KRAMERS-SMOLUCHOWSKI APPROXIMATION FOR STOCHASTIC EVOLUTION EQUATIONS WITH FBM

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Let $\{B_t^H, t \in [0, \tau]\}$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$. We prove Kramers- Smoluchowski approximation for the solution of the equation $X_t = x + B_t^H + \int_0^t b(X_s)ds$. The case $H = 1/2$ is the classical situation, which may describe the motion of particles in a fluid.
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1. INTRODUCTION

Let $(\Omega, \mathcal{F}, I\!\!P)$ be a probability space and $\tau > 0$ a fixed time. Consider $\{B_t^H, t \in [0, \tau]\}$ to be a fractional Brownian motion with Hurst parameter $H \in (0,1)$. That is B^H is a centered Gaussian process with covariance

$$R(t,s) = E(B_s^H B_t^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

Consider the following stochastic differential equation

$$X_t = x + B_t^H + \int_0^t b(X_s)ds, \tag{1.1}$$

where $b : R \to R$ is a measurable function satisfying a global Lipschitz condition. In this case one can prove, by the classical Picard iterations method, the existence and the uniqueness of a strong solution of (1.1).

For every $\alpha \in (0, \infty)$, consider the system (1.2)-(1.3)

$$dX_t^{(\alpha)} = Y_t^{(\alpha)} dt, \; X_0^{(\alpha)} = x \in R \tag{1.2}$$

$$dY_t^{(\alpha)} = \alpha b(X_t^{(\alpha)})dt - \alpha Y_t^{(\alpha)} dt + \alpha dB_t^H, \; Y_0^{(\alpha)} = y \in R. \tag{1.3}$$

In this paper we investigate the asymptotic behavior of X^α, as $\alpha \to \infty$ and will prove that the process X^α converges a.s uniformly to X solution of (1.1). Note that if $H = 1/2$, then $B^{1/2}$ is the standard Brownian motion, the problem is then classic and it describe a physical phenomena. To be more precise, by a change of scaling, the system (1.2)-(1.3) may describe the motion of a particle in a force field. (Originally, the model has been introduced by Langevin [9]. He proposed to use the equation

$$dv(t) = -\frac{f}{m}dt + \frac{F(t)}{m}dt$$

where $m > 0$ is the mass of the particle, $f > 0$ the friction coefficient and F a fluctuating force resulting from the impacts of the molecules of the surrounding medium). The process X^α is the position of the particle and Y^α is the velocity process. We refer to [10] for an other example concerning the motion of tagged particle corresponding to large number of oscillators of Liénard type. The use of Smoluchowski-Kramers approximation to treat problems of chemical reaction can be found in [7] and [6].

The paper is organized as follows: In Section 2 we give some preliminaries on fractional Brownian motion. Section 3 is devoted to investigate the asymptotic behavior of the position and velocity process (X^α and Y^α) when α is large enough.

2. PRELIMINARIES

Consider $T = [0, \tau]$ a time interval with arbitrary fixed horizon τ, and let $(B_t^H)_{t\in T}$ be the one-dimensional fractional Brownian motion with Hurst parameter $H \in (0, 1)$. This means by definition that B^H is a centered Gaussian process with covariance

$$R(t, s) = E(B_s^H B_t^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

Note that $B^{1/2}$ is a standard Brownian motion. Moreover B^H has the following Wiener integral representation:

$$B_t^H = \int_0^t K^H(t, s)dW_s,$$

where $W = (W_t)_{t\in T}$ is a Wiener process, and $K^H(t, s)$ is the kernel given by

$$K^H(t, s) = c_H(t - s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}} F\left(\frac{t}{s}\right) \tag{2.1}$$

c_H being a constant and

$$F(z) = c_H\left(\frac{1}{2} - H\right) \int_0^{z-1} r^{H-\frac{3}{2}} \left(1 - (1 + r)^{H-\frac{1}{2}}\right) dr.$$

