The Euler scheme for a class of anticipating stochastic differential equations

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Abstract

Using the techniques of the Malliavin calculus and standard Itô calculus methods, we give an Euler scheme to approximate the solution of a class of anticipating stochastic differential equations.

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1 Introduction

The purpose of this work is to study the Euler Scheme in order to approximate solutions for a class of anticipating stochastic differential equation (in short ASDE).

The anticipating, or Skorohod, integral introduced in [7] represents an extension of the standard Itô integral for non-adapted integrands and coincides with this one if the integrand is adapted. The need of studying ASDE was followed in a natural way.

The existence and uniqueness of the solution of stochastic differential equation with the stochastic integral taken in the anticipating sense are not known in the general case. The difficulty with these equations is notorious: the Picard iterations method involves Malliavin derivative of successive orders and the procedure cannot be closed. Nevertheless, in particular linear cases, existence and unique are given. We refer to [1], [2] and [3] for anticipating equations in the Skorohod sense and to [6] for Stratonovich anticipating equations.

On the other hand, it was proved in [8] that the class of Skorohod integral processes $X = (X(t))_{t\in[0,1]}$, $X(t) = \int_0^t u_s dW(s)$, coincides, for smooth enough integrands, with the class of processes Y of the form $Y_t = \int_0^t \mathbb{E}\left[u_s/\mathbb{F}_{[s,t]^c}\right] dW(s)$. The last integral is an anticipating integral, it is not a martingale, but it enjoys similar properties with the classical Itô integrals. This fact leads to the introduction of the class of stochastic equations (6) as an intermediary step between the theory of Itô stochastic equations and ASDE. Moreover, in the particular case of linear coefficients, the solution of the equation can be explicitly obtained and a Black-Scholes market model with price dynamic following a such equation can be introduced (see [9]). Standard arguments give the existence and uniqueness of the solution of the equation (6) and the basic properties of the solution (see [8]).

Our aim is to give an Euler scheme to approximate the solution of (6). We combine standard Itô methods and techniques of the Malliavin calculus. Using this scheme, simulations for a special type of anticipating integral can be done.

Our paper is organized as follows: Section 2 contains some preliminaries on the Malliavin calculus and in Section 3 we give the Euler scheme to approximate the non-adapted solution of the equation (6). The speed of convergence of the Euler scheme is estimated in the cases of linear and non-linear coefficients.

2 Preliminaries

We start with some elements of the Malliavin calculus. We refer to [4] for a complete presentation of this topic. Let $(W(t))_{t\in[0,T]}$ be a standard Wiener process on the canonical Wiener space $(\Omega, \mathbb{F}, P)$ and let $(\mathbb{F}_t)_{t\in[0,T]}$ the filtration generated by W. A functional of the Brownian motion of the form

$$F = f\left(W(t_1), \dots, W(t_n)\right) \tag{1}$$

with $t_1, \dots, t_n \in [0,T]$ and $f \in C_b^\infty(R^n)$, is called a smooth random variable and this class is denoted by $\mathcal{S}$. The Malliavin derivative is defined on $\mathcal{S}$ as

$$D_t F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\left(W(t_1), \dots, W(t_n)\right) 1_{[0,t_i]}(t),\ t \in [0,T]$$

if F has the form (1). The operator D is closable and it can be extended to the closure of $\mathcal{S}$ with respect to the seminorm

$$\|F\|_{k,p}^p = \mathbb{E}\,|F|^p + \sum_{j=1}^{k} \mathbb{E}\|D^{(j)}F\|_{L^2(T)}^p$$

where $D^{(i)}$ denotes the i th iterated derivative. The adjoint of D is denoted by δ and it is called the Skorohod integral. That is, δ is defined on its domain

$$Dom(\delta) = \left\{ u \in L^2([0,T]\times\Omega) / \left| \mathbb{E}\int_0^T u_s D_s F ds \right| \leq C\|F\|_{L^2(\Omega)} \right\}$$

and it is given by the duality relationship

$$\mathbb{E}(F\delta(u)) = \mathbb{E}\int_0^T u_s D_s F ds, \quad u \in Dom(\delta), F \in \mathcal{S}.$$

Recall that the variance of the Skorohod integral is

$$\mathbb{E}(\delta^2(u)) = \mathbb{E}\int_0^T u_\alpha v_\alpha d\alpha + \mathbb{E}\int_0^T\int_0^T D_\beta u_\alpha D_\alpha u_\beta d\alpha d\beta \tag{2}$$

By $\mathbb{L}^{k,p}$ we denote the set $L^2([0,T];\mathbb{D}^{k,p})$, for $k \geq 1$ and $p \geq 2$ and we note that $\mathbb{L}^{k,p}$ is a subset of the domain of δ. The following generalized version of the Ocone-Clark formula was given in [5].

$$F = \mathbb{E}\left(F/\mathbb{F}_{[s,t]^c}\right) + \int_s^t \mathbb{E}\left(D_\alpha F/\mathbb{F}_{[\alpha,t]^c}\right) dW(\alpha) \text{ , for } F \in \mathbb{D}^{1,2}. \tag{3}$$

We will need the commutativity relationship between the derivative operator and the Skorohod integral

$$D_t\delta(u) = u_t + \delta(D_t u) \tag{4}$$

if all above terms are defined. Recall also that, if F is random variable Malliavin differentiable, measurable with respect to a σ-algebra $\mathbb{F}_A$, $A \in \mathcal{B}([0,T])$, then

$$DF = 0, \text{ on } A^c \times \Omega. \tag{5}$$

Consider now the stochastic differential equation

$$X(t) = X(0) + \int_0^t \sigma\left(s, \mathbb{E}\left(X(s)/\mathbb{F}_{[s,t]^c}\right)\right) dW(s) + \int_0^t b\left(s, \mathbb{E}\left(X(s)/\mathbb{F}_{[s,t]^c}\right)\right) ds. \tag{6}$$

Using the method of Picard's iterations and taking account that, by (2) and (5) it holds that

$$\mathbb{E}\left|\int_0^t \sigma\left(s, \mathbb{E}\left(X(s)/\mathbb{F}_{[s,t]^c}\right)\right) dW(s)\right|^2 = \mathbb{E}\int_0^t \left|\sigma\left(s, \mathbb{E}\left(X(s)/\mathbb{F}_{[s,t]^c}\right)\right)\right|^2 ds, \tag{7}$$

the existence and uniqueness of the solution of (6) can be obtained if we assume the functions σ and b satisfy the regularity conditions (A1), (A3)-(A5) below.

