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Existence and Uniqueness of Pseudo Almost Automorphic Solutions to Some Classes of Partial Evolution Equations

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ABSTRACT

We are concerned in this paper with the partial differential equation $\frac{du}{dt}(t) = Au(t)\ \ t \in \mathbb{R}$, where A is a (generally unbounded) linear operator which generates a semigroup of bounded linear operators $(T(t))_{t\geq 0}$. Under appropriate sufficient conditions, we prove the existence and uniqueness of a pseudo almost automorphic mild solution to the equation.

RESUMEN

Nosotros consideramos en este artículo la ecuación diferencial parcial $\frac{du}{dt}(t) = Au(t)\ \ t \in \mathbb{R}$, donde A es un (generalmente no acotado) operador lineal que genera un semigrupo de operadores lineales acotados $(T(t))_{t\geq 0}$. Bajo condiciones suficientes apropriados, provamos la existencia y unicidad de una solución blonda casi automorfica para tal equación.

Key words and phrases: *Pseudo Almost automorphic function, exponentially stable semigroup, partial differential equations.*

Math. Subj. Class.: *34G10, 47A55.*

1 Introduction

Since the publication of the monograph [12], the study of almost automorphic function (a concept introduced by S. Bochner in the literature in the mid sixties as a generalization of almost periodicity in the sense of Bohr) has regained great interest. Several extensions of the concept were introduced including asymptotic almost automorphy by N'Guérékata ([10]), p-almost automorphy by Diagana ([2]), and Stepanov-like almost automorphy by N'