¿From (2.1) we obtain

$$\frac{\partial K^H}{\partial t}(t,s) = c_H(H-\frac{1}{2})(t-s)^{H-\frac{3}{2}}\left(\frac{s}{t}\right)^{\frac{1}{2}-H}. \tag{2.2}$$

We will denote by $\mathcal{E}_{\mathcal{H}}$ the linear space of step functions on T of the form

$$\varphi(t) = \sum_{i=1}^{n} a_i 1_{(t_i,t_{i+1}]}(t) \tag{2.3}$$

where $t_1, \ldots, t_n \in T, n \in \mathbf{N}, \mathbf{a_i} \in \mathbf{R}$ and by $\mathcal{H}$ the closure of $\mathcal{E}_{\mathcal{H}}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R(t,s).$$

For $\varphi \in \mathcal{E}_{\mathcal{H}}$ of the form (2.3) we define its Wiener integral with respect to the fractional Brownian motion as

$$\int_T \varphi_s dB^H(s) = \sum_{i=1}^{n} a_i \left(B^H_{t_{i+1}} - B^H_{t_i}\right). \tag{2.4}$$

Obviously, the mapping

$$\varphi = \sum_{i=1}^{n} a_i 1_{(t_i,t_{i+1}]} \to \int_T \varphi_s dB^H(s) \tag{2.5}$$

is an isometry between $\mathcal{E}_{\mathcal{H}}$ and the the linear space $span\{B^H_t, t \in R\}$ viewed as a subspace of $L^2(\Omega)$ and it can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos of the fractional Brownian motion $\overline{span}^{L^2(\Omega)}\{B^H_t, t \in R\}$. The image on an element $\Phi \in \mathcal{H}$ by this isometry is called the Wiener integral of Φ with respect to B^H. For every $s < \tau$, let us consider the operator K^* in $L^2(T)$

$$(K^*_\tau \varphi)(s) = K(\tau,s)\varphi(s) + \int_s^\tau (\varphi(r) - \varphi(s)) \frac{\partial K}{\partial r}(r,s)dr.$$

When $H > \frac{1}{2}$, the operator K^* has the simpler expression

$$(K^*_\tau \varphi)(s) = \int_s^\tau \varphi(r) \frac{\partial K}{\partial r}(r,s)dr.$$

We refer to [2] for the proof of the fact that K^* is an isometry between $\mathcal{H}$ and $L^2(T)$ and, as a consequence, we will have the following relationship between the Wiener integral with respect to fBm and the Wiener integral with respect to the Wiener process W

$$\int_0^t \varphi(s)dB^H(s) = \int_0^t (K^*_t \varphi)(s)dW(s) \tag{2.6}$$

for every $t \in T$ and we have $\varphi 1_{[0,t]} \in \mathcal{H}$ if and only if $K_t^* \varphi \in L^2(T)$. We also recall that, if $\phi, \chi \in \mathcal{H}$ are such that $\int_T \int_T |\phi(s)||\chi(t)||t-s|^{2H-2} ds dt < \infty$, their scalar product in $\mathcal{H}$ is given by

$$\langle \phi, \chi \rangle_{\mathcal{H}} = H(2H-1) \int_0^\tau \int_0^\tau \phi(s)\chi(t)|t-s|^{2H-2} ds dt. \tag{2.7}$$

3. APPROXIMATION

Let us consider the equation

$$X_t = X_0 + \int_0^t b(X_s) ds + B_t^H \ , \ t \in [0, \tau]. \tag{3.1}$$

Remark 3.1. • In [13] the authors proved the existence and uniqueness of a strong solution of the equation (3.1) under the following assumptions on the coefficient b:

- if $H < \frac{1}{2}$, b satisfies a linear growth condition.
- if $H > \frac{1}{2}$, b is Hölder continuous of order $1 > \alpha > 1 - \frac{1}{2H}$.