Remark 1 1. *Suppose that the initial condition $X(0) = x \in \mathbb{R}$ (or X_0 is adapted). In this case it is easy to see that the standard Picard iterations X_t^n are adapted to the filtration $\mathcal{F}_t$. Then the equation (6) is nothing else that the classical Itô equation.*

2. *If the initial value $X(0)$ is anticipating in this case the solution of (6) is also anticipating.*

3. *In (6) we can replace the deterministic integral by $\int_0^t b(s, X(s))$ and we will have again the existence and the uniqueness of the solution under usual conditions on b.*

4. *Using the identity $\int_0^t \mathbb{E}\left(v(s)/\mathbb{F}_{[s,t]^c}\right) dW(s) = \int_0^t (v(s) - r(s)) dW(s)$, where $r(s) = \delta\left(\mathbb{E}\left(D_s v(\cdot)/\mathcal{F}_{[\cdot,s]^c}\right) 1_{[0,s]}(\cdot)\right)$, assuming that the coefficient σ is linear, we can write (6) as*

$$\begin{aligned} X(t) &= X(0) + \int_0^t \sigma(s)X(s)dW(s) - \int_0^t \delta\left(\sigma(\cdot)\mathbb{E}\left(D_s X(\cdot)/\mathcal{F}_{[\cdot,s]^c}\right) 1_{[0,s]}(\cdot)\right) dW(s) \\ &\quad + \int_0^t b\left(s, \mathbb{E}\left(X(s)/\mathbb{F}_{[s,t]^c}\right)\right) ds \end{aligned}$$

and it can be seen that the above stochastic integrals are anticipating.

3 The Euler Scheme

Let $0 = t_0 \leq t_1 \ldots \leq t_n = T$ be a discretization of $[0,T]$ and δ the time step such that $\delta = \delta_n = \frac{T}{n}$. The process $Y^\delta = \{Y^\delta(t), 0 \leq t \leq T\}$ defined below will be considered to approximate the solution X. First we define Y^δ at t_k recursively as follows: $Y^\delta(0) = Y(0) \in$ $mathbbR$,

$$Y^\delta(t_{k+1}) = Y^\delta(t_k)+b(t_k, \mathbb{E}\left(Y^\delta(t_k))/\mathbb{F}_{[t_k,t_{k+1}]^c}\right)\delta+\sigma(t_k, \mathbb{E}\left(Y^\delta(t_k)/\mathbb{F}_{[t_k,t_{k+1}]^c}\right)(W(t_{k+1})-W(t_k))$$

for $k = 0, ..., n-1$. Next, $Y^\delta(t)$ can be defined for each $t \in [t_k, t_{k+1}[, k = 0, 1, \ldots, n-1$ as the following linear interpolation

$$Y^\delta(t) = Y^\delta(t_k) + \int_{t_k}^t b(t_k, \mathbb{E}\left(Y^\delta(t_k)/\mathbb{F}_{[t_k,t]^c}\right)ds + \int_{t_k}^t \sigma(t_k, \mathbb{E}\left(Y^\delta(t_k)/\mathbb{F}_{[t_k,t]^c}\right)dW(s). \quad (8)$$

We will make use of the following standing assumptions throughout the paper.

(A1) $\mathbb{E}|X(0)|^2 < \infty$

(A2) $\mathbb{E}|X(0) - Y^\delta(0)|^2 \leq K_1 \cdot \delta$

(A3) $|\sigma(t,x) - \sigma(t,y)|^2 + |b(t,x) - b(t,y)|^2 \leq K_2 \cdot |x-y|^2$

(A4) $|\sigma(t,x)|^2 + |b(t,x)|^2 \leq K_3 \cdot (1 + |x|^2)$

(A5) $|\sigma(s,x) - \sigma(t,x)|^2 + |b(s,x) - b(t,x)|^2 \leq K_4 \cdot (1+|x|^2) \cdot \mid s-t \mid$ for all $x, y \in \mathbb{R}, s, t \in [0,T]$ where the constants $K_1, \ldots, K_4$ do not depend on δ.

Theorem 1 *Assume (A1)-(A5) hold. Then there exists two positive constants A and B not depending on δ such that*

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |X(t) - Y^\delta(t)|^2\right) \leq \delta A \cdot e^{BT}.$$

Proofs: The proof of Theorem 1 is based upon the following lemmas 1-7.

