# Stochastic evolution equations with fractional Brownian motion

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**Abstract**

In this paper linear stochastic evolution equations driven by infinite-dimensional fractional Brownian motion are studied. A necessary and sufficient condition for the existence and uniqueness of the solution is established and the spatial regularity of the solution is analyzed; separate proofs are required for the cases of Hurst parameter above and below 1/2. The particular case of the Laplacian on the circle is discussed in detail.

**Key words and phrases:** fractional Brownian motion, stochastic partial differential equation, Hurst parameter.

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# 1 Introduction

The recent development of stochastic calculus with respect to fractional Brownian motion (fBm) has led to various interesting mathematical applications, and in particular, several types of stochastic differential equations driven by fBm have been considered in finite dimensions (see among others [8], [7] or [2]). The question of infinite dimensional equations has emerged very recently (see [5], [6]). The purpose of this article is to provide a detailed study of the existence and regularity properties of the stochastic evolution equations with linear additive fractional Brownian noise. Before providing a complete summary of the contents of this article, we comment on the fact that, as in the few published works ([5], [6]) on infinite-dimensional fBm-driven equations, we study only equations in which noise enters linearly. The difficulty with non-linear fBm-driven equations is notorious: the Picard iteration technique involves Malliavin derivatives in such a way that the equations for estimating these derivatives cannot be closed. The preprint [10] treats an equation with fBm multiplied by a nonlinear term; however the noise term has a trace-class correlation, and moreover they treat only the case $H > 1/2$, which allows one to solve the equation using stochastic integrals understood in a pathwise way, not in the Skorohod sense. The general non-linearity issue remains unsolved.

Let $B^H = (B^H_t)_{t\in[0,1]}$ be a fractional Brownian motion on a real and separable Hilbert space $U$. That is, $B^H$ is a $U$-valued centered Gaussian process, starting from zero, defined by its covariance

$$E(B^H(t)B^H(s)) = R(s,t)Q, \qquad \text{for every } s,t \in [0,1]$$

where $Q$ is a self-adjoint and positive operator from $U$ to $U$ and $R$ is the standard covariance structure of one-dimensional fractional Brownian motion (as in (2)). We consider the following stochastic differential equation

$$X(dt) = AX(t)dt + F(X(t))\Phi dB^H(t) \tag{1}$$

and we study the existence, uniqueness, and regularity properties of the solution in several particular cases. The goal is to formulate necessary and sufficient conditions for these properties as conditions on the equations' input parameters $A$, $\Phi$, and $Q$. It is always possible, and usually convenient, to assume that $B^H$ is *cylindrical*, i.e. that $Q$ is the identity operator. We will also translate the conditions for regularity as necessary and sufficient conditions on the almost-sure regularity of $B^H$ itself.

In Section 3 we let $F(u) \equiv 1$ and $A$ a linear operator from another Hilbert space $V$ to $V$ with $\Phi \in \mathcal{L}(U;V)$ a deterministic linear operator not depending on $t$. We give a necessary and sufficient condition for the existence of the solution. The stochastic integral appearing in (1) is a Wiener integral over Hilbert spaces. Our context is more general than the one studied in [6], or in [5], since we consider both cases $H > \frac{1}{2}$ and $H < \frac{1}{2}$. Our study goes further since we prove the sufficiency *and* the necessity of the condition for the existence of the solution. Section 4 contains a study of the space-time regularity of the solution using the so-called factorization method.

# 2 Preliminaries

## 2.1 The Wiener integral with respect to one-dimensional fractional Brownian motion

Consider $T = [0,\tau]$ a time interval with arbitrary fixed horizon $\tau$, and let $(B^H_t)_{t\in T}$ be the one-dimensional fractional Brownian motion with Hurst parameter $H \in (0,1)$. This means by definition that $B^H$ is a centered Gaussian process with covariance

$$R(t,s) = E(B_s^H B_t^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}). \tag{2}$$

Note that $B^{1/2}$ is standard Brownian motion. Moreover $B^H$ has the following Wiener integral representation:

$$B_t^H = \int_0^t K^H(t,s)dW_s, \tag{3}$$

where $W = \{W_t : t \in T\}$ is a Wiener process, and $K^H(t,s)$ is the kernel given by

$$K^H(t,s) = c_H(t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}} F\left(\frac{t}{s}\right) \tag{4}$$

$c_H$ being a constant and

$$F(z) = c_H \left(\frac{1}{2} - H\right) \int_0^{z-1} r^{H-\frac{3}{2}} \left(1 - (1+r)^{H-\frac{1}{2}}\right) dr. \tag{5}$$

From (4) we obtain

$$\frac{\partial K^H}{\partial t}(t,s) = c_H(H - \frac{1}{2})(t-s)^{H-\frac{3}{2}} \left(\frac{s}{t}\right)^{\frac{1}{2}-H}. \tag{6}$$

We will denote by $\mathcal{E}_H$ the linear space of step functions on $T$ of the form

$$\varphi(t) = \sum_{i=1}^n a_i 1_{(t_i, t_{i+1}]}(t) \tag{7}$$

where $t_1, \dots, t_n \in T, n \in \mathbf{N}, a_i \in \mathbb{R}$ and by $\mathcal{H}$ the closure of $\mathcal{E}_H$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R(t,s).$$

For $\varphi \in \mathcal{E}_H$ of the form (7) we define its Wiener integral with respect to the fractional Brownian motion as

$$\int_T \varphi_s dB^H(s) = \sum_{i=1}^n a_i \left(B_{t_{i+1}}^H - B_{t_i}^H\right). \tag{8}$$

Obviously, the mapping

$$\varphi = \sum_{i=1}^n a_i 1_{(t_i, t_{i+1}]} \to \int_T \varphi_s dB^H(s) \tag{9}$$

is an isometry between $\mathcal{E}_H$ and the the linear space $span\{B_t^H, t \in \mathbb{R}\}$ viewed as a subspace of $L^2(\Omega)$ and it can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos of the fractional Brownian motion $\overline{span}^{L^2(\Omega)}\{B_t^H, t \in \mathbb{R}\}$. The image on an element $\Phi \in \mathcal{H}$ by this isometry is called the Wiener integral of $\Phi$ with respect to $B^H$.