- In [3] an existence and uniqueness result for (3.1) is given when $H > \frac{1}{2}$ under the hypothesis $b(x) = b_1(x) + b_2(x)$, b_1 satisfying the above conditions and b_2 being a bounded nondecreasing left (or right) continuous function. For the case of discontinuous drift we refer to [12].
- The case of the Hölder continuous drift is obvious. It is not difficult to show that the method of the usual Picard iterations can be used to prove the existence and uniqueness of strong solution even in the multidimensional case.

For every $\alpha \in (0, \infty)$, consider the system (3.2)-(3.3),

$$dX_t^{(\alpha)} = Y_t^{(\alpha)} dt, \quad X_0^{(\alpha)} = x \tag{3.2}$$

$$dY_t^{(\alpha)} = \alpha b(X_t^{(\alpha)}) dt - \alpha Y_t^{(\alpha)} dt + \alpha dB_t^H, \ \ Y_0^{(\alpha)} = y. \tag{3.3}$$

where $b : R \to R$ is a measurable function satisfying the global Lipschitz condition

$$|b(x) - b(y)| \leq K\,|x - y| \ \text{ for every } x, y \in R\,. \tag{3.4}$$

We will prove that the position process $X^{(\alpha)}$, solution of (3.2)-(3.3) converges a.s as $\alpha \to \infty$ to the solution of (3.1). We start with the following

LEMMA 3.2 *Consider the equation*

$$dX_t = -aX_t dt + f(t)dt + dB_t^H\,, \quad X_0 = x,$$

where a is a positive real number and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable function such that $\int_0^T e^{as} |f(s)|\, ds < \infty$. Then its unique solution is given by

$$X_t = e^{-at}x + \int_0^t e^{-a(t-s)} f(s)ds + \int_0^t e^{-a(t-s)} dB_s^H. \tag{3.5}$$

Proof. Let us prove first that for every t, the function $1_{[0,t]}e^{-a(t-s)} \in \mathcal{H}$ (or $K^*\left(1_{[0,t]}e^{-a(t-s)}\right) \in L^2(T)$). Given the expression of the operator K^*, we will separate the proof into two cases: $H > \frac{1}{2}$ and $H < \frac{1}{2}$.

Suppose first that $H > \frac{1}{2}$. By (2.8), we need to prove that

$$I_t = \int_0^t \int_0^t e^{-a(t-u)} e^{-a(t-v)} |v-u|^{2H-2}\, dvdu < \infty.$$

We can write

$$\begin{aligned} I_t &= 2\int_0^t \int_0^t e^{-a(t-u)} e^{-a(t-v)} |v-u|^{2H-2}\, dvdu \\ &= \int_0^t \left(n \int_0^u e^{-a(2t-2u+x)} x^{2H-2} dx \right) du \\ &= \int_0^t e^{-a(2t-2u)} \left(\int_0^u e^{-ax} x^{2H-2} dx \right) du \le a^{1-2H}\Gamma(2H-1) \int_0^t e^{-a(2t-2u)} < \infty. \end{aligned}$$

Suppose now that $H < \frac{1}{2}$. We will prove that $K^*\left(1_{[0,t]}e^{-a(t-s)}\right) \in L^2(T)$. It is clear that the first term appearing in the definition of K^* is finite, so it suffices to check the second one. More precisely, we have to show that

$$J_t = \int_0^t \left(\int_s^t \frac{\partial K}{\partial r}(r,s) \left(e^{-a(t-r)} - e^{-a(t-s)} \right) dr \right)^2 ds < \infty.$$

