_n(\omega) = \frac{k}{2^n}$ on the set $\{\omega / \frac{k-1}{2^n} \leq T(\omega) < \frac{k}{2^n}\}$ et $T(\omega) = T(\omega)$ on $\{T = \infty\}$ for $n \geq 1$, $k \geq 1$.

   a. Show that $T_n \geq T_{n+1} \geq T$ for every n.

   b. Show that T_n is a $\mathcal{F}_t$ stopping time for every n.

   c. Show that $\lim_{n\to\infty} T_n = T$.

6. Let B be a Brownian motion. We define

$$L = \{(t,\omega) \in [0,\infty[\times \Omega; B_t(\omega) = 0\}.$$

For fixed ω we set

$$L_\omega = \{0 \leq t < \infty; B_t(\omega) = 0\}.$$

   a. Show that L is included in the σ algebra $\mathcal{B}_{[0,\infty[} \otimes \mathcal{F}$.

   b. Show that L_ω is of zero Lebesque measure.

7. (Exam December 2006) Consider the centered Gaussian process $(S_t^H)_{t\geq 0}$ with covariance

$$R(t,s) = s^H + t^H - \frac{1}{2}\left((s+t)^H + |t-s|^H\right), \qquad s,t \geq 0$$

avec $H \in]0,2[$.

   a. Show that if $H = 1$ the process S^H is a.s. a BM for its natural filtration.

   b. Show that S^H is a self-similar process of order $\frac{H}{2}$.

c. Calculate $E\left(S_t^H - S_s^H\right)^2$. Show that

$$E\left(S_t^H - S_s^H\right)^2 \leq |t-s|^H, \qquad \text{si } H > 1$$

and

$$E\left(S_t^H - S_s^H\right)^2 \leq (2 - 2^{H-1})|t-s|^H, \qquad \text{si } H < 1.$$

d. Show that the trajectories of the process S^H are almost surely continuous. Are these trajectories holderian? with which order?

e. Compute the quadratic variation of the process S^H, i.e. for every $t \geq 0$ find the limit in probability when $|\Delta| \to 0$, of the sequence

$$T_t^{\Delta}(S^H) = \sum_{i=0}^{n-1} \left(S_{t_{i+1}}^H - S_{t_i}^H\right)^2$$

where $\Delta : 0 = t_0 < t_1 < \ldots < t_n = t$ is a partition of $[0,t]$. Distinguish the cases $H > 1$, $H = 1$ and $H < 1$.

f. Deduce that if $H > 1$ the process S^H is not a martingale.

Martingales and Itô integral

1. (the Poisson process) *A Poisson process with parameter (intensity)* $\lambda > 0$ is an adapted cadlag stochastic process $(N_t)_{t\geq 0}$ such that $N_0 = 0$ a.s. and for every $0 \leq s \leq t$, $N_t - N_s$ is independent by $\mathcal{F}_s$ and follows a Poisson law with parameter $\lambda(t-s)$. *The compensated Poisson process* is given by, for every $t \geq 0$

$$\tilde{N}_t = N_t - \lambda t.$$

a. Prove that $\tilde{N}$ is a martingale.

b. Prove that $\tilde{N}_t^2 - \lambda t$ is a martingale.

2. Show that, if X is a $\mathcal{F}_t$ -adapted process in L^1, then X is a martingale if and only if for every stopping time T, we have

$$E(X_T) = E(X_0).$$

3. Prove the following properties of the semimartingale's bracket:

i. if $Y = \int_0^{\cdot} b_s ds$ where $b \in H_1^{loc}$, then $\langle Y, Y \rangle = 0$.

i. if $X = \int_0^{\cdot} a_s dB_s$ where $a \in H_2^{loc}$, then $T^{\Delta}(X,Y) \to 0$ in probability. (i.e. $\langle X, Y \rangle = 0$).

iii. if $X = X_0 + \int_0^{\cdot} a_t dB_t + \int_0^{\cdot} b_t dt$ is a semi-martingale, then $\langle X, X \rangle_t = \int_0^t a_s^2 ds$.

iv. We have $\langle X, Y \rangle = \frac{1}{4}(\langle X+Y, X+Y \rangle + \langle X-Y, X-Y \rangle)$ and the angle bracket $\langle X, Y \rangle$ of two semi-martingales $X = X_0 + \int_0^{\cdot} a_t dB_t + \int_0^{\cdot} b_t dt$ and $Y = Y_0 + \int_0^{\cdot} a_t' dB_t + \int_0^{\cdot} b_t' dt$ is given by the bracket of the "two martingale parts"

$$\langle X, Y \rangle_t = \int_0^t a_s a_s' ds.$$

v. Suppose that B_1 et B_2 are two $\{\mathcal{F}_t\}$-independent Brownian motions. Compute $\langle B_1 + B_2, B_1 + B_2 \rangle$, $\langle B_1 - B_2, B_1 - B_2 \rangle$ and finnally show that $\langle B_1, B_2 \rangle = 0$.

