Solutions TD 4

1. Estimation paramétrique

1) Oui, parce que $f_\theta(x) \geq 0$ et $\int f_\theta(x)dx = 1$.
2) $F_X(x) = x^2/\theta^2, x \in [0, \theta]$,
$F_{max}(x) = \mathbb{P}(\max(X_1, X_2) \leq x) = \mathbb{P}(X_1 \leq x, X_2 \leq x) = (\mathbb{P}(X_1 \leq x))^2 = x^4/\theta^4, x \in [0, \theta]$,
$g_\theta(x) = 4x^3/\theta^4, x \in [0, \theta]$.
3) $\mathbb{E}(X_1) = 2\theta/3, \mathbb{E}(\max(X_1, X_2)) = 4\theta/5$, donc $\mathbb{E}(\hat{\theta}_1) = \mathbb{E}(\hat{\theta}_2) = \theta$.
4) $\mathbb{E}(X_1^2) = \theta^2/2, \mathbb{E}(\max(X_1, X_2)^2) = 2\theta^2/3$, donc $\mathrm{Var}\,(X_1) = \theta^2/18$ et $\mathrm{Var}\,(\max(X_1, X_2)) = (2/3 - 16/25)\theta^2$.
$RQM(\hat{\theta}_1) = \theta^2/16$ et $RQM(\hat{\theta}_2) = \theta^2/24$.
5) $\hat{\theta}_2$ est meilleur.

2. Loi normale

1) $b = (an - 1)\sigma^2$.
2) $RQM(Q_a) = (2na^2 + (an - 1)^2)\sigma^4$.
3) Le RQM est minimum implique $a = 1/(n + 2)$. L'estimateur est sans biais implique $a = 1/n$.

3. Machine outil

1) $\hat{m} = \frac{1}{n}\sum X_i = \bar{X}$, $\hat{\sigma^2} = \frac{1}{n}\sum(X_i - \bar{X})^2$.
2) Rappelons que si $X_n \xrightarrow{p} X$ et $Y_n \xrightarrow{p} Y$ alors $aX_n + bY_n \xrightarrow{p} aX + bY$ et $X_nY_n \xrightarrow{p} XY$. Par la LGN, on a $\hat{m} \xrightarrow{p} m$ et $\frac{1}{n}\sum X_i^2 \xrightarrow{p} \mathbb{E}(X^2)$. On a ainsi $\bar{X}^2 = \hat{m}^2 \xrightarrow{p} m^2$ et $\hat{\sigma^2} = \frac{1}{n}\sum X_i^2 - \bar{X}^2 \xrightarrow{p} \mathbb{E}(X^2) - m^2 = \sigma^2$.
3) $S^2 = \frac{1}{n-1}\sum(X_i - \bar{X})^2$, $\mathrm{Var}\,(S^2) = \frac{2}{n-1}\sigma^4$, $RQM(S^2) = \frac{2}{n-1}\sigma^4$, $\mathrm{Var}\,(\hat{\sigma^2}) = \frac{2(n-1)}{n^2}\sigma^4$, $RQM(\hat{\sigma^2}) = \frac{2n-1}{n^2}\sigma^4$. On a $RQM(S^2) > RQM(\hat{\sigma^2})$ car $\frac{2}{n-1} > \frac{2n-1}{n^2} \iff 3n - 1 > 0$.
4) $\hat{m} = 399,54$, $\hat{\sigma^2} = 0,56$, $S^2 = 0,64$.

4. Variables de Poisson

1) $\hat{\theta}_{MV} = \bar{X}$.
2) $b = 0$, $\hat{\theta}_{MV} \xrightarrow{p} \theta$ d'après la loi des grands nombres.

5. Chaîne de Markov

1) $\mu = ((1 - q)/(2 - p - q), (1 - p)/(2 - p - q))^T$.
2) $L(p, q) = \mathbb{P}(X_1 = x_1, \ldots, X_n = x_n)$
$= \Pi_{i=1}^{n-1}\mathbb{P}(X_{i+1} = x_{i+1}|X_i = x_i)$

$= p^{\sum \mathbb{1}_{\{x_i=e_1,x_{i+1}=e_1\}}}(1-p)^{\sum \mathbb{1}_{\{x_i=e_1,x_{i+1}=e_2\}}}(1-q)^{\sum \mathbb{1}_{\{x_i=e_2,x_{i+1}=e_1\}}} q^{\sum \mathbb{1}_{\{x_i=e_2,x_{i+1}=e_2\}}}$.
3) $\hat{p} = \sum \mathbb{1}_{\{x_i=e_1,x_{i+1}=e_1\}}/(\sum \mathbb{1}_{\{x_i=e_1,x_{i+1}=e_1\}} + \sum \mathbb{1}_{\{x_i=e_1,x_{i+1}=e_2\}})$,
$\hat{q} = \sum \mathbb{1}_{\{x_i=e_2,x_{i+1}=e_2\}}/(\sum \mathbb{1}_{\{x_i=e_2,x_{i+1}=e_1\}} + \sum \mathbb{1}_{\{x_i=e_2,x_{i+1}=e_2\}})$.

6. Variables exponentielles

1) $\hat{\lambda}_{MV} = \bar{X}$.
2) $I_\alpha = [\overline{X} - \lambda q_{(1+\alpha)/2}/\sqrt{n}, \overline{X} + \lambda q_{(1+\alpha)/2}/\sqrt{n}]$.
3) $I_{90\%} = [1000 - 164, 1000 + 164]$

7. Tirage avec remise

1) $\mathbb{P}(X_1 = 1) = k/100, \mathbb{P}(X_1 = 0) = 1 - k/100$,
$\mathbb{P}(X_1 = 0, X_2 = 0) = (1 - k/100)^2, \mathbb{P}(X_1 = 0, X_2 = 1) = (1 - k/100)k/100$,
$\mathbb{P}(X_1 = 1, X_2 = 0) = (1 - k/100)k/100, \mathbb{P}(X_1 = 1, X_2 = 1) = (k/100)^2$.
2) $L(k) = (1 - k/100)^{n-\sum x_i}(k/100)^{\sum x_i}$.
3) $\hat{k}_{MV} = 100\bar{X}$.
4) $\mathbb{E}(\hat{k}_{MV}) = k$, $\hat{k}_{MV} \xrightarrow{p} k$ d'après la LGN.
5) $\hat{k}_{MV} \sim Bin(100, k/100)$. $\hat{k}_{MV}$ suit approximativement la loi $\mathcal{N}(k, k(1 - k/100))$, avec la majoration de la variance on a $\hat{k}_{MV} \sim \mathcal{N}(k, 25)$, $I_{90\%} = [\hat{k}_{MV} - 5 \times 1.64, \hat{k}_{MV} + 5 \times 1.64]$.