Note that (see (2.2) $\frac{\partial K}{\partial r}(r,s)$ can be written as $\frac{\partial K}{\partial r}(r,s) = a(H)f(s,t)$ where $a(H)$ is a negative constant and $f(s,t)$ is a positive function lesser than $(r-s)^{H-\frac{3}{2}}$. Therefore

$$\begin{aligned} J_t &\le a(H)^2 \int_0^t e^{-2a(t-s)} \left(\int_s^t (r-s)^{H-\frac{3}{2}} \left(e^{a(r-s)} - 1 \right) dr \right)^2 ds \\ &= a(H)^2 \int_0^t e^{-2a(t-s)} \left(\int_0^{t-s} y^{H-\frac{3}{2}} \left(e^{ar} - 1 \right) dr \right)^2 ds \end{aligned}$$

and it suffices to observe that the quantity $\int_0^\infty y^{H-\frac{3}{2}}\left(e^{ar}-1\right) dr$ is finite.

Concerning (3.5), let us note that the following integration by parts holds

$$e^{at}B_t^H = \int_0^t e^{as}dB_s^H + a\int_0^t B_s^H e^{as}ds. \tag{3.6}$$

The rest of the proof is standard. ■

We state now our main result.

THEOREM 3.3. *Assume that* (3.4) *holds. Then, for every value of* $\alpha \in (0,\infty)$, *the system* (3.2)-(3.3) *admits an unique solution and we have*

$$\lim_{\alpha\to\infty} \sup_{0\le t\le\tau} \left| X_t^{(\alpha)} - X_t \right| = 0 \text{ a.s. } . \tag{3.7}$$

where $X = (X_t)_{t\in[0,\tau]}$ *is the unique solution of* (3.1).

Proof. Let us denote by $Z_t^{(\alpha)}$ the transposed vector $\left(X_t^{(\alpha)}, X_t^{(\alpha)}\right)^T$. The system (3.2)-(3.3) can be expressed as

$$Z_t^{(\alpha)} = b\left(Z_t^{(\alpha)}\right)dt + \sigma dB_t^H \tag{3.8}$$

where $B : R^2 \to R$ is a 2×1 vector which components are $B_1(x,y) = y$ and $B_2(x,y) = \alpha b(x) - \alpha y$ and σ is the constant vector $(0,\alpha)^T$. It is not difficult to see that B satisfies the Lipschitz condition

$$||B(u) - B(v)|| \le K'\|u - v\| \text{ for every } x, y \in R^2,$$

where $\|\cdot\|$ denotes the euclidian norm and $K' = 1 + \alpha \vee \alpha K$. Therefore, standard arguments apply to obtain the existence of a unique (strong) solution of (3.8) (see the last point of Remark 3.1).

Let us choose $t_0 = 0 < t_1 < \ldots < t_N = \tau$, a partition of $[0,\tau]$ such that $K\left(t_{n+1} - t_n\right) \le \frac{1}{2}$, K being the Lipschitz constant of b . Consider the equations (3.2) and (3.3) on $I_n = [t_n, t_{n+1}]$, $n = 0, 1, 2, \ldots N-1$. It holds

$$X_t^{(\alpha)} = X_{t_n}^{(\alpha)} + \int_{t_n}^t Y_s^{(\alpha)}ds \tag{3.9}$$

and

$$Y_t^{(\alpha)} = Y_{t_n}^{(\alpha)} - \alpha\int_{t_n}^t Y_s^{(\alpha)}ds + \alpha\int_{t_n}^t b(X_s^{(\alpha)})\,ds + \alpha\left(B_t^H - B_{t_n}^H\right). \tag{3.10}$$

Combining (3.9) and (3.10) we obtain

$$X_t^{(\alpha)} = X_{t_n}^{(\alpha)} + \frac{Y_{t_n}^{(\alpha)}}{\alpha} - \frac{Y_t^{(\alpha)}}{\alpha} + \int_{t_n}^t b(X_s^{(\alpha)})\, ds + \left(B_t^H - B_{t_n}^H\right). \tag{3.11}$$