For every $s < \tau$, let us consider the operator $K^*$ in $L^2(T)$

$$(K_\tau^* \varphi)(s) = K(\tau, s)\varphi(s) + \int_s^\tau (\varphi(r) - \varphi(s)) \frac{\partial K}{\partial r}(r,s)dr. \tag{10}$$

When $H > \frac{1}{2}$, the operator $K^*$ has the simpler expression

$$(K_\tau^* \varphi)(s) = \int_s^\tau \varphi(r) \frac{\partial K}{\partial r}(r,s)dr.$$

We refer to [1] for the proof of the fact that $K^*$ is a isometry between $\mathcal{H}$ and $L^2(T)$ and, as a consequence, we will have the following relationship between the Wiener integral with respect to fBm and the Wiener integral with respect to the Wiener process $W$

$$\int_0^t \varphi(s)dB^H(s) = \int_0^t (K_t^*\varphi)(s)dW(s) \tag{11}$$

for every $t \in T$ and $\varphi 1_{[0,t]} \in \mathcal{H}$ if and only if $K_t^*\varphi \in L^2(T)$. We also recall that, if $\phi, \chi \in \mathcal{H}$ are such that $\int_T \int_T |\phi(s)||\chi(t)|t-s|^{2H-2}dsdt < \infty$, their scalar product in $\mathcal{H}$ is given by

$$\langle \phi, \chi \rangle_{\mathcal{H}} = H(2H-1)\int_0^\tau \int_0^\tau \phi(s)\chi(t)|t-s|^{2H-2}dsdt. \tag{12}$$

Note that in the general theory of Skorohod integration with respect to fBm with values in a Hilbert space $V$, a relation such as (11) requires careful justification of the existence of its right-hand side (see [11], Section 5.1). But we will work only with Wiener integrals over Hilbert spaces; in this case we note that, if $u \in L^2(T;V)$ is a deterministic function, then relation (11) holds, the Wiener integral on the right-hand side being well defined in $L^2(\Omega;V)$ if $K^*u$ belongs to $L^2(T \times V)$.

## 2.2 Infinite dimensional fractional Brownian motion and stochastic integration

Let $U$ be a real and separable Hilbert space and let $Q$ be a self-adjoint and positive operator on $U$ ($Q = Q^* > 0$). It is typical and usually convenient to assume moreover that $Q$ is nuclear ($Q \in L_1(U)$). In this case it is well-known that $Q$ admits a sequence $(\lambda_n)_{n\geq 0}$ of eigenvalues with $0 < \lambda_n \searrow 0$ and $\sum_{n\geq 0}\lambda_n < \infty$. Moreover, the corresponding eigenvectors $(e_n)_{n\geq 0}$ form an orthonormal basis in $U$. We define the infinite dimensional fBm on $U$ with covariance $Q$ as

$$B^H(t) = B_Q^H(t) = \sum_{n=0}^{\infty} \sqrt{\lambda_n} e_n \beta_n^H(t) \tag{13}$$

where $\beta_n^H$ are real, independent fBm's. This process is a $U$-valued Gaussian process, it starts from 0, has zero mean and covariance

$$E(B_Q^H(t)B_Q^H(s)) = R(s,t)Q, \text{ for every } s,t \in T \tag{14}$$

(see [5], [16], [6]). We will encounter below cases in which the assumption that $Q$ is nuclear is not convenient. For example one may wish to consider the case of a genuine *cylindrical* fractional Brownian motion on $U$ by setting $\lambda_n \equiv 1$, i.e.

$$B^H(t) = \sum_{n=0}^{\infty} e_n \beta_n^H(t).$$

More generally we state the following.

**Remark 1** *Following the standard approach as in [3] for $H = 1/2$, it is possible to define a generalized fractional Brownian motion on $U$ (e.g. in the sense of generalized functions if $U$ is a space of functions) by the right-hand side of formula (13) for any fixed complete orthonormal system $(e_n)_n$ in $U$, and any fixed sequence of positive numbers $(\lambda_n)_n$, even if $\sum_{n\geq 0}\lambda_n = \infty$. Although for any fixed $t$ the series (13) is not convergent in $L^2(\Omega \times U)$, we can always consider a Hilbert space $U_1$ such that $U \subset U_1$ and such that this inclusion is a Hilbert-Schmidt operator. In this way, $B^H(t)$ given by (13) is a well-defined $U_1$-valued Gaussian stochastic process.*

Let now $V$ be another real separable Hilbert space, $B^H$ the process defined above, defined as a $U_1$-valued process if necessary (see Remark 1), and $(\Phi_s)_{s\in T}$ a deterministic function with values in $\mathcal{L}_2(U;V)$, the space of Hilbert-Schmidt operators from $U$ to $V$. The stochastic integral of $\Phi$ with respect to $B^H$ is defined by

$$\int_0^t \Phi_s dB^H(s) = \sum_{n=0}^{\infty} \int_0^t \Phi_s e_n d\beta_n^H(s) = \sum_{n=0}^{\infty} \int_0^t (K^*(\Phi e_n))_s d\beta_n(s) \tag{15}$$

where $\beta_n$ is the standard Brownian motion used to represent $\beta_n^H$ as in (3), and the above sum is finite when

$$\sum_n \|K^*(\Phi e_n)\|^2_{L^2(T;V)} = \sum_n |\|\Phi e_n\|_{\mathcal{H}}|_V^2 < \infty.$$

In this case the integral (15) is well defined as a $V$-valued Gaussian random variable. However, as we are about to see, the linear additive equation in its evolution form can have a solution even if $\int_0^t \Phi_s dB^H(s)$ is not properly defined as a $V$-valued Gaussian random variable. A remark similar to Remark 1 applies in order to define this stochastic integral in a larger Hilbert space than $V$. In particular, there is no reason to assume that $\Phi \in \mathcal{L}_2(U,V)$.

# 3 Linear stochastic evolution equations with fractional Brownian motion

We will work in this section with a cylindrical fBm $B^H$ on a real separable Hilbert space $U$, $\Phi$ a linear operator in $\mathcal{L}(U,V)$ that is not necessarily Hilbert-Schmidt, and $A : Dom(A) \subset V \to V$ the infinitesimal generator of the strongly continuous semigroup $(e^{tA})_{t\in T}$. We study the equation

$$dX(t) = AX(t)dt + \Phi dB^H(t),\ X(0) = x \in V \tag{16}$$

As previously noted, the stochastic integral $\int_0^t \Phi dB^H(s)$ is only well-defined as a $V$-valued random variable if $\Phi \in \mathcal{L}_2(U,V)$ since

$$E\left|\int_0^t \Phi dB^H(s)\right|_V^2 = \sum_n E\left|\int_0^t \Phi e_n d\beta_n^H(s)\right|_V^2 = \sum_n E\left|\int_0^t d\beta_n^H(s)\right|^2 |\Phi e_n|_V^2 = t^{2H}\|\Phi\|_{HS}^2$$

where here and in the sequel, $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm. However, the operator $A$ may be irregular enough that no strong solution to (16) exists even if $\int_0^t \Phi dB^H(s)$ exists. We then consider the so-called *mild* form (a.k.a. *evolution* form) of the equation, whose unique solution, if it exists, can be written in the evolution form

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A}\Phi dB^H(s),\ t \in T. \tag{17}$$

Our aim is to find necessary and sufficient conditions on $A$ and $\Phi$ for this solution to exist in $L^2(\Omega)$ for each $t \geq 0$. For this goal, we will see that it is no longer necessary to even assume that $\int_0^t \Phi dB^H(s)$ exists; in contrast, we only need to guarantee the existence of the stochastic integral in (17). This is the reason for dropping the hypothesis that $\Phi$ is Hilbert-Schmidt. Note that, in the case where $V$ is a space of functions, the so-called *weak* form of (16), using test functions, is another alternative formulation which is morally equivalent to the mild form. We will use this form below in Proposition 1 to formulate a slightly stronger existence result than is possible with the mild form. Proposition 1 excluded, this article deals only with the mild form. We assume throughout that $A$ is a self-adjoint operator on $V$. In this situation, it is well known that (see [13], Section 8.3 for a classical account on this topic) there exists a uniquely defined projection-valued measure $dP_\lambda$ on the real line such that, for every $\phi \in V$, $d\langle \phi, P_\lambda \phi \rangle$ is a Borel measure on $R$ and for every $\phi \in Dom(A)$, we have

$$\langle \phi, A\phi \rangle = \int_R \lambda d\langle \phi, P_\lambda \phi \rangle.$$

Furthermore, for any real-valued Borel function $g$ on $R$, we can define a self-adjoint operator $g(A)$ by setting

$$\langle \phi, g(A)\phi \rangle = \int_R g(\lambda) d\langle \phi, P_\lambda \phi \rangle \tag{18}$$

for $\phi \in D_g$ with

$$D_g = \{x; \int_R |g(\lambda)|^2 \, d\langle x, P_\lambda x \rangle < \infty\}.$$

The statement of our main existence and uniqueness theorem follows.