Lemma 1 *Let $X(t)$ be the process satisfying equation (6). Then under (A1)-(A5) there exists two positive constants C_1 and C_2 such that*

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |X(t)|^2\right) \leq C_1(1 + |X(0)|^2)e^{C_2 T}.$$

Proof: Since $X(t)$ satisfies the equation (6) we have

$$\begin{aligned}\mathbb{E}\left|X(t)\right|^2 &\leq 3\left(\mathbb{E}\left|X(0)\right|^2 + \mathbb{E}\left(\int_0^t \sigma\left(s, \mathbb{E}\left(X(s)/\mathbb{F}_{[s,t]^c}\right)\right) dW(s)\right)^2\right.\\ &+ \left.\mathbb{E}\left(\int_0^t b\left(s, \mathbb{E}\left(X(s)/\mathbb{F}_{[s,t]^c}\right)\right) ds\right)^2\right).\end{aligned}$$

From hypothesis (A1) and growth bound (A4) the following inequality holds.

$$\begin{aligned}\mathbb{E}\left|X(t)\right|^2 &\leq 3\left(\mathbb{E}\left|X_0\right|^2 + \mathbb{E}\left(\int_0^t \sigma\left(s, E\left(X(s)/\mathbb{F}_{[s,t]^c}\right)\right) dW(s)\right)^2 + \mathbb{E}\left(\int_0^t b\left(s, \mathbb{E}\left(X(s)/\mathbb{F}_{[s,t]^c}\right)\right) ds\right)^2\right)\\ &\leq 3\left(\mathbb{E}\left|X_0\right|^2 + \int_0^t \mathbb{E}\left(\sigma\left(s, \mathbb{E}\left(X(s)/\mathbb{F}_{[s,t]^c}\right)\right)\right)^2 ds + t\int_0^t \mathbb{E}\left(b\left(s, \mathbb{E}\left(X(s)/\mathbb{F}_{[s,t]^c}\right)\right)\right)^2 ds\right)\\ &\leq 3\left(\left|X(0)\right|^2 + K_3 t(1+t) + K_3(1+t)\int_0^t \mathbb{E}\left|X(s)\right|^2 ds\right).\end{aligned}$$

By applying Gronwall's inequality we conclude that

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|X(t)\right|^2\right) \leq C_1(1+|X(0)|^2)e^{C_2 T},$$

where $C_1 = \max\left\{3, 3K_3T\left(1+T\right)\right\}$ and $C_2 = 3K_3\left(1+T\right)$. ■

The following lemma is proved in the same way, which proof is left to the reader.

Lemma 2 *Under (A1)-(A5), there exists two positive constants C_3 and C_4 such that*

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|Y^\delta(t)\right|^2\right) \leq C_3(1+|Y^\delta(0)|^2)e^{C_4 T}.$$

The next step is to prove a bound for the increment of the solution $X(t) - X(s)$. We will treat two cases; first we will consider the case when the coefficients of (6) are linear and is this situation the estimation follows easier assuming the boundnesss of the Malliavin derivative of the initial value. Then, under supplementary conditions, we will treat the case of nonlinear coefficients.

Lemma 3 *Suppose that the coefficients of (6) are linear functions and assume that*

(A6) There exists $K_6 > 0$ such that $\mathbb{E}\left|D_\alpha X(0)\right|^2 \leq K_6$, for every $\alpha \in [0,T]$.

Under (A1)-(A6), there exists a positive constant C_5 such that the solution $X(t)$ of (6) satisfies

$$\mathbb{E}\left|X(t) - X(s)\right|^2 \leq C_5(t-s).$$

Proof: Since X satisfies the equation (6) we have

$$\begin{aligned}
\mathbb{E}\left|X(t)-X(s)\right|^2 &\leq 4\Bigg(\mathbb{E}\left(\int_0^s b\left(u,\mathbb{E}\left(X(u)/\mathbb{F}_{[u,t]^c}\right)\right)-b\left(u,\mathbb{E}\left(X(u)/\mathbb{F}_{[u,s]^c}\right)\right)du\right)^2 \\
&+\mathbb{E}\left(\int_s^t b\left(u,\mathbb{E}\left(X(u)/\mathbb{F}_{[u,s]^c}\right)\right)du\right)^2 \\
&+\mathbb{E}\left(\int_0^s \sigma\left(u,\mathbb{E}\left(X(u)/\mathbb{F}_{[u,t]^c}\right)\right)-\sigma\left(u,\mathbb{E}\left(X(u)/\mathbb{F}_{[u,s]^c}\right)\right)dW(u)\right)^2 \\
&+\mathbb{E}\left(\int_s^t \sigma\left(u,\mathbb{E}\left(X(u)/\mathbb{F}_{[u,s]^c}\right)\right)dW(u)\right)^2\Bigg).
\end{aligned}$$

From hypothesis (A1), the Lipschitz condition (A3) and growth bound (A4) the following inequality holds

$$\begin{aligned}
\mathbb{E}\left|X(t)-X(s)\right|^2 &\leq 4\Bigg(K_2(1+s)\int_0^s \mathbb{E}\left(\mathbb{E}\left(X(u)/\mathbb{F}_{[u,t]^c}\right)-\mathbb{E}\left(X(u)/\mathbb{F}_{[u,s]^c}\right)\right)^2 du \\
&+ (1+t-s)K_3\int_s^t\left(1+\mathbb{E}\left|X(u)\right|^2\right)du\Bigg). \qquad (9)
\end{aligned}$$

By applying Lemma 1 we get that

$$\begin{aligned}
\mathbb{E}\left|X(t)-X(s)\right|^2 &\leq 4\Bigg(K_2(1+s)\int_0^s \mathbb{E}\left(\mathbb{E}\left(X(u)/\mathbb{F}_{[u,s]^c}\right)-\mathbb{E}\left(X(u)/\mathbb{F}_{[u,t]^c}\right)\right)^2 du \\
&+ (t-s)(1+t-s)K_3\left(1+C_1\left(1+\mathbb{E}|X(0)|^2\right)e^{C_2T}\right)\Bigg). \qquad (10)
\end{aligned}$$