On the other hand the equation (3.1) can be written on I_n as

$$X_t = X_{t_n} + \int_{t_n}^t b(X_s)\, ds + \left(B_t^H - B_{t_n}^H\right)$$

and therefore, for every $t \in I_n$,

$$\begin{aligned} X_t^{(\alpha)} - X_t &= X_{t_n}^{(\alpha)} - X_{t_n} + \frac{Y_{t_n}^{(\alpha)}}{\alpha} - \frac{Y_t^{(\alpha)}}{\alpha} \\ &\quad + \int_{t_n}^t \left[b(X_s^{(\alpha)}) - b(X_s)\right] ds \end{aligned}$$

and using the Lipschitz condition (3.4), we get

$$\left|X_t^{(\alpha)} - X_t\right| \le \left|X_{t_n}^{(\alpha)} - X_{t_n}\right| + 2 \sup_{t\in I_n} \frac{\left|Y_t^{(\alpha)}\right|}{\alpha} + (t - t_n) K \sup_{t \in I_n} \left|X_t^{(\alpha)} - X_t\right|.$$

Since $(t - t_n)K \le \frac{1}{2}$, we have

$$\sup_{t\in I_n} \left|X_t^{(\alpha)} - X_t\right| \le 2 \left|X_{t_n}^{(\alpha)} - X_{t_n}\right| + 4 \sup_{t\in I_n} \frac{\left|Y_t^{(\alpha)}\right|}{\alpha}. \tag{3.12}$$

Denote by

$$a_n(\alpha) = \sup_{t\in I_n} \left|X_t^{(\alpha)} - X_t\right| ,$$

then we get

$$a_n(\alpha) \le 2\, a_{n-1}(\alpha) + 4 \sup_{t\in I_n} \frac{\left|Y_t^{(\alpha)}\right|}{\alpha}, \quad n = 1, \ldots, N,$$

Since $X_0^{(\alpha)} = X_0 = 0$, clearly $a_0(\alpha) \to 0$ as $\alpha \to 0$ by (3.12) and (3.13) proved in the Lemma 3.4 below. An induction argument will finish the proof. ■

LEMMA 3.4. *For every* $0 \le n \le N - 1$,

$$\sup_{t\in I_n} \frac{\left|Y_t^{(\alpha)}\right|}{\alpha} \to 0 \tag{3.13}$$

almost surely as $\alpha \to 0$.

Proof. We follow the lines of [11]. We show first the bound

$$\sup_{t\in I_n}\left|X_t^{(\alpha)}-X_{t_n}^{(\alpha)}\right| \le 4\sup_{t\in I_n}\frac{\left|Y_t^{(\alpha)}\right|}{\alpha}+\frac{1}{K}\left|b\left(X_{t_n}^{(\alpha)}\right)\right|+2\sup_{t\in I_n}\left|B_t^H-B_{t_n}^H\right|. \tag{3.14}$$

Indeed, by (3.11) and the inequality

$$\left|b(X_s^{(\alpha)})\right| \le \left|b(X_{t_n}^{(\alpha)})\right| + K\left|X_{t_n}^{(\alpha)}-X_s^{(\alpha)}\right|$$

we obtain

$$\begin{aligned}\sup_{t\in I_n}\left|X_t^{(\alpha)}-X_{t_n}^{(\alpha)}\right| \le{}& \frac{\left|Y_t^{(\alpha)}\right|}{\alpha}+\frac{\left|Y_{t_n}^{(\alpha)}\right|}{\alpha}+(t-t_n)\left|b(X_{t_n}^{(\alpha)})\right| \\ &+K(t-t_n)\sup_{t\in I_n}\left|X_t^{(\alpha)}-X_{t_n}^{(\alpha)}\right|+\sup_{t\in I_n}|B_t^H-B_{t_n}^H|,\end{aligned}$$

and since $t-t_n\le\frac{1}{2K}$, we obtain (3.14).