**Theorem 1** *Let $B^H$ be a cylindrical fBm in a Hilbert space $U$ and let $A : Dom(A) \subset V \to V$ be a self-adjoint operator on a Hilbert space $V$. Assume that $A$ is a negative operator, and more specifically that there exists some $l > 0$ such that $dP_\lambda$ is supported on $(-\infty, -l]$. Then for any fixed $\Phi \in \mathcal{L}_2(U, V)$, there exists a unique mild solution $(X(t))_{t \in T}$ of (16) belonging to $L^2(\Omega; V)$ if and only if $\Phi^* G_H(-A)\Phi$ is a trace class operator, where*

$$G_H(\lambda) = (\max(\lambda, 1))^{-2H}. \tag{19}$$

This theorem is valid for both $H < 1/2$ and $H > 1/2$. However, separate proofs are required in each case: Theorems 2 and 3. Several technical calculations, although they be interesting in their own right as well as elementary, are given in the Appendix in order to increase the article's readability.

**Remark 2** *Theorem 1 holds for those operators $A$ satisfying only a "spectral gap" condition, i.e. such that $dP_\lambda$ is supported on $(-\infty, -l]$ except for an atom at $\{0\}$, as long as one assumes that the kernel of $A$ is finite-dimensional. To check this one only needs to include the terms corresponding to $\lambda = 0$ in the proofs of Theorems 2 and 3.*

**Remark 3** *When $Supp(P_\lambda) \subset (-\infty, -l)$, with $l > 0$, we can replace $G_H(-A)$ in Theorem 1 by $(-A)^{-2H}$. Seeing this is obvious, for example, in the proof of the case $H > 1/2$ (see Lemma 1 below, and its usage). When $A$ is non-positive with a spectral gap, one can instead replace by $G_H(-A)$ by $(-A+I)^{-2H}$ for example. The spectral gap situation occurs for example in the case of the Laplace-Beltrami operator on compact Lie groups; in this situation, with $H = 1/2$, the trace condition with $(-A+I)^{-2H}$ was proved to be optimal in [14]. This condition is equivalent to conditions presented in work done in [12] for both the stochastic heat and wave equations in Euclidean space $\mathbf{R}^d$ with $d \geq 2$; therein, the authors even treat non-linear equations under a non-degeneracy assumption on the nonlinearity function $F$ ($F$ bounded above and below by positive numbers). Proposition 1 below shows that we can have existence of a weak solution to (16) even if $P_\lambda$ charges all of $(-\infty, a)$ for some $a \geq 0$. In this case, using $(-A)^{-2H}$, or even $(-A+I)^{-2H}$, instead of $G_H(-A)$ for a trace condition for existence is too strong to be necessary.*

### 3.1 A fundamental example: the Laplacian on the circle

Before proving the theorem we discuss its consequences for the fundamental example in which the operator $A$ is the Laplacian $\Delta$ on the circle. This means that with $e_n(x) = (2\pi)^{-1} \cos nx$ and $f_n(x) = (2\pi)^{-1} \sin nx$ for each $n \in N$, the set of functions $\{e_n, f_n : n \in N\}$ is not only an orthogonal basis for $U = L^2(S^1, dx)$ where $dx$ is the normalized Lebesgue measure on $[-\pi, \pi)$, this set is exactly the set of eigenfunctions of $\Delta$. An infinite-dimensional fractional Brownian motion $B^H$ in $L^2(S^1)$ can be defined by

$$B^H(t,x) = \sum_{n=0}^{\infty} \sqrt{q_n} e_n(x) \beta_n^H(t) + \sum_{n=1}^{\infty} \sqrt{q_n} f_n(x) \bar{\beta}_n^H(t).$$

where $\left\{\beta_n^H, \bar{\beta}_n^H : n \in N\right\}$ is a family of IID standard fractional Brownian motions with common parameter $H$. If $\sum q_n < \infty$ then $B^H$ is a bonafide $L^2(S^1)$-valued process. Otherwise we can consider that it is a generalized-function-valued process in $L^2(S^1)$, as in remark 1. Note that $B^H$ defined in this way is a Gaussian field on $T \times S^1$ that is fBm in time for fixed $x$ and that is homogeneous in space for fixed $t$. The spatial covariance function calculates to

$$Q(x-y) = E\left[B^H(1,x) B^H(1,y)\right] = \sum_{n=0}^{\infty} q_n \cos(n(x-y)).$$

To apply Theorem 1, we only need to represent $B^H$ as $\Phi \tilde{B}^H$ where $\tilde{B}^H$ is cylindrical on $L^2(S^1)$. This is obviously achieved using $\Phi e_n = \sqrt{q_n} e_n$, yielding the following immediate Corollary.

**Corollary 1** *Let $B^H$ be the fBm on $L^2(S^1)$ with $H \in (0,1)$ and the assumptions above. Then there exists a square integrable solution of (17) if and only if*

$$\sum_{n=1}^{\infty} q_n n^{-4H} < \infty. \tag{20}$$

This corollary clearly shows that many generalized-function-valued fBm's on $L^2(S^1)$ yield a solution. More precisely, if we define a fractional "antiderivative" of order $2H$ of $B^H$ by $Y = (I - \Delta)_x^{-H} B$, we have existence if and only if $Y$ is a bonafide $L^2(S^1)$-valued process. The following examples may be enlightening, in view of the well-known results for standard Brownian motion.

- Let $B^H$ be fBm in time and white-noise in space, i.e. let $q_n \equiv 1$. Then equation (16) has a unique mild solution in $L^2(S^1)$ if and only if $H > 1/4$.

- More generally consider the equation (16) with space-time fractional noise as a generalization of the well-known space-time white noise. This would mean that $B^H$ is the space derivative of a field $Z$ that is fBm in time *and* in space. Call $H'$ the Hurst parameter of $Z$ in space. To translate this on the behavior of the $q_n$'s we can say that, by analogy with the standard white-noise, and at least up to universal multiplicative constants, we can take $\sqrt{q_n} = n^{1/2-H'}$. Then equation (16) has a unique mild solution in $L^2(S^1)$ if and only if $H' > 1 - 2H$. Thus if $B^H$ is fractional Brownian in time with $H \geq 1/2$, existence holds for any fractional noise behavior in space, while if $B^H$ is fractional Brownian in time with $H < 1/2$, existence holds if and only if the fractional noise behavior in space exceeds $1 - 2H$.