Let now for simplicity $\sigma(x)=\sigma x$ and $b(x)=bx$ where σ and b are real numbers. In this case it is easy to see that

$$\begin{aligned}
&\mathbb{E}\left(X(r)/\mathbb{F}_{[r,t]^c}\right) \\
=\ &\mathbb{E}\left(X(0)/\mathbb{F}_{[r,t]^c}\right)+\sigma\mathbb{E}\left(\int_0^r \mathbb{E}\left(X(\alpha)/\mathbb{F}_{[\alpha,r]^c}\right)dW(\alpha)/\mathbb{F}_{[r,t]^c}\right)+b\mathbb{E}\left(\int_0^r \mathbb{E}\left(X(\alpha)/\mathbb{F}_{[\alpha,r]^c}\right)d\alpha/\mathbb{F}_{[r,t]^c}\right) \\
=\ &\mathbb{E}\left(X(0)/\mathbb{F}_{[r,t]^c}\right)+\sigma\int_0^r \mathbb{E}\left(X(\alpha)/\mathbb{F}_{[\alpha,t]^c}\right)dW(\alpha)+b\int_0^r \mathbb{E}\left(X(\alpha)/\mathbb{F}_{[\alpha,t]^c}\right)d\alpha
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left(X(u)/\mathbb{F}_{[u,s]^c}\right)-\mathbb{E}\left(X(u)/\mathbb{F}_{[u,t]^c}\right) \qquad (11) \\
&=\mathbb{E}\left(X(0)/\mathbb{F}_{[u,s]^c}\right)-\mathbb{E}\left(X(0)/\mathbb{F}_{[u,t]^c}\right)+\sigma\int_0^u\left(\mathbb{E}\left(X(\alpha)/\mathbb{F}_{[\alpha,s]^c}\right)-\mathbb{E}\left(X(\alpha)/\mathbb{F}_{[\alpha,t]^c}\right)\right)dW(\alpha) \\
&+b\int_0^u\left(\mathbb{E}\left(X(\alpha)/\mathbb{F}_{[\alpha,s]^c}\right)-\mathbb{E}\left(X(\alpha)/\mathbb{F}_{[\alpha,t]^c}\right)\right)d\alpha. \qquad (12)
\end{aligned}$$

But, by (3) we have

$$\mathbb{E}\left(X(0)/\mathbb{F}_{[u,s]^c}\right)-\mathbb{E}\left(X(0)/\mathbb{F}_{[u,t]^c}\right)=\mathbb{E}\left(X(0)/\mathbb{F}_{[u,s]^c}\right)-\mathbb{E}\left(\mathbb{E}\left(X(0)/\mathbb{F}_{[u,s]^c}\right)/\mathbb{F}_{[u,t]^c}\right)$$
$$=\int_u^t \mathbb{E}\left(D_\alpha \mathbb{E}\left(X(0)/\mathbb{F}_{[u,s]^c}\right)/\mathbb{F}_{[\alpha,t]^c}\right)dW(\alpha)=\int_s^t \mathbb{E}\left(\mathbb{E}\left(D_\alpha X(0)/\mathbb{F}_{[u,s]^c}\right)/\mathbb{F}_{[\alpha,t]^c}\right)dW(\alpha)$$

and that implies, by (A6),

$$\mathbb{E}\left|\mathbb{E}\left(X(0)/\mathbb{F}_{[u,s]^c}\right)-\mathbb{E}\left(X(0)/\mathbb{F}_{[u,t]^c}\right)\right|^2 \le \int_s^t \mathbb{E}\left|D_\alpha X(0)\right|^2 d\alpha \le K_6(t-s). \tag{13}$$

Denote by $A_{s,t}(u)=\left|\mathbb{E}\left(X(u)/\mathbb{F}_{[u,s]^c}\right)-\mathbb{E}\left(X(u)/\mathbb{F}_{[u,t]^c}\right)\right|^2$. By (12) and (13) we have

$$A_{s,t}(u) \le 3\mathbb{E}\left|\mathbb{E}\left(X(0)/\mathbb{F}_{[u,s]^c}\right)-\mathbb{E}\left(X(0)/\mathbb{F}_{[u,t]^c}\right)\right|^2+3(1+u)\int_0^u A_{s,t}(\alpha)d\alpha$$
$$\le 3K_6(t-s)+3(1+T)\int_0^u A_{s,t}(\alpha)d\alpha$$

and we conclude by Gronwall that for every u,s,t

$$A_{s,t}(u) \le 3K_6 e^{3T(1+T)}(t-s). \tag{14}$$

By combining (10) and (14) we obtain

$$\mathbb{E}\left|X(t)-X(s)\right|^2 \le 4(t-s)(1+T)\left(K_2 3K_6 e^{3T(1+T)}+K_3\left(1+C_1\left(1+\mathbb{E}|X(0)|^2\right)e^{C_2T}\right)\right). \tag{15}$$

■

Our next goal is to prove the above lemma for nonlinear coefficients. But in this we need case we need to make to supplementary hypothesis. That is, there exist two positive constants K_7 and K_8 such that for every t,x,y

(A7) $\quad |\partial_2\sigma(t,x)-\partial_2\sigma(t,y)|^2+|\partial_2 b(t,x)-\partial_2 b(t,y)|^2 \le K_7|x-y|,$

(A8) $\quad |\partial_2\sigma(t,x)|+|\partial_2 b(t,x)| \le K_8.$

Note that (A8) implies obviously (A2).

We need first two Lemmas who study the Malliavin differentiability of the solution.