Let us apply now Lemma 3.2 to the equation (3.3) ($X^{(\alpha)}$ is considered as known). We obtain

$$Y_t^{(\alpha)} = e^{-\alpha(t-t_n)}Y_{t_n}^{(\alpha)}+\alpha\int_{t_n}^t e^{-\alpha(t-s)}b(X_s^\alpha)ds+\alpha\int_{t_n}^t e^{-\alpha(t-s)}dB_s^H$$

and therefore

$$\begin{aligned}\sup_{t\in I_n}\left|Y_t^{(\alpha)}\right| &\le \left|Y_{t_n}^{(\alpha)}\right|+\sup_{t\in I_n}\left|b(X_t^{(\alpha)})\right|+\sup_{t\in I_n}\left|\alpha\int_{t_n}^t e^{-\alpha(t-s)}dB_s^H\right| \\ &\le \left|Y_{t_n}^{(\alpha)}\right|+\left|b(X_{t_n}^{(\alpha)})\right|+K\sup_{t\in I_n}\left|X_t^{(\alpha)}-X_{t_n}^{(\alpha)}\right|+\sup_{t\in I_n}\left|\alpha\int_{t_n}^t e^{-\alpha(t-s)}dB_s^H\right| \\ &\le \left|Y_{t_n}^{(\alpha)}\right|+4K\sup_{t\in I_n}\frac{\left|Y_t^{(\alpha)}\right|}{\alpha}+2\left|b\left(X_{t_n}^{(\alpha)}\right)\right| \\ &\quad +2K\sup_{t\in I_n}\left|B_t^H-B_{t_n}^H\right|+\sup_{t\in I_n}\left|\alpha\int_{t_n}^t e^{-\alpha(t-s)}dB_s^H\right|.\end{aligned}$$

Since

$$|b(X_{t_n}^{(\alpha)})|\le|b(0)|+K\,|X_{t_n}^{(\alpha)}|$$

and if we apply equation (3.9) for t_n and t_{n-1}, we get

$$|X_{t_n}^{(\alpha)}|\le|X_{t_{n-1}}^{(\alpha)}|+\sup_{t\in I_{n-1}}\left|Y_t^{(\alpha)}\right|(t_n-t_{n-1})$$

$$\leq |X_0^\alpha| + \sum_{j=1}^{n} \sup_{t\in I_{j-1}} \left|Y_t^{(\alpha)}\right| (t_j - t_{j-1})\,. \tag{3.15}$$

On the other hand if we choose α large enough ($\alpha > 8K$), we obtain

$$\sup_{t\in I_n} \frac{\left|Y_t^{(\alpha)}\right|}{\alpha} \leq \frac{2}{\alpha}\left|Y_{t_n}^{(\alpha)}\right| + \frac{4}{\alpha}\left|b\left(X_{t_n}^{(\alpha)}\right)\right|$$

$$+\frac{4K}{\alpha}\sup_{t\in I_n}\left|B_t^H - B_{t_n}^H\right| + 2\sup_{t\in I_n}\left|\int_{t_n}^t e^{-\alpha(t-s)}dB_s^H\right|\,, \tag{3.16}$$

and using the bounds (3.15) and (3.16) we get

$$\begin{aligned}
\sup_{t\in I_n}\frac{\left|Y_t^{(\alpha)}\right|}{\alpha} &\leq \frac{4}{\alpha}\left(K|x| + |b(0)|\right) + \sum_{j=1}^{n}\sup_{t\in I_{j-1}}\frac{\left|Y_t^{(\alpha)}\right|}{\alpha}\,4\,K\,(t_j - t_{j-1}) + \frac{2}{\alpha}\left|Y_{t_n}^{(\alpha)}\right| \\
&\quad +\frac{4K}{\alpha}\sup_{t\in I_n}\left|B_t^H - B_{t_n}^H\right| + 2\sup_{t\in I_n}\left|\int_{t_n}^t e^{-\alpha(t-s)}dB_s^H\right|\,, \\
&\leq \frac{4}{\alpha}\left(K|x| + |b(0)|\right) + 4\sum_{j=0}^{n-1}\sup_{t\in I_j}\frac{\left|Y_t^{(\alpha)}\right|}{\alpha} \\
&\quad +\frac{4K}{\alpha}\sup_{t\in I_n}\left|B_t^H - B_{t_n}^H\right| + 2\sup_{t\in I_n}\left|\int_{t_n}^t e^{-\alpha(t-s)}dB_s^H\right|.
\end{aligned}$$