- In particular, for $dB^H$ that is space-time fractional noise with the same parameter $H$ in time and space, existence holds if and only if $H > 1/3$.

**Remark 4** *The thresholds obtained in the three situations above for the circle should also hold in any non-degenerate one-dimensional situation. This can be easily established for the Laplace-Beltrami on a smooth compact one-dimensional manifold. We also believe it should hold in non-compact situations such as for the Laplacian on R.*

### 3.2 The case $H > \frac{1}{2}$

**Theorem 2** *Assume $H \in (1/2, 1)$. Then the result of Theorem 1 holds.*

**Proof:** Let us estimate the mean square of the Wiener integral of ( 17). For every $t \in T$, it holds ($C(H)$ denoting a generic constant throughout this proof)

$$\begin{aligned}
I_t = E\left|\int_0^t e^{(t-s)A}\Phi dB^H(s)\right|_V^2 &= E\left|\sum_n \int_0^t e^{(t-s)A}\Phi e_n d\beta_n^H(s)\right|_V^2 \\
&= \sum_n C(H)\int_0^t\int_0^t \langle e^{(t-u)A}\Phi e_n, e^{(t-v)A}\Phi e_n\rangle_V |u-v|^{2H-2}dudv \\
&= C(H)\sum_n \int_0^t\int_0^t \langle e^{(2t-u-v)A}\Phi e_n, \Phi e_n\rangle_V |u-v|^{2H-2}dudv \\
&= 2C(H)\sum_n \int_0^t \left(\int_0^u \langle e^{(2t-2u+v)A}\Phi e_n, \Phi e_n\rangle_V v^{2H-2}dv\right) du. \qquad (21)
\end{aligned}$$

Consider now the measure $d\mu_n(\lambda)$ defined as

$$d\mu_n(\lambda) = d\langle \Phi e_n, P_\lambda \Phi e_n\rangle_V \qquad (22)$$

where $P_\lambda$ is the spectral measure of the operator $-A$. We have

$$\langle e^{(2t-2u+v)A}\Phi e_n, \Phi e_n\rangle_V = \int_R e^{(2t-2u+v)\lambda}d\mu_n(\lambda) = \int_0^\infty e^{-(2t-2u+v)\lambda}d\mu_n(\lambda)$$

because, since $A \leq 0$, $P_\lambda$ vanishes for $\lambda > 0$. The expression (21) becomes, using Fubini theorem

$$\begin{aligned}
I_t &= C(H)\sum_n \int_0^t\int_0^u v^{2H-2}\left(\int_0^\infty e^{-(2t-2u+v)\lambda}d\mu_n(\lambda)\right) dvdu \\
&= C(H)\sum_n \int_0^\infty e^{-2t\lambda}\int_0^t e^{2u\lambda}\left(\int_0^u v^{2H-2}e^{-v\lambda}dv\right) dud\mu_n(\lambda)
\end{aligned}$$

and doing the change of variables $v\lambda = v'$ in the integral with respect to $dv$, and integrating by parts with respect to $u$, we get

$$\begin{aligned}
I_t &= C(H)\sum_n \int_0^\infty e^{-2t\lambda}\lambda^{1-2H}\int_0^t e^{2u\lambda}\left(\int_0^{\lambda u} v^{2H-2}e^{-v}dv\right) dud\mu_n(\lambda) \\
&= C(H)\sum_n \int_0^\infty \lambda^{-2H}\left(\int_0^{\lambda t} v^{2H-2}e^{-v}\left[\frac{e^{2\lambda t}-e^{2v}}{e^{2\lambda t}}\right] dv\right) d\mu_n(\lambda) \qquad (23)
\end{aligned}$$

Denote by

$$A(\lambda, t) = \int_0^{\lambda t} v^{2H-2}e^{-v}\left[\frac{e^{2\lambda t}-e^{2v}}{e^{2\lambda t}}\right] dv. \qquad (24)$$

At this point we need the following technical lemma whose proof is given in the Appendix.

**Lemma 1** *For every $t \in T$, there exist positive constants $c(H,t)$ and $C(H,t)$ depending only on $H$ and $t$ such that*

**(i)** *If $\lambda > 1$, $c(H,t) \leq A(\lambda,t) \leq C(H,t)$, and*

**(ii)** *if $\lambda \leq 1$ , $c(H,t) \leq A(\lambda,t)\,\lambda^{-2H} \leq C(H,t)$.*

Using the notation $A \asymp B$ for two quantities whose ratio is bounded above and below by positive constants (in which case we say the quantities are *commensurate*), putting the two estimations of $A(\lambda)$ together we obtain

$$\begin{aligned} I_t &\asymp \sum_n \int_0^1 d\mu_n(\lambda) + \int_1^\infty \lambda^{-2H} d\mu_n(\lambda) \\ &\asymp \sum_n \int_0^\infty (\max(\lambda;1))^{-2H}\, d\mu_n(\lambda), \end{aligned}$$

where the constants needed in the $\asymp$ relations depend only on $H$ and $t$. This yields the theorem. □

### 3.3 The case $H < \frac{1}{2}$

**Theorem 3** *Let $H \in (0,\frac{1}{2})$, and let $P_\lambda$ denote the spectral measure of $-A$. If there exists a positive constant $l$ such that*

$$Supp\,(P_\lambda) \subset (l;\infty), \tag{25}$$

*then Theorem 1 holds.*

**Proof.** We let $P_\lambda$ denote the spectral measure of $-A$, and $\mu_n$ the corresponding scalar measures as before. Denoting $I_t = E\left|X(t) - e^{tA}x\right|_V^2$, it is sufficient to estimate $I_t$ optimally from above and below. We have

$$I_t = E\left|\int_0^t e^{(t-s)A}\Phi dB^H(s)\right|_V^2 = E\left|\sum_n \int_0^t e^{(t-s)A}\Phi e_n d\beta_n^H(s)\right|_V^2$$

**Step 1** *(Upper bound).* We prove first the sufficient condition for the existence of a square integrable mild solution of equation (16). We start with the following technical Lemma (its proof is given in the Appendix).