4. (exam december 2007) Let $T > 0$ and let $(B_t)_{t\in[0,T]}$ be a $\mathcal{F}_t$ Brownian motion.

a. Express the process $(tB_t)_{t\in[0,T]}$ as the sum of an Itô integral and a finite variation stochastic process.

b. Show that the stochastic process $(X_t)_{t\in[0,T]}$ given by

$$X_t = B_t^3 - 3tB_t, \qquad t \in [0,T]$$

is a $\mathcal{F}_t$ martingale. Write, for every $t \in [0,T]$, X_t in the form $X_t = \int_0^t \alpha_s dB_s$, for every $t \in [0,T]$, where the process $(\alpha_t)_{t\in[0,T]}$ will be written.

c. Consider the stochastic process $(Y_t)_{t\in[0,T]}$ given by

$$Y_t^u = X_t + \int_0^t u_s ds, \qquad t \in [0,T].$$

where the stochastic process $(u_t)_{t\in[0,T]}$ is such that $E\int_0^T |u_s|ds < \infty$. Show that Y is a martingale if and only if $u_t(\omega) = 0$ for almost all (t,ω).

d. Consider the process $(Z_t)_{t\in[0,T]}$ defined by $Z_t = e^{-B_t - \frac{t}{2}}$. Show that Z is a martingale.

e. Prove that the process $(Y_t^\alpha Z_t)_{t\in[0,T]}$ is a martingale, where α is the stochastic process appearing in point 2. above.

f. Show that the process $(B_t^4 - 6tB_t^2 + 3t^2)_{t\in[0,T]}$ is also a martingale and it can be written as $B_t^4 - 6tB_t^2 + 3t^2 = 4\int_0^t X_s dB_s$, for every $t \in [0,T]$.

5. (exam december 2007)For fixed $a, b \in \mathbb{R}$ and for fixed $0 < T < 1$ we define the stochastic process $(Y_t)_{t\in[0,T]}$ by

$$Y_t = a(1-t) + bt + (1-t)\int_0^t \frac{1}{1-s} dB_s, \qquad t \in [0,T]$$

where $(B_t)_{t\in[0,1]}$ is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$.

a. Show that for every $0 \le t \le T$ the random variable $\int_0^t \frac{1}{1-s} dB_s$ is well-defined and it is Gaussian.

b. Show that the process $(X_t)_{t\in[0,T]}$ is Hölder continuous of order δ for any $0 < \delta < \frac{1}{2}$.

c. Prove that the following convergences holds in $L^2(\Omega)$:

$$Y_T \to_{T\to 1} b \text{ and } Y_T \to_{T\to 0} a.$$

d. By using Doob's inequality, show that for any n integer

$$E\left(\sup_{t\in[1-2^{-n},1-2^{-n-1}]} |M_t|\right)^2 \le 4\left(2^{n+1}-1\right)$$

e. Show that for every $\varepsilon > 0$,

$$P\{\sup_{t\in[1-2^{-n},1-2^{-n-1}]} |M_t| > \varepsilon\} \le 2^{-2n}\varepsilon^{-2}4\left(2^{n+1}-1\right).$$

f. Use the Borel-Cantelli Lemma to deduce that $Y_T \to_{T\to 1} b$ almost surely.

6. (exam juin 2008) Let $(W_t)_{t\ge 0}$ be a standard Brownian motion and define for every $t \ge 0$

$$Y_t = \cos(B_t).$$

a. Explain why the integral $\int_0^t \sin(B_s)dB_s$ exists. Is the process $\left(\int_0^t \sin(B_s)dB_s\right)_{t\ge 0}$ a martingale?

b. Show that for every $t \ge 0$ we have

$$Y_t = 1 - \frac{1}{2}\int_0^t Y_s ds - \int_0^t \sin(B_s)dB_s. \tag{1}$$

c. We define the function $u : [0,\infty) \to \mathbf{R}$ by $u(t) = E(\cos(B_t))$. Deduce from the above question that u is differentiable and that for every $t \ge 0$

$$u'(t) = -\frac{1}{2}u(t). \tag{2}$$

d. Prove the relation (2) by direct calculation.

7. We defined the application $J : H_2^2 \to M^2$ as the stochastic integral with respect to a Brownian motion B arbitrary.

a. If B^1 and B^2 are two independent Brownian motions, show that $B^3 := \frac{1}{\sqrt{2}}(B^1 + B^2)$ is still a Brownian motion.

b. To each of these Brownian motion, one can associate a different application $J : H_2^2 \to M^2$ that we will denote by J_1, J_2 et J_3 respectively. Show that $J_3(\phi) = \frac{1}{\sqrt{2}}(J_1(\phi) + J_2(\phi))$. Using the integral notation

$$\int_0^{\cdot} \phi_t dB_t^3 = \frac{1}{\sqrt{2}}(\int_0^{\cdot} \phi_t dB_t^1 + \int_0^{\cdot} \phi_t dB_t^2).$$

8. (exam January 2006) Let $(X_t)_{t\geq 0}$ a stochastic process on a probability space $(\Omega, \mathcal{F}, P)$. We define for every $\alpha \in \mathbb{R}$

$$X_\alpha(t) = e^{\alpha X_t - \frac{\alpha^2}{2}t}, \qquad \forall t \geq 0.$$

a. Show that if $X_t \sim N(0, t)$ for every $t > 0$ then $E(X_\alpha(t)) = 1$ for every $t > 0$.