We will use the following notations:

$$x_n(\alpha) = \sup_{t\in I_n}\frac{\left|Y_t^{(\alpha)}\right|}{\alpha}$$

and

$$z_n(\alpha) = \frac{4K}{\alpha}\sup_{t\in I_n}\left|B_t^H - B_{t_n}^H\right| + 2\,\sup_{t\in I_n}\left|\int_{t_n}^t e^{-\alpha(t-s)}dB_s^H\right|.$$

Then we obtain the following inductive formulae

$$x_n(\alpha) \leq \frac{4}{\alpha}\left(K|x| + |b(0)|\right) + 4\,\sum_{j=0}^{n-1} x_j(\alpha) + z_n(\alpha)\,.$$

We prove first that

$$z_n \to 0 \text{ as } \alpha\to\infty, \text{ a.s. }. \tag{3.17}$$

Clearly, the first summand of z_n tends a.s. to 0 as $\alpha \to \infty$ due to the continuity of the paths of B^H. Concerning the second summand of z_n, we can write, using the integration by parts (3.6)

$$\int_{t_n}^{t} e^{-\alpha(t-s)} dB_s^H = -\alpha \int_{-\infty}^{t} e^{-\alpha(t-s)} Z_s ds + Z_t = \alpha \int_{-\infty}^{t} e^{-\alpha(t-s)} (Z_t - Z_s) ds$$

where we used the notation: $Z_t = B_t^H - B_{t_n}^H$ if $t \geq t_n$ and $Z_t = 0$ if $t < t_n$. Therefore (3.17) holds, noting that $xe^{-ax} \to_{x\to\infty} 0$ for $a > 0$ and applying the dominated convergence theorem. Finally we obtain the convergence of $x_n(\alpha)$ to 0 by induction. ■

Now we will investigate, after a suitable change of scaling, the asymptotic behavior of the velocity term Y^α when $\alpha \to \infty$. For this purpose, define $\widetilde{Y}_t^\alpha = \frac{1}{\alpha^{1-H}} Y_{t/\alpha}^\alpha$ and $\widetilde{B}_t^H = \alpha^H B_{\frac{t}{\alpha}}^H$. Clearly $\widetilde{B}^H$ is a fractional Brownian motion with Hurst parameter H and $\widetilde{Y}^\alpha$ satisfies the following equation

$$d\widetilde{Y}_t^{(\alpha)} = -\widetilde{Y}_t^{(\alpha)} dt + \alpha^{H-1} b(X_{\frac{t}{\alpha}}^{(\alpha)}) dt + d\widetilde{B}_t^H , \quad \widetilde{Y}_0^{(\alpha)} = \frac{y}{\alpha^{1-H}} . \tag{3.18}$$

PROPOSITION 3.5. *The velocity process $\widetilde{Y}^\alpha$ given by* (3.18) *converges a.s uniformly to the process* $\left(\widetilde{Y}_t = \int_0^t e^{(s-t)} d\widetilde{B}_s^H , \quad t \in [0, \tau] \right)$.

