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# Time Series Tutorial $n^0$ 2 : How to identify a white noise and generate ARMA and GARCH processes?

The aims of this tutorial is a to provide a first tool to identify a white noise and to generate both the classical ARMA and GARCH processes.

## Identification of a white noise: test of portemanteau

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| | ———————————— |
| | Vizualization of the independance |
| | ———————————— |
| x=rnorm(50) | Generate a Gaussian white noise. |
| r=acf(x) | Correlogram of $x$ |
| | It represents empirical auto-correlations for consecutive lags |
| | $0, 1, 2, \ldots$. Dots represent 95% confidence intervals of $Z/\sqrt{n}$, |
| | where $Z$ is $\mathcal{N}(0, 1)$ r.v. and $n$ is the length of $x$. |
| | Why? Which test is associated to the confidence interval? Can we consider |
| | $x$ as a white noise? |
| y=x[1:49]+x[2:50] | Generation of a new trajectory. Which kind of process is $y$? |
| | What is the distribution of $y$? Compute the correlations of $y$ for any lag. |
| ry=acf(y) | Correlogram of $y$. Is it a white noise? |
| rho=ry$acf[2] | Computation of the empirical correlation for lag 1. |
| | Compare with the theoretical correlation. |
| | ——————————————— |
| | Goodness-of-fit tests |
| | ——————————————— |
| n=50; x=5*rnorm(n)-2 | What is the law of $x$? |
| hist(x,nclass=11) | Histogram which is a first estimation of the density of $x$. Why? |
| | Note the choice of numbers of classes of histogram. |
| qqnorm(x) | QQplot test (what is it?). Conclusion? |
| y=3*runif(100)+1 | what is the law of $y$? |
| qqnorm(y) | Conclusion? |
| dn=ks.test(x,"pnorm",-2,5) | Kolmogorov-Smirnov test where the distribution of $x$ is compared to $\mathcal{N}(-2, 5^2)$. |
| | The $p-value$ provides a quantitative way for deciding from a test. |
| | Classicaly $H_0$ is accepted when $p-value \geq 0.05$. |
| ddn=ks.test(y,"pexp",3) | Test on $y$. What could we expect? |
| | Test now if $y$ follow a uniform law on $[-1, 1]$. |
| ks.test(x,y) | Test the similarity of the distributions of $x$ and $y$. Conclusion? |
| Box.test(x, lag = 8) | Use of a portemanteau test for testing the whiteness. |
| | As usual, if $p-value \geq 0.05$, accept $H_0$: White noise. |
| Box.test(x, lag = 8,"Ljung-Box") | Use of another portemanteau test, Ljung-Box, which is numerically more accurate. |
| X=x(2:n)-0.5*x(1:(n-1)) | Which kind of process is generated? Apply the portemanteau tests. Result? |

**Exercice 1**: Generate 1000 trajectories of $x$ and $X$, for $n = 50$, $n = 200$, $n = 1000$ and $n = 5000$. For each trajectory, apply both the portemanteau tests, for `lag=5`, `lag=10`, `lag=log(n)` and `lag=sqrt(n)`, and save the $p-values$. Compare the results for finding a most efficient portemanteau test.

## ARMA process

In the sequel, two different ways are followed for generating a trajectory of a ARMA process.

```
n=100; m=100; X=0
epsi=3*rnorm(n+m)
for (j in c(1:(n+m)))
X[j+1]=-0.3*X[j]+epsi[j+1]+0.7*epsi[j]
x=X[c((m+1):(n+m+1))]; tsplot(x)
```

Explain what is done here (in particular why using $m$?). Another way could be by writting these commands:

```
XX=arima.sim(100,model=list(ar=-.3,ma=.7))
```

Write the recurrent equation followed by this process. Draw this trajectory. Representation of its correlogram. Conclusion? Generate another trajectory using directly the recurrent equation. Generate a trajectory of a ARMA[2,2] process (chose the coefficients). Generate the same trajectory with a noise following a uniform distribution on $[-1, 1]$.

# Correlograms

```
X=arima.sim(100,model=list(ar=-.3,ma=.7))
R=acf(X); R[1]
```

What is the theoretical formula of `R[1]`? Explain the obtained value from its asymptotic value (compute it from the equation of the ARMA process!). We are going to replicate this computation of the sample correlation for lag 1 many times for observing the asymptotic behavior of this estimator. This is the aim of the following Monte-Carlo experiment:

```
Res=c()
for (j in c(1:200))
{X=arima.sim(1000,model=list(ar=-.3,ma=.7))
R=acf(X)
Res=c(Res,as.numeric(R[1]$acf))}
hist(Res)
```

Explain what is done with this program and connect the result with theoretical results. Replace the law of the innovation by a uniform law on $[-1, 1]$. Conclusion?
In the sequel, we define and use an estimator of the parameters of a AR[1] process. If $X_{t+1} = \theta\, X_t + \xi_t$ for $t \in \mathbf{Z}$, with $|\theta| < 1$, then $\theta = \mathrm{cov}(X_0, X_1)/\mathrm{var}(X_0)$. Hence, an estimator of $\theta$ could be obtained:

$$\widehat{\theta} = \frac{\frac{1}{n} \sum_{i=1}^{n-1} X_i X_{i+1}}{\frac{1}{n} \sum_{i=1}^{n} X_i^2}.$$

Use the previous program for exhibiting the asymptotic behavior of $\widehat{\theta}$. Consider the cases $n = 100$, $n = 500$ and $n = 1000$. How to check the $\sqrt{n}$ convergence rate of this estimator?

For estimating the variance $\sigma_\xi^2$ of $\xi_t$, a natural estimator is:

$$\widehat{\sigma}_\xi^2 = \frac{1}{n} \sum_{i=1}^{n-1} \big(X_{i+1} - \widehat{\theta}\, X_i\big)^2.$$

Exhibit also the asymptotic behavior of this estimator. Which convergence rate could be expected?

Write the conditional log-likelihood of $(X_t)$ and compare the QMLE with the previous estimators.

# GARCH process

Let $X$ be a trajectory of length 100 of the following process:

$$X_t = \varepsilon_t \times \sigma_t \quad \text{and} \quad \sigma_t^2 = 4 + 0.3 \times X_{t-1}^2 + 0.4 \times \sigma_{t-1}^2,$$

where $(\varepsilon_t)$ is a Gaussian standard white noise.

Generate directly this trajectory (as previously for ARMA process let run the routine $m = 100$ before for being close to a stationary process). Draw this trajectory and the correlogram.
Could you imagine the same kind of estimator than with AR process?