In the following we will assume that $X = B$ is a $\mathcal{F}_t$ Brownian motion.

b. Prove that the process $(X_\alpha(t))_{t\geq 0}$ is a $\mathcal{F}_t$ martingale.

c. We will accept that $t \to \infty$ $X_\alpha(t)$ converges almost surely to a r.v. x_α. Show that $x_\alpha = 0$ a.s. .

d. Show why $X_\alpha(t)$ does not converge in $L^1(\Omega)$ when $t \to \infty$.

e. Show that the family $(X_\alpha(t))_{t\geq 0}$ is bounded in L^1. In this family uniformly integrable ?

9. (exam January 2006) Let $B = (B_t)_{t\geq 0}$ be a Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$ et $a \in \mathbb{R}$.

a. May we apply the Itô formula to B for $f(x) = \max(x, 0)$?

b. Let us consider the function

$$\rho(x) = c \exp\left[\frac{1}{(x-1)^2 - 1}\right], \qquad \text{pour } 0 < x < 2$$

and $\rho(x) = 0$ otherwise. The constant c is chosen such that $\int_{\mathbb{R}} \rho(s)ds = 1$. We define for every $n \geq 1$,

$$\rho_n(x) = n\rho(nx)$$

and

$$u_n(x) = \int_{-\infty}^{x} \int_{-\infty}^{y} \rho_n(z - a)dzdy, \qquad x \in \mathbb{R}.$$

Show that the functions ρ and ρ_n are continuous and that the function u_n id of class C^2.

c. Write the Itô formula for $u_n(B_t)$.

We will admit in the following that $x \in \mathbb{R}$,

$$\lim_{n\to\infty} u_n'(x) = 1_{]a,\infty[}(x) \text{ et } \lim_{n\to\infty} u_n(x) = \max(x - a, 0).$$

d. Show that when $n \to \infty$, the sequence $\int_0^t u_n'(B_s)dB_s$ converges in $L^2(\Omega)$ to $\int_0^t 1_{]a,\infty[}(B_s)dB_s$ for every $t \geq 0$.

e. We suppose that for every $t \geq 0$, when $n \to \infty$ the sequence $\int_0^t \rho_n(B_s - a)ds$ converges almost surely to a limit $L_t(a)$. For every $t \geq 0$, show that a.s.

$$\max(B_t - a, 0) = \max(-a, 0) + \int_0^t 1_{]a,\infty[}(B_s)dB_s + \frac{1}{2}L_t(a).$$

10. (exam June 2006) Let $T > 0$ and $(W_t)_{t\in[0,T]}$ a $\mathcal{F}_t$-Brownian motion.

a. Explain why

$$A_t = \int_0^t W_s dW_s \tag{3}$$

exists for every $t \in [0, T]$. Which property is satisfied fby $(A_t)_{t\in[0,T]}$?

b. May we have

$$A_t = \int_0^t u_s ds, \qquad \forall t \in [0, T]$$

where $(u_s)_{s\in[0,T]}$ is a process L^1? Justify.

let $\Delta_n, n \geq 1$ be the sequence of simple stochastic processes

$$\Delta_n(t) = \sum_{k=0}^{n-1} W(\frac{k}{n}T) 1_{[\frac{k}{n}T, \frac{k+1}{n}T[}(t), \qquad \forall t \in [0, T[.$$

and $\Delta_n(T) = W_T$. We admit that

$$E \int_0^T |\Delta_n(t) - W_t|^2 \, dt = 0.$$

c. Fix $n \geq 1$ and denote by $W_j = W_{\frac{j}{n}T}$, $j = 0, 1, \ldots n$. Prove that

$$\frac{1}{2} \sum_{j=0}^{n-1} (W_{j+1} - W_j)^2 = \frac{1}{2} W_n^2 + \sum_{j=0}^{n-1} W_j (W_j - W_{j+1}).$$

d. Deduce another expression for the integral A_t given by (3).

e. Write the Itô formula for A_t^2.

11. (exam June 2006)

a. Recall the Itô formula of the type $f(t, X_t)$ where $(X_t)_{t\geq 0}$ is a semimartingale.

Let $(W_t)_{\in[0,1]}$ a $\mathcal{F}_t$ Brownian motion and Z the process given by

$$Z_t = \frac{1}{\sqrt{1-t}} \exp\left(-\frac{W_t^2}{2(1-t)}\right), \qquad t \in [0, 1[.$$

b. Show that Z is a martingale. Calculate $E(Z_t)$ for every $t \in [0, 1[$.

c. Show that Z_t tends to 0 when t tends to 1. Discuss in which sense this convergence holds.

d. Write Z_t of the form

$$Z_t = \exp\left(\int_0^t g(s) dW_s - \frac{1}{2}\int_0^t g(s)^2 ds\right)$$

where the process g is to be written.