**Lemma 2** *Let*

$$B(a,A) = \int_0^1 ds \exp(-2as)\left[\int_0^s (\exp ar - 1)\, r^{A-1} dr\right]^2$$

*where $a \geq 0$ and $A \in (-1/2, 0]$. Then it holds*

$$B(a,A) \leq K_A a^{-2A-1}$$

*with $K_A$ a positive constant depending only on $A$.*

Using (10) and the representation (11), we have

$$
\begin{aligned}
I_t &\leq 2\sum_n \int_0^t \left|e^{(t-s)A}\Phi e_n\right|_V^2 K^2(t,s)ds \\
&+ 2\sum_n \int_0^t \left|\int_s^t \left(e^{(t-r)A}\Phi e_n - e^{(t-s)A}\Phi e_n\right)\frac{\partial K}{\partial r}(r,s)dr\right|_V^2 ds \\
&= \sum_n (I_1(n) + I_2(n))
\end{aligned}
$$

Using the following inequality (see [4], Th. 3.2),

$$
K(t,s) \leq c(H)(t-s)^{H-\frac{1}{2}} s^{H-\frac{1}{2}}
$$

the first sum above can be majorized in the following way

$$
\begin{aligned}
\sum_n I_1(n) &\leq c(H)\sum_n \int_0^t \langle e^{2(t-s)A}\Phi e_n, \Phi e_n\rangle_V (t-s)^{2H-1} s^{2H-1} ds \\
&= c(H)\sum_n \int_0^\infty \lambda^{-2H}\left(\int_0^{2\lambda t} e^{-v}v^{2H-1}(t-\frac{v}{2\lambda})^{2H-1}dv\right)d\mu_n(\lambda) \\
&\leq c(H)\sum_n \int_0^\infty \lambda^{-2H} C\,(t,H)\, d\mu_n(\lambda) \\
&= C\,(t,H)\, Tr(\Phi^*(-A)^{-2H}\Phi)
\end{aligned} \tag{26}
$$

where $C\,(t,H)$ depends only on $t$ and $H$. Here we used the fact that

$$
\begin{aligned}
&\int_0^{2\lambda t} e^{-v}v^{2H-1}(t-\frac{v}{2\lambda})^{2H-1}dv \\
&\leq (t/2)^{2H-1}\int_0^\infty e^{-v}v^{2H-1}dv + (\lambda t)^{2H-1}\int_{\lambda t}^{2\lambda t} e^{-v}(t-\frac{v}{2\lambda})^{2H-1}dv \\
&\leq C(t,H) + (\lambda t)^{2H-1}\int_0^{\lambda t} e^{-(2\lambda t - v')}\left(v'/(2\lambda)\right)^{2H-1}dv' \\
&\leq C\,(t,H) + C\,(t,H)\, e^{-\lambda t}\,(\lambda t)^{2H} = C\,(t,H)\,.
\end{aligned}
$$

For the second sum from above, we can write

$$
\begin{aligned}
\sum_n I_2(n) &= \sum_n \int_0^t ds \int_s^t dr_1 \int_s^t dr_2 \frac{\partial K}{\partial r_1}(r_1,s)\frac{\partial K}{\partial r_2}(r_2,s) \\
&\times \langle\left(e^{(t-r_1)A} - e^{(t-s)A}\right)\Phi e_n, \left(e^{(t-r_2)A} - e^{(t-s)A}\right)\Phi e_n\rangle_V \\
&= \sum_n \int_0^t ds \int_s^t dr_1 \int_s^t dr_2 \frac{\partial K}{\partial r_1}(r_1,s)\frac{\partial K}{\partial r_2}(r_2,s) \\
&\times \langle\left(e^{(t-r_1)A} - e^{(t-s)A}\right)\left(e^{(t-r_2)A} - e^{(t-s)A}\right)\Phi e_n, \Phi e_n\rangle_V
\end{aligned}
$$

and, by the fact that $\frac{\partial K}{\partial r}(r,s) \leq 0$ for every $r,s \in T$ and $|\frac{\partial K}{\partial r}(r,s)| \leq C(H)(r-s)^{H-\frac{3}{2}}$, we get

$$
\begin{aligned}
\sum_n I_2(n) &= \sum_n \int_0^t ds \int_s^t dr_1 \int_s^t dr_2 \frac{\partial K}{\partial r_1}(r_1,s)\frac{\partial K}{\partial r_2}(r_2,s) \\
&\times \int_0^{+\infty} \left(e^{-\lambda(t-r_1)} - e^{-\lambda(t-s)}\right)\left(e^{-\lambda(t-r_2)} - e^{-\lambda(t-s)}\right) d\mu_n \\
&\leq C(H) \sum_n \int_0^t du \int_0^u dv_1 \int_0^u dv_2 (u-v_1)^{H-\frac{3}{2}}(u-v_2)^{H-\frac{3}{2}} \\
&\times \int_0^\infty \left(e^{-\lambda(v_1+v_2)} - e^{-\lambda(u+v_2)} - e^{-\lambda(v_1+u)} + e^{-2\lambda u}\right) d\mu_n
\end{aligned}
$$

where we used the change of variables $t-s=u$, $t-r_1=v_1$, $t-r_2=v_2$ and the symmetry of $A$. Let us note that the above quantities are positive and therefore we can apply Fubini theorem, obtaining

$$
\begin{aligned}
\sum_n I_2(n) &\leq C(H) \sum_n \int_0^\infty d\mu_n \int_0^t du \int_0^u dv_1 \int_0^u dv_2 (u-v_1)^{H-\frac{3}{2}}(u-v_2)^{H-\frac{3}{2}} \\
&\times \left(e^{-\lambda(v_1+v_2)} - e^{-\lambda(u+v_2)} - e^{-\lambda(v_1+u)} + e^{-2\lambda u}\right) \\
&= \sum_n \int_0^\infty d\mu_n \int_0^t du \left(\int_0^u (u-v)^{H-\frac{3}{2}}(e^{-\lambda u} - e^{-\lambda v}) dv\right)^2 \\
&= \sum_n \int_0^\infty d\mu_n \int_0^t e^{-2\lambda s}\left(\int_0^s (e^{\lambda r}-1) r^{H-\frac{3}{2}} dr\right)^2 ds \\
&= \sum_n \int_0^\infty I_2(\lambda,t) d\mu_n(\lambda) \qquad (27)
\end{aligned}
$$

where on the last line we came back to the initial variables. Now, applying (27) and Lemma 2 to

$$
I_2(\lambda,t) = \int_0^t e^{-2\lambda s}\left(\int_0^s (e^{\lambda r}-1) r^{H-\frac{3}{2}} dr\right)^2 ds,
$$

with (26), we have the upper bound

$$
I_t \leq C(t,H) \sum_n \int_0^\infty \lambda^{-2H} d\mu_n(\lambda) = C(t,H)\, Tr(\Phi^*(-A)^{-2H}\Phi).
$$

**Step 2** *(Lower bound).* To prove the necessity, note that

$$
\begin{aligned}
I_t = E\Bigg[\Bigg|&\sum_n \int_0^t \left(e^{(t-s)A}\Phi e_n\right) K(t,s) d\beta_n(s) \\
&+ \int_0^t \left(\int_s^t \frac{\partial K}{\partial r}(r,s)\left(e^{(t-r)A} - e^{(t-s)A}\right)\Phi e_n dr\right) d\beta_n(s)\Bigg|_V^2\Bigg]
\end{aligned}
$$

and this equals

$$
\begin{aligned}
I_t &= \sum_n \int_0^t \left| e^{(t-s)A} \Phi e_n \right|_V^2 K^2(t,s) ds \\
&+ 2 \sum_n \int_0^t K(t,s) \int_s^t \frac{\partial K}{\partial r}(r,s) \langle e^{(t-s)A} \Phi e_n, \left( e^{(t-r)A} - e^{(t-s)A} \right) \Phi e_n \rangle_V dr ds \\
&+ \sum_n \int_0^t \left| \int_s^t \frac{\partial K}{\partial r}(r,s) \left( e^{(t-r)A} - e^{(t-s)A} \right) \Phi e_n dr \right|_V^2 ds.
\end{aligned}
$$