Lemma 4 *Assume that (A1)-(A8) hold and define, for every $t \le T$, the processes*

$$X^n(t)=X(0)+\int_0^t \sigma\left(s,\mathbb{E}\left(X^{n-1}(s)/\mathbb{F}_{[s,t]^c}\right)\right)dW(s)+\int_0^t b\left(s,\mathbb{E}\left(X^{n-1}(s)/\mathbb{F}_{[s,t]^c}\right)\right)ds \tag{16}$$

with $X^0(t)=X(0)$. Then for every $t\in[0,T]$ and $n\ge 1$ the process X^n is Malliavin differentiable and there exists a constant $C_6=C_6(n)>0$ such that

$$\mathbb{E}\left|D_\beta X^n(t)\right|^2 \le C_6 \text{ for every } \beta,t\in[0,T]\text{ , } \beta\le t,\ \forall n\ge 1. \tag{17}$$

Proof: We will use the induction on n. We have, by (4), that for $\beta \le t$,

$$\begin{aligned} D_\beta X^1(t) &= D_\beta X(0) + \sigma\left(\beta, \mathbb{E}\left(X(0)/\mathbb{F}_{[\beta,t]^c}\right)\right) \\ &\quad + \int_0^t D_\beta \sigma\left(s, \mathbb{E}\left(X(0)/\mathbb{F}_{[s,t]^c}\right)\right) dW(s) + \int_0^t D_\beta b\left(s, \mathbb{E}\left(X(0)/\mathbb{F}_{[s,t]^c}\right)\right) ds \\ &= D_\beta X(0) + \sigma\left(\beta, \mathbb{E}\left(X(0)/\mathbb{F}_{[\beta,t]^c}\right)\right) \\ &\quad + \int_\beta^t \partial_2 \sigma\left(s, \mathbb{E}\left(X(0)/\mathbb{F}_{[s,t]^c}\right)\right) \mathbb{E}\left(D_\beta X(0)/\mathbb{F}_{[s,t]^c}\right) dW(s) \\ &\quad + \int_\beta^t \partial_2 b\left(s, \mathbb{E}\left(X(0)/\mathbb{F}_{[s,t]^c}\right)\right) \mathbb{E}\left(D_\beta X(0)/\mathbb{F}_{[s,t]^c}\right) ds \end{aligned}$$

and thus

$$\begin{aligned} \mathbb{E}\left|D_\beta X^1(t)\right|^2 &\le 4\Big(\mathbb{E}\left|D_\beta X(0)\right|^2 + \left|\sigma\left(\beta, \mathbb{E}\left(X(0)/\mathbb{F}_{[\beta,t]^c}\right)\right)\right|^2 \\ &\quad + \mathbb{E}\left|\int_\beta^t \partial_2 \sigma\left(s, \mathbb{E}\left(X(0)/\mathbb{F}_{[s,t]^c}\right)\right) \mathbb{E}\left(D_\beta X(0)/\mathbb{F}_{[s,t]^c}\right) dW(s)\right|^2 \\ &\quad + \mathbb{E}\left|\int_\beta^t \partial_2 b\left(s, \mathbb{E}\left(X(0)/\mathbb{F}_{[s,t]^c}\right)\right) \mathbb{E}\left(D_\beta X(0)/\mathbb{F}_{[s,t]^c}\right) ds\right|^2\Big). \end{aligned}$$

Using the properties (2), (5) and hypothesis (A6) , (A8) we get

$$\mathbb{E}\left|D_\beta X^1(t)\right|^2 \le 4\left(K_6 + K_3(1 + \mathbb{E}\left|X_0\right|^2) + K_8 K_6 (t-\beta)(1+t-\beta)\right).$$

The induction step is similar with the case $n = 1$. ■

Lemma 5 *Let X^n be the processes given by (16). Then, for every $t \in [0,T]$, the sequence of random variables $X^n(t)$ converges to $X(t)$ in the Sobolev space $\mathbb{D}^{1,2}$.*

Proof: Throughout this proof C will denotes a generic constant, its value can be different from a line to another. Note first that

$$\begin{aligned} D_\beta X^1(t) - D_\beta X(0) &= \sigma\left(\beta, \mathbb{E}\left(X(0)/\mathbb{F}_{[\beta,t]^c}\right)\right) 1_{[0,t]}(\beta) \\ &+ \left(\int_\beta^t \partial_2 \sigma\left(s, \mathbb{E}\left(X(0)/\mathbb{F}_{[s,t]^c}\right)\right) \mathbb{E}\left(D_\beta X(0)/\mathbb{F}_{[s,t]^c}\right) dW(s)\right) 1_{[0,t]}(\beta) \\ &+ \left(\int_0^t \partial_2 \sigma\left(s, \mathbb{E}\left(X(0)/\mathbb{F}_{[s,t]^c}\right)\right) \mathbb{E}\left(D_\beta X(0)/\mathbb{F}_{[s,t]^c}\right) dW(s)\right) 1_{[t,T]}(\beta) \\ &+ \left(\int_\beta^t \partial_2 b\left(s, \mathbb{E}\left(X(0)/\mathbb{F}_{[s,t]^c}\right)\right) \mathbb{E}\left(D_\beta X(0)/\mathbb{F}_{[s,t]^c}\right) ds\right) 1_{[0,t]}(\beta) \\ &+ \left(\int_0^t \partial_2 b\left(s, \mathbb{E}\left(X(0)/\mathbb{F}_{[s,t]^c}\right)\right) \mathbb{E}\left(D_\beta X(0)/\mathbb{F}_{[s,t]^c}\right) ds\right) 1_{[t,T]}(\beta). \end{aligned}$$

Therefore

$$\mathbb{E}\left|D_\beta X^1(t) - D_\beta X(0)\right|^2 \le C\left(1 + \mathbb{E}\left|X(0)\right|^2\right) 1_{[0,t]}(\beta) + Ct$$

where we used the set of conditions (A1)-(A8) and the properties of the Skorohod integral. We obtain

$$\mathbb{E}\int_0^T \left|D_\beta X^1(t) - D_\beta X(0)\right|^2 d\beta \le Ct. \tag{18}$$