Proof. By Lemma 3.2, we have

$$\widetilde{Y}_t^{(\alpha)} = e^{-t} y\alpha^{H-1} + \alpha^{H-1} \int_0^t e^{(s-t)} b(X_{\frac{s}{\alpha}}^{(\alpha)}) ds + \int_0^t e^{(s-t)} d\widetilde{B}_s^H .$$

Since

$$|b(X_{\frac{s}{\alpha}}^{(\alpha)})| \leq K \sup_{s\in[0,\tau]} |X_s^{(\alpha)} - X_s| + |b(X_{\frac{s}{\alpha}})| ,$$

then

$$|\widetilde{Y}_t^{(\alpha)} - \widetilde{Y}_t| \leq |y| \alpha^{H-1} + \alpha^{H-1} (1 - e^t) \left(K \sup_{s\in[0,\tau]} |X_s^{(\alpha)} - X_s| + \sup_{0\leq s\leq\tau} |b(X_s)| \right) ,$$

where the supremum of $|b(X_s)|$ is finite since b is continuous and we can show, by Kolmogorov criterion, that X is a.s continuous. Since $0 < H < 1$, the result is just a consequence of Theorem 3.3. ■

Remark 3.6. If $H = \frac{1}{2}$ and $b = 0$, the process $\widetilde{Y}$ is the classical Ornstein-Uhlenbeck process satisfying the following one-dimensional stochastic differential equation:

$$d\widetilde{Y}_t = -\widetilde{Y}_t \, dt \, + \, d\widetilde{B}_t \, , \quad \widetilde{Y}_0 = 0 \, .$$

and for arbitrary $H \in (0,1)$, $\widetilde{Y}^{(\alpha)}$ is the fractional Ornstein-Uhlenbeck process as defined, e.g., in [4].

4. GENERALIZATION: APPROXIMATION OF STRATONOVICH EQUATIONS WITH FRACTIONAL BROWNIAN MOTION

We will consider the equation

$$X_t = x + \int_0^t \sigma(X_s) d^\circ B_s^H + \int_0^t b(X_s) ds. \tag{4.1}$$

It has been proved in [1] that, if $H > \frac{1}{4}$, the equation (4.1) has an unique strong solution if σ is of class C^2 with bounded first and second derivative and b is a Lipschitz continuous function. The stochastic integral is taken in the symmetric (Stratonovich) sense (see [14] or [1] for the definition of the symmetric integral).

We can prove the following approximation result.

PROPOSITION 4.1 *Let V_n be a sequence of continuous processes having a finite total variation on compact intervals. For every n,consider the equation*

$$X_t^{(n)} = x + \int_0^t \sigma(X_s^{(n)}) dV_s^n + \int_0^t b(X_s^{(n)}) ds \tag{4.2}$$

where the integral dV_s^n is understood in the Lebesque-Stieltjes sense. Let B^H be a fBm such that

$$\lim_{n\to\infty} \sup_{0\le s\le t} \left| V_s^n - B_t^H \right| = 0 \text{ a.s.} \tag{4.3}$$

Then the sequence of unique solutions of (4.2) converges almost surely, uniformly on bounded intervals, to the unique solution of (4.1).

Proof. The standard arguments (see e.g. [8], pag. 295-299) can be used in this case. ■

Remark 4.2. Let us give an example of sequence of processes satisfying (4.3). Let $\{\chi_n \, , \; n \ge 0\}$ be the orthonormal Haar basis of $L^2([0,1])$ (for the definition we refer to [5] and the reference therein). Let $\varphi_n(.) = \int_0^{\cdot} \chi_n(s)\, ds\,, \;\; n \geq 0$ be the classical family of Schauder functions and define

$$V_t^n = \sum_{j=0}^{n} g_j^H \, \varphi_j(t)\,, \quad \forall t \in [0\,,\,1]\,,$$

where g_j are the gaussian variables given by $g_j^H = \int_0^1 \chi_j(s)\, dB_s^H$. One can prove easily that V^n is a.s of finite variations and since B^H is a.s continuous the uniform convergence (4.3) holds. In the case of $H = \frac{1}{2}$ this is just the classical Lévy's representation of the Brownian motion.

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