We let $t = 1$ for simplicity and we use the measure $d\mu_n(\lambda) = d\langle \Phi e_n, P_\lambda \Phi e_n \rangle_V$. Taking account that $P_\lambda = 0$ outside $(-\infty, -l)$, we get

$$
\begin{aligned}
I_t &= \sum_n \int_l^\infty \left( \int_0^1 \exp(-2\lambda(1-s)) K^2(1,s) ds \right) d\mu_n(\lambda) \\
&+ 2 \sum_n \int_l^\infty \int_0^1 ds \exp(-2\lambda(1-s)) K(1,s) \\
&\times \left( \int_s^1 (\exp((r-s)\lambda) - 1) \frac{\partial K}{\partial r}(r,s) dr \right) d\mu_n(\lambda) \\
&+ \sum_n \int_l^\infty \left( \int_0^1 \exp(-2\lambda(1-s)) \left( \int_s^1 (\exp((r-s)\lambda) - 1) \frac{\partial K}{\partial r}(r,s) dr \right)^2 ds \right) d\mu_n(\lambda) \\
&= \int_l^\infty J(\lambda) d\mu_n(\lambda).
\end{aligned}
$$

The conclusion of the theorem follows from the next lemma. □

**Lemma 3** *Let*

$$
\begin{aligned}
J(\lambda) &= \int_0^1 \exp(-2\lambda(1-s)) K^2(1,s) ds \\
&+ 2 \int_0^1 \exp(-2\lambda(1-s)) K(1,s) \left( \int_s^1 (\exp((r-s)\lambda) - 1) \frac{\partial K}{\partial r}(r,s) dr \right) ds \\
&+ \int_0^1 \exp(-2\lambda(1-s)) \left( \int_s^1 (\exp((r-s)\lambda) - 1) \frac{\partial K}{\partial r}(r,s) dr \right)^2 ds.
\end{aligned}
$$

*Then $J(\lambda) \geq c(H)\lambda^{-2H}$ for every $\lambda > l > 0$ with $l$ arbitrary small.*

**Proof:** See the Appendix. □

### 3.4 Extended existence for the weak equation

Assume now that $V$ is a Hilbert space of functions on finite-dimensional Euclidean space $E$, and assume $A$ is a self-adjoint operator on $V$. One can interpret the noise term $\Phi B^H(t)$ directly as a Gaussian field on $T \times E$ that is fBm in time and possibly a generalized function in space. For the formulation of an existence result, we keep using representation of this field via the operator $\Phi \in \mathcal{L}(V,V)$ operating on a cylindrical $B^H(t)$ in $V$. Equation (16) now reads,

$$
X(dt,x) = [AX(t,\cdot)](x)\, dt + \left[\Phi B^H\right](dt,x),\ X(0) = X_0 \in V, t \geq 0, x \in E
$$

and its *weak* version is

$$\int_E \phi(x) X(t,x)\,dx = \int_E \phi(x) X_0(x)\,dx + \int_E \int_0^t X(t,x) A\phi(x)\,dxdt + \int_E \left[\Phi B^H\right](t,x)\phi(x)\,dx, \tag{28}$$

for all $t \geq 0$, $x \in E$, $\phi \in Dom(A)$. If it happens that the Gaussian field $\Phi B^H$ on $T \times E$ is generalized-function-valued in the parameter $x$, the last term in (28) must be interpreted as

$$\left[\Phi B^H\right](t,\phi)$$

for all test functions $\phi$ in $Dom(A) \cap dom\left[\Phi B^H(1)\right]$. More generally, we can formulate a weak equation in an abstract separable Hilbert space $V$. We assume that $A$ is a self-adjoint operator on $V$, that $B^H$ is a cylindrical fBm in $V$, and that $\Phi \in \mathcal{L}(V,V)$. The generalization of (28) is

$$\langle X(t),\phi\rangle = \langle X(0),\phi\rangle + \int_0^t \langle X(s), A\phi\rangle ds + \int_0^t \langle \Phi^*\phi, dB^H(s)\rangle, \tag{29}$$

for all $t \geq 0$ and all test functions $\phi$ in $Dom(A)$, where $\langle\ ,\ \rangle$ denotes the scalar product in $V$. The following proposition shows that the spectral gap condition for existence can be eliminated when dealing only with the weak equation.

**Proposition 1** *Let $H \in (0,1)$. Let $B^H$ be a cylindrical fBm in $V$, a separable Hilbert space, and let $A : Dom(A) \subset V \to V$ be a self-adjoint operator on $V$ such that for some $\lambda_0 > 0$, $A - \lambda_0 I$ is a negative operator. Then for any fixed $\Phi \in \mathcal{L}(V,V)$, there exists a solution $(X(t,\cdot))_{t\in T}$ of (28) belonging to $L^2(\Omega;V)$ as long as $\Phi^* G_H(-A)\Phi$ is a trace class operator.*

**Proof.** By hypothesis we can find positive numbers $\mu$ and $\varepsilon$ such that $A - \mu I < -\varepsilon I$, that is to say, the operator $\bar{A} = A - \mu I$ satisfies the hypotheses of both Theorem 2 and Theorem 3. Therefore, in both the cases $H < 1/2$ and $H > 1/2$, we have existence and uniqueness of a mild solution in $L^2(\Omega;V)$ to the following equation:

$$dY_t = (A - \mu I) Y_t dt + \Phi dB_t^H$$

if and only if $\Phi(\mu I - A)^{-2H}\Phi^*$ is trace class. Indeed, one should require, rather, that $\Phi G_H(\mu I - A)\Phi^*$ be trace class, but here the strict negativity of $\bar{A}$ allowed us to replace the function $G_H$ by the function $F_H(\lambda) = \lambda^{-2H}$. Now a simple repetition of arguments of Da Prato and Zabczyk in [3] shows that for any Lipschitz function $F$ on $V$, the equation

$$dZ_t = (A - \mu I) Z_t dt + F(Z_t)\,dt + \Phi dB_t^H$$

also has a unique mild solution formed by considering the semigroup of the operator $A - \mu I$. By taking $F(z) = \mu z$ we see that the following mild equation has a unique solution $Z$:

$$Z(t) = e^{t(A-\mu I)}x + \int_0^t e^{(t-s)(A-\mu I)}\Phi dB^H(s) + \mu \int_0^t e^{(t-s)(A-\mu I)} Z(s)\,ds. \tag{30}$$