The relation (4) gives

$$\begin{aligned} D_\beta X^{n+1}(t) &= D_\beta X(0) + \sigma\left(\beta, \mathbb{E}\left(X^n(\beta)/\mathbb{F}_{[\beta,t]^c}\right)\right) 1_{[0,t]}(\beta) \\ &+ \left(\int_\beta^t \partial_2\sigma\left(s, \mathbb{E}\left(X^n(s)/\mathbb{F}_{[s,t]^c}\right)\right) \mathbb{E}\left(D_\beta X^n(s)/\mathbb{F}_{[s,t]^c}\right) dW(s)\right) 1_{[0,t]}(\beta) \\ &+ \left(\int_0^t \partial_2\sigma\left(s, \mathbb{E}\left(X^n(s)/\mathbb{F}_{[s,t]^c}\right)\right) \mathbb{E}\left(D_\beta X^n(s)/\mathbb{F}_{[s,t]^c}\right) dW(s)\right) 1_{[t,T]}(\beta) \\ &+ \left(\int_\beta^t \partial_2 b\left(s, \mathbb{E}\left(X^n(s)/\mathbb{F}_{[s,t]^c}\right)\right) \mathbb{E}\left(D_\beta X^n(s)/\mathbb{F}_{[s,t]^c}\right) ds\right) 1_{[0,t]}(\beta) \\ &+ \left(\int_0^t \partial_2 b\left(s, \mathbb{E}\left(X^n(s)/\mathbb{F}_{[s,t]^c}\right)\right) \mathbb{E}\left(D_\beta X^n(s)/\mathbb{F}_{[s,t]^c}\right) ds\right) 1_{[t,T]}(\beta). \end{aligned}$$

We will have, using Lemma 4 and the conditions imposed on the coefficients, that

$$\begin{aligned} \mathbb{E}\left|D_\beta X^{n+1}(t) - D_\beta X^n(t)\right|^2 &\le C\Big(\mathbb{E}\left|X^n(\beta) - X^{n-1}(\beta)\right|^2 1_{[0,t]}(\beta) \\ + \int_0^t \mathbb{E}\left|X^n(s) - X^{n-1}(s)\right|^2 ds &+ \int_0^t \mathbb{E}\left|D_\beta X^n(s) - D_\beta X^{n-1}(s)\right|^2 ds.\Big) \end{aligned}$$

We refer to [8] for the following bound

$$\mathbb{E}\left|X^n(t) - X^{n-1}(t)\right|^2 \le \frac{(Ct)^n}{n!} \text{ for every } t \in [0,T] \tag{19}$$

and (19) implies

$$\mathbb{E}\left|D_\beta X^{n+1}(t) - D_\beta X^n(t)\right|^2 \le \frac{(Ct)^{n+1}}{(n+1)!} + \int_0^t \mathbb{E}\left|D_\beta X^n(s) - D_\beta X^{n-1}(s)\right|^2 ds. \tag{20}$$

Combining (18) and (20) it not difficult to prove by induction that

$$\mathbb{E}\left|D_\beta X^{n+1}(t) - D_\beta X^n(t)\right|^2 \le \frac{(Ct)^{n+1}}{(n+1)!}. \tag{21}$$

Consequently, by (19) and (21), we get

$$\|X^{n+1}(t) - X^n(t)\|_{1,2}^2 = \mathbb{E}\left|X^{n+1}(t) - X^n(t)\right|^2 + \mathbb{E}\int_0^T \left(D_\alpha(X^{n+1}(t) - X^n(t))\right)^2 d\alpha \le \frac{(CT)^{(n+1)}}{(n+1)!}$$

and the sequence $X(0) + \sum_{n=0}^{\infty}\left(X^{n+1}(t) - X^n(t)\right)$ converges in the Hilbert space $\mathbb{D}^{1,2}$ to a limit which cannot be anything else that $X(t)$. ■

Lemma 6 *For every $\beta, t \in [0,T]$ with $\beta \le t$ we have that*

$$\mathbb{E}\left|D_\beta X(t)\right|^2 \le C_7 \tag{22}$$

where C_7 is a positive constant not depending on β, t.

Proof: The prior Lemma shows that $X(t)$ is differentiable in the Malliavin sense for every t. Let $\beta \le t$. We have

$$\begin{aligned} D_\beta X(t) &= D_\beta X(0) + \sigma\left(\beta, \mathbb{E}\left(X(\beta)/\mathbb{F}_{[\beta,t]^c}\right)\right) \\ &\quad + \int_\beta^t \partial_2\sigma\left(s, \mathbb{E}\left(X(s)/\mathbb{F}_{[s,t]^c}\right)\right)\mathbb{E}\left(D_\beta X(s)/\mathbb{F}_{[s,t]^c}\right) dW(s) \\ &\quad + \int_\beta^t \partial_2 b\left(s, \mathbb{E}\left(X(s)/\mathbb{F}_{[s,t]^c}\right)\right)\mathbb{E}\left(D_\beta X(s)/\mathbb{F}_{[s,t]^c}\right) ds. \end{aligned}$$

Therefore, applying (A3) and (A8)

$$\mathbb{E}\left|D_\beta X(t)\right|^2 \le 4(\mathbb{E}\left|D_\beta X(0)\right|^2 + K_3(1+\sup_\beta \mathbb{E}\left|X_\beta\right|^2) + K_8(1+t-\beta)\int_0^t \mathbb{E}\left|D_\beta X(s)\right|^2 ds.$$

An usual application of Gronwall lemma and condition (A6) will give the conclusion. ■

Lemma 7 *Let $X = (X(t))_{t\in[0,T]}$ be the solution of equation (6) and assume that hypothesis (A1)-(A8) are satisfied. Then*

$$\mathbb{E}\left|X(t)-X(s)\right| \le C_8|t-s| \tag{23}$$

where C_8 denotes a positive constant.