The next step in the proof is to show that $Z$ defined by (30) also satisfies (28). This can be checked by a classical calculation for all test functions $\phi \in Dom(A - \mu I)$. However this domain is defined as the set of all functions $\phi \in V$ such that $(A - \mu I)\phi \in V$. Thus it coincides with $Dom(A)$, and the weak equation (28) is satisfied by $Z$. The last step in the proof is to show that the trace condition on $\Phi\left(\mu I-A\right)^{-2H}\Phi^{*}$ is equivalent to the condition that $\Phi G_{H}\left(-A\right)\Phi^{*}$ be trace class. Recall that for any function $F$ we have

$$tr\left[\Phi F\left(-A\right)\Phi^{*}\right]=\sum_{n}\int_{-\infty}^{\infty}F\left(\lambda\right)d\mu_{n}\left(\lambda\right)$$

where $\mu_n$, defined in (22), is a positive measure for any $n$. Therefore it is sufficient to show that the function $G_{H}\left(\lambda\right)=\left(\max\left(1,\lambda\right)\right)^{-2H}$ is commensurable with the function $\bar{G}_{H}\left(\lambda\right)=\left(\lambda+\mu\right)^{-2H}$. For $\lambda>1$ this is clear. For $\lambda<1$, we use the fact that the support of all measures $d\mu_n$ is in $[-\lambda_0;+\infty)$. Since it is no restriction to require that $\mu>\lambda_0+\varepsilon$, we have that for $\lambda\in[-\lambda_0;1]$, $\bar{G}_{H}\left(\lambda\right)$ is bounded above by $\varepsilon^{-2H}$ and below by $\left(1+\mu\right)^{-2H}$; in this sense it is commensurable with $G_{H}\left(\lambda\right)$ since the latter is equal to 1 in that interval.□

# 4 Space-time regularity of the solution

In this section, we give some general results on the spatial regularity of the solution to our linear additive equation. As in Theorem 1, we assume that:

**(R)** the operator $A$ is self adjoint and there exist $\varepsilon>0$ such that $A\leq-\varepsilon I$.

As in Remark 2 we could also allow $A$ to have 0 as an eigenvalue, with a finite dimensional eigenspace, and then a spectral gap up to $-\varepsilon$. We omit these details. Our regularity result is based on a proposition taken from [3], which we enunciate here for sake of completeness: let $A$ be an unbounded operator satisfying condition (R). For $\alpha,\gamma\in(0,1)$, $p>1$ and $\psi\in L^p([0,T];V)$, set

$$R_{\alpha,\gamma}\psi(t)=\frac{\sin(\alpha\pi)}{\pi}\int_0^t(t-\sigma)^{\alpha-1}(-A)^{\gamma}e^{(t-\sigma)A}\psi(\sigma)d\sigma,$$

where $A^{\gamma}$ has to be interpreted as in (18). It is a known fact (see [3, Proposition A.1.1]) that, if $\alpha>\gamma+\frac{1}{p}$, then

$$R_{\alpha,\gamma}\in\mathcal{L}\left(L^p([0,T];V);C^{\alpha-\gamma-\frac{1}{p}}\left([0,T];D((-A)^{\gamma})\right)\right). \tag{31}$$

Let now $X$ be the process defined by relation (17) with $x=0$, that is the usual stochastic convolution of $B^H$ by $A$. The main result of this section is the following:

**Theorem 4** *Let $H\in(0,1)$, and suppose that for $\alpha\in(0,H)$, the operator*

$$\Phi^{*}(-A)^{-2(H-\alpha)}\Phi$$

*is trace class. Then, for any $\gamma<\alpha$ and any $\varepsilon<(\alpha-\gamma)$, almost surely,*

$$X\in C^{\alpha-\gamma-\varepsilon}\left([0,T];D\left((-A)^{\gamma}\right)\right).$$

*In particular, for any fixed $t>0$, $X(t)\in D((-A)^{\gamma})$.*

**Proof:** Under our assumptions, it can be shown by the usual factorization method (see e.g. [3, Theorem 5.2.6]) that the process $(-A)^{\gamma}X$ can be written as

$$(-A)^{\gamma}X(t)=\left[R_{\alpha,\gamma}Y_{\alpha}\right](t),$$

where the process $Y_\alpha$ is defined by

$$Y_\alpha(s) = \int_0^s (s-\sigma)^{-\alpha} e^{(s-\sigma)A} \phi dB^H(\sigma).$$

Then, using relation (31), we are reduced to showing that $Y_\alpha \in L^p([0,T];V)$, and since $Y_\alpha$ is a Gaussian process, it is sufficient to prove that $Y_\alpha \in L^2([0,T];V)$. We first treat the case of $H > \frac{1}{2}$: along the same lines as in the proof of Theorem 2, and taking up the notations introduced therein, it can be seen that

$$E\left[|Y_\alpha(t)|_V^2\right] = C(H) \sum_n \int_0^\infty \lambda^{-2(H-\alpha)} M_\alpha(\lambda,t) d\mu_n(\lambda),$$

where

$$M_\alpha(\lambda,t) = \int_0^{\lambda t} x^{-\alpha} e^{-x} \left( \int_0^x y^{-\alpha} (x-y)^{2H-2} e^{-y} dy \right) dx.$$

Since $M_\alpha$ is obviously bounded by a constant for all $t,\lambda > 0$, whenever $\alpha < H$, we get the desired result. Let us now turn to the case $H < \frac{1}{2}$. Following again the proof of Theorem 3, we can decompose $E[|Y_\alpha(t)|_V^2]$ as

$$E\left[|Y_\alpha(t)|_V^2\right] = \sum_n I_1(n) + I_2(n),$$

where $I_2(n)$, that contains the main part of the contribution to the norm of $Y_\alpha(t)$, is defined by

$$I_2(n) = \int_0^t \left| \int_s^t \left( (t-r)^{-\alpha} e^{(t-r)A} \phi e_n - (t-s)^{-\alpha} e^{(t-s)A} \phi e_n \right) \frac{\partial K}{\partial r}(r,s) \right|_V^2 ds.$$

Now, the same computations as in the proof of Theorem 3 yield

$$\begin{aligned} I_2(n) &\le C(H) \int_0^\infty d\mu_n(\lambda) \int_0^t du \int_0^u dv_1 \int_0^u dv_2 (u-v_1)^{H-3/2} (u-v_2)^{H-3/2} \\ &\quad \times \left( (v_1 v_2)^{-\alpha} e^{-\lambda(v_1+v_2)} - (v_1 u)^{-\alpha} e^{-\lambda(v_1+u)} - (u v_2)^{-\alpha} e^{-\lambda(u+v_2)} + u^{-2\alpha} e^{-2\lambda u} \right) \\ &= C(H) \int_0^\infty d\mu_n(\lambda) \int_0^t \left( \int_0^u (u-v)^{H-3/2} \left( u^{-\alpha} e^{-\lambda u} - v^{-\alpha} e^{-\lambda v} \right) dv \right)^2 du \\ &= C(H) \int_0^\infty \lambda^{-2(H-\alpha)} N(\lambda t) d\mu_n(\lambda), \end{aligned}$$

where $N(\tau)$ is given by

$$N(\tau) = \int_0^\tau \left( \int_0^x (x-y)^{H-3/2} \left( y^{-\alpha} e^{-y} - x^{-\alpha} e^{-x} \right) dy \right)^2 dx.$$