Proof: The increments of the process X can be written as

$$\begin{aligned} X(t)-X(s) &= \int_s^t \sigma\left(r, \mathbb{E}\left(X(r)/\mathbb{F}_{[r,t]^c}\right)\right) dW(r) + \int_s^t b\left(r, \mathbb{E}\left(X(r)/\mathbb{F}_{[r,t]^c}\right)\right) dr \\ &+ \int_0^s \left(\sigma\left(r, \mathbb{E}\left(X(r)/\mathbb{F}_{[r,t]^c}\right)\right) - \sigma\left(r, \mathbb{E}\left(X(r)/\mathbb{F}_{[r,s]^c}\right)\right)\right) dW(r) \\ &+ \int_0^s \left(b\left(r, \mathbb{E}\left(X(r)/\mathbb{F}_{[r,t]^c}\right)\right) - b\left(r, \mathbb{E}\left(X(r)/\mathbb{F}_{[r,s]^c}\right)\right)\right) dr \end{aligned}$$

and its square mean can be majorized by

$$\begin{aligned} &\mathbb{E}\left|X(t)-X(s)\right|^2 \\ \le\ & C\mathbb{E}\int_s^t \left|\sigma\left(r, \mathbb{E}\left(X(r)/\mathbb{F}_{[r,t]^c}\right)\right)\right|^2 ds + C(t-s)\mathbb{E}\int_s^t \left|b\left(r, \mathbb{E}\left(X(r)/\mathbb{F}_{[r,t]^c}\right)\right)\right|^2 ds \\ & + C\int_0^s \mathbb{E}\left|\sigma\left(r, \mathbb{E}\left(X(r)/\mathbb{F}_{[r,t]^c}\right)\right) - \sigma\left(r, \mathbb{E}\left(X(r)/\mathbb{F}_{[r,s]^c}\right)\right)\right|^2 dr \\ & + C(\int_0^s \mathbb{E}\left|b\left(r, \mathbb{E}\left(X(r)/\mathbb{F}_{[r,t]^c}\right)\right) - b\left(r, \mathbb{E}\left(X(r)/\mathbb{F}_{[r,s]^c}\right)\right)\right|^2 dr. \end{aligned} \tag{24}$$

The first two terms of the right side are bounded by $C(t-s)$ by hypothesis and Jensen inequality. The last two expressions from above can be treated in a similar way. First, note that

$$\left|\sigma\left(r,\mathbb{E}\left(X(r)/\mathbb{F}_{[r,t]^c}\right)\right)-\sigma\left(r,\mathbb{E}\left(X(r)/\mathbb{F}_{[r,s]^c}\right)\right)\right|^2\leq K_2\left|\mathbb{E}\left(X(r)/\mathbb{F}_{[r,t]^c}\right)-\mathbb{E}\left(X(r)/\mathbb{F}_{[r,s]^c}\right)\right|^2$$

and by Ocone-Clark formula (3)

$$\begin{aligned}&\mathbb{E}\left(X(r)/\mathbb{F}_{[r,s]^c}\right)-\mathbb{E}\left(X(r)/\mathbb{F}_{[r,t]^c}\right)=\mathbb{E}\left(X(r)/\mathbb{F}_{[r,s]^c}\right)-\mathbb{E}\left(\mathbb{E}\left(X(r)/\mathbb{F}_{[r,s]^c}\right)/\mathbb{F}_{[r,t]^c}\right)\\&=\int_r^t\mathbb{E}\left(D_\beta\mathbb{E}\left(X(r)/\mathbb{F}_{[r,s]^c}\right)/\mathbb{F}_{[\beta,t]^c}\right)dW(\beta)=\int_s^t\mathbb{E}\left(\mathbb{E}\left(D_\beta X(r)/\mathbb{F}_{[r,s]^c}\right)/\mathbb{F}_{[\beta,t]^c}\right)dW(\beta)\end{aligned}$$

where at the last line we used the property (5) of the Malliavin derivative. Therefore we obtain by using Lemma 6

$$E\left|\mathbb{E}\left(X(r)/\mathbb{F}_{[r,s]^c}\right)-\mathbb{E}\left(X(r)/\mathbb{F}_{[r,t]^c}\right)\right|^2\leq C\int_s^t\mathbb{E}(D_\beta X(r))^2d\beta\leq C(t-s)$$

and combining the last relation and (24) we finish the proof. ■

Proof of Theorem 1 Let $Z(T)=\mathbb{E}\left\{\sup_{0\leq t\leq T}\left(|X(t)-Y^\delta(t)|^2\right)\right\}$ and $c(s)=\frac{[sn]}{n}$, $s\in[0,T]$. By (6) and (8) we can write

$$Z(T)\leq 4\mathbb{E}\left(|X(0)-Y^\delta(0)|^2+\sup_{0\leq t\leq T}(\mathcal{I}(t))+\sup_{0\leq t\leq T}(\mathcal{J}(t))+\sup_{0\leq t\leq T}(\mathcal{K}(t))+\sup_{0\leq t\leq T}(\mathcal{L}(t))\right)$$

where

$$\begin{aligned}\mathcal{I}(t)&=\left|\int_0^t\left(\sigma\left(c(s),\mathbb{E}\left(X(c(s))/\mathbb{F}_{[s,t]^c}\right)\right)-\sigma\left(c(s),\mathbb{E}\left(Y^\delta(c(s))/\mathbb{F}_{[s,t]^c}\right)\right)W(s)\right)\right.\\&\quad\left.+\int_0^t\left(b\left(c(s),\mathbb{E}\left(X(c(s))/\mathbb{F}_{[s,t]^c}\right)\right)-b\left(c(s),\mathbb{E}\left(Y^\delta(c(s))/\mathbb{F}_{[s,t]^c}\right)\right)ds\right)\right|^2,\end{aligned}$$

$$\begin{aligned}\mathcal{J}(t)&=\left|\int_0^t\left(\sigma\left(c(s),\mathbb{E}\left(X(s)/\mathbb{F}_{[s,t]^c}\right)\right)-\sigma\left(c(s),\mathbb{E}\left(X(c(s))/\mathbb{F}_{[s,t]^c}\right)\right)dW(s)\right)\right.\\&\quad\left.+\int_0^t\left(b\left(c(s),\mathbb{E}\left(X(s)/\mathbb{F}_{[s,t]^c}\right)\right)-b\left(c(s),\mathbb{E}\left(X(c(s))/\mathbb{F}_{[s,t]^c}\right)\right)ds\right)\right|^2,\end{aligned}$$

$$\begin{aligned}\mathcal{K}(t)&=\left|\int_0^t\left(\sigma\left(s,\mathbb{E}\left(X(s)/\mathbb{F}_{[s,t]^c}\right)\right)-\sigma\left(c(s),\mathbb{E}\left(X(s)/\mathbb{F}_{[s,t]^c}\right)\right)dW(s)\right)\right.\\&\quad\left.+\int_0^t\left(b\left(s,\mathbb{E}\left(X(s)/\mathbb{F}_{[s,t]^c}\right)\right)-b\left(c(s),\mathbb{E}\left(X(s)/\mathbb{F}_{[s,t]^c}\right)\right)ds\right)\right|^2\end{aligned}$$

and

$$\mathcal{L}(t) = \left| \int_0^t \left(\sigma\left(c(s), \mathbb{E}\left(Y^\delta(c(s))/\mathbb{F}_{[s,t]^c}\right)\right) - \sigma\left(c(s), \mathbb{E}\left(Y^\delta(c(s))/\mathbb{F}_{[c(s),t]^c}\right)\right)\right) dW(s) \right.$$
$$\left. + \int_0^t \left(b\left(c(s), \mathbb{E}\left(Y^\delta(c(s))/\mathbb{F}_{[s,t]^c}\right)\right) - b\left(c(s), \mathbb{E}\left(Y^\delta(c(s))/\mathbb{F}_{[c(s),t]^c}\right)\right)\right) ds \right|^2.$$

We have the following estimates

$$\mathbb{E}\left(\sup_{0\le t\le T} (\mathcal{I}(t))\right) \le 2(T+1)K_2 \int_0^T Z(s)ds \ , \ \mathbb{E}\left(\sup_{0\le t\le T} (\mathcal{J}(t))\right) \le \delta 2(T+1)TK_2C_5,$$

$$\mathbb{E}\left(\sup_{0\le t\le T} (\mathcal{K}(t))\right) \le \delta 2(T+1)TK_4\left(1 + C_1\left(1+|X(0)|^2\right)e^{C_2T}\right)$$

and using the techniques used in the proof of Lemma 3, together with Lemmas 6 and 7, we get $\mathbb{E}\left(\sup_{0\le t\le T} (\mathcal{I}(t))\right) \le K_9\delta$. By combining these estimates and using hypothesis $(A2)$ we obtain

$$Z(T) \le A\delta + B\int_0^T Z(s)ds, \tag{25}$$

where $A = 4K_1 + 8\,(T+1)\,K_2C_5T + 8\,(T+1)\,K_4T\left(1 + C_1\left(1+\mathbb{E}|X(0)|^2\right)e^{C_2T}\right)$ and $B = 8(T+1)K_2$. Applying Gronwall inequality to (25) we have $Z(T) \le \delta A^{BT}$. ■

Remark 2 *In some particular situation, the approximations Y^δ can be explicitly computed. Suppose $\sigma = 1$, $b = 0$ and let the initial value be $W(1) - W(t)$. Then $Y^\delta(0) = W(1)$ and*

$$Y^\delta(t_1) = W(1) + \mathbb{E}\left(W(1)/\mathbb{F}_{[0,t_1]^c}\right)W(1) = W(1) + \left(W(1) - W(t_1)\right)W(1).$$

Using the independence of increments of the Wiener process, it's clear that at any step the process Y^δ can be concretely found without any conditional expectation appearing in its expression. Therefore, numerical solution of the equation can be obtained.

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References

[1] R. Buckdahn (1991): *Linear stochastic Skorohod differential equations.* Prob. Theory Rel. Fields, 90: 223-240.

[2] R. Buckdahn (1992): *Skorohod stochastic differential equations of diffusion type.* Prob. Theory Rel. Fields, 92: 297-234.

[3] R. Buckdahn, D. Nualart (1994): *Linear stochastic differential equations and Wick product.* Prob. Theory Rel. Fields, 99: 501-525.

[4] D. Nualart (1995): *The Malliavin calculs and related topics.* Springer-Verlag.

[5] D. Nualart, E. Pardoux (1988): *Stochastic calculus with anticipating integrands.* Probab. Theory Relat. Fields, 78: 535-581.

[6] D. Ocone, E. Pardoux (1989): *A generalized Itô-Ventzell formula. Application to a class of anticipating stochastic differential equations.* Ann. Inst. Henri Poincaré, 25: 39-71.

[7] A.V. Skorohod (1975): *On a generalization of a stochastic integral.* Th. Probab. Applic., XX: 219-233.

[8] Tudor C.A. (2002): *Martingale type stochastic calculus for anticipating integral processes.* To appear in Bernoulli.

[9] Tudor C.A. (2003): *Itô-Skorohod stochastic equations and applications to finance.* Preprint, submitted.