The following lemma ends the proof. □

**Lemma 4** *If $a < H$, then* $\sup_{\tau \ge 0} N(\tau) < \infty$

**Proof:** Left to the reader. □

## 5 Appendix: Proofs of Lemmas 1, 2 and 3.

**Proof of Lemma 1.**

If $\lambda \geq 1$, note that, by (24),

$$A(\lambda, t) \leq \left( \int_0^{\infty} v^{2H-2} e^{-v} dv \right) = C(H)$$

and also

$$\begin{aligned} A(\lambda, t) &\geq \left(1 - e^{-\lambda t}\right) \int_0^{\frac{\lambda t}{2}} v^{2H-2} dv \\ &\geq \left(1 - e^{-\lambda t}\right) \int_0^{\frac{t}{2}} v^{2H-2} e^{-v} dv \end{aligned}$$

and this a positive constant denoted generically by $c(H,t)$. The assertion (i) is proved. Suppose now that $\lambda \leq 1$. We let $t = 1$ for simplicity and we use the following facts: for $0 \leq x \leq 1$,

$$\begin{aligned} &2x \geq 1 - e^{-2x} \geq 2x/3 \\ &1 \geq e^{-x} \geq 1/3. \end{aligned}$$

We use the notation $A \asymp [c, C]B$ to mean $c < A/B < C$. We obtain

$$\begin{aligned} A(\lambda, 1) &\asymp [1/3, 1] \int_0^{\lambda} v^{2H-2} e^{-v} 2(\lambda - v)\, dv \\ &\asymp [2/9, 2] \cdot \int_0^{\lambda} v^{2H-2} (\lambda - v)\, dv \\ &= \lambda^{2H} \cdot [c(H); C(H)]. \end{aligned}$$

□

**Proof of Lemma 2.**

Doing the changes of variables $ar = y$ and $as = x$ we get

$$B(a, A) = a^{-2a-1} \int_0^a \left( \int_0^x (e^y - 1) y^{A-1} dy \right)^2 dx$$

and it suffices to observe that the quantity $K_A = \int_0^{\infty} \left( \int_0^x (e^y - 1) y^{A-1} dy \right)^2 dx$ is finite. □

**Proof of Lemma 3**.

First, let us replace the kernel $K$ by its singular part $c_H (t-s)^{H-\frac{1}{2}}$. In this case, with the notation $e(\lambda, s) = \exp\left(-2\lambda(1-s)\right)$, it holds that

$$\begin{aligned}
J(\lambda) &= c_H^2 \int_0^1 e(\lambda, s) \left( (1-s)^{H-\frac{1}{2}} + \left(H - \frac{1}{2}\right) \int_s^1 \left(e^{(r-s)\lambda} - 1\right) (r-s)^{H-\frac{3}{2}} dr \right)^2 ds \\
&= c_H^2 \int_0^1 e(\lambda, s) \left( (1-s)^{H-\frac{1}{2}} + \left(H - \frac{1}{2}\right) \lambda^{\frac{1}{2}-H} \int_0^{\lambda(1-s)} (e^v - 1) v^{H-\frac{3}{2}} dv \right)^2 ds \\
&= c_H^2 \int_0^1 e^{-2\lambda x} \left( x^{H-\frac{1}{2}} + \left(H - \frac{1}{2}\right) \lambda^{\frac{1}{2}-H} \int_0^{\lambda x} (e^v - 1) v^{H-\frac{3}{2}} dv \right)^2 dx \\
&= c_H^2 \lambda^{-2H} \int_0^{\lambda} e^{-2y} \left( y^{H-\frac{1}{2}} + \left(H - \frac{1}{2}\right) \int_0^{y} (e^v - 1) v^{H-\frac{3}{2}} dv \right)^2 dy \\
&\geq c_H^2 \lambda^{-2H} \int_0^{l} e^{-2y} \left( y^{H-\frac{1}{2}} + \left(H - \frac{1}{2}\right) \int_0^{y} (e^v - 1) v^{H-\frac{3}{2}} dv \right)^2 dy = \lambda^{-2H} c(l, H)
\end{aligned}$$

where for every $H \in (0, 1/2)$ and every $l > 0$ the constant $c(H, l)$ is also positive. Note that $c(H, l) \to 0$ when $l \to 0$, which indicates that our bound is of decaying quality for decreasing spectral gap... To finish, we recall that by (4), the kernel $K(t, s)$ can be written as $c_H(t-s)^{H-\frac{1}{2}}$ plus a function without singularities. Adding the second part does change the final estimation. But the proof is much longer and we prefer to omit it. □

# References


[1] Alos, E.; Mazet, O.; Nualart, D. Stochastic calculus with respect to Gaussian processes. *Ann. Probab.*, **29** (1999), no 2, 766-801.

[2] Coutin, L.; Decreusefond, L. *Stochastic differential equations driven by a fractional Brownian motion.* Preprint, 1999.

[3] DaPrato, G.; Zabczyk, J. *Stochastic equations in infinite dimensions*. Cambridge Univ. Press, 1992.

[4] Decreusefond, L.; Ustunel, A.-S. Stochastic analysis of the fractional Brownian motion. *Potential Analysis*, **10** (1997), 177-214.

[5] Grecksch , W. ;Ahn, V.V. A parabolic stochastic differential equation with fractional Brownian motion input. *Statistics and Probability Letters*, **41** (1999), 337-346.

[6] Duncan, T.E.; Maslowski, B.; Pasik-Duncan, B. *Fractional Brownian motion and stochastic equations in Hilbert spaces.* Preprint, 2001.

[7] Kleptsyna, M.L.; Kloeden, P.E.; Ahn, V.V. Existence and uniqueness theorems for stochastic differential equations with fractal Brownian motion. (Russian) Problemy Peredachi Informatsii **34** (1998), no. 4, 51–61; translation in *Problems Inform. Transmission 34 (1998)*, no. 4, 332–341 (1999).

[8] Lin, S.J. Stochastic calculus of fractional Brownian motion. *Stochastics and Stochastic Reports*, **55** (1995), 121-140.

[9] Mandelbrot, B.B.; Van Ness, J.W.; Fractional Brownian motion, fractional noises and application. *SIAM Review* **10** (1968), no. 4, 422-437.

[10] Maslowski, B.; Nualart, D. Stochastic evolution equations driven by fBm. Preprint, 2002.

[11] Nualart, D. *Malliavin Calculus and Related topics.* Springer Verlag, 1995.

[12] Peszat, S.; Zabczyk, J. Nonlinear stochastic wave and heat equations. *Probab. Theory Related Fields* **116** (2000), no. 3, 421–443.

[13] Reed, M.; Simon, B. *Methods of modern mathematical physics. 2nd ed.* Academic Press, Inc., 1980.

[14] Tindel, S., Viens, F. On space-time regularity for the stochastic heat equation on Lie groups. *J. Func. Analy.* **169** (1999), no. 2, 559-603.

[15] Tindel, S.; Viens, F. Regularity conditions for the stochastic heat equation on some Lie groups. *Seminar on Stochastic Analysis, Random Fields and Applications III, Centro Stefano Franscini, Ascona, September 1999.* Progress in Probability, **52** Birkhäuser (2002), 275-297.

[16] Tudor, C.A. *Stochastic calculus with respect to the infinite dimensional fractional Brownian motion.* Preprint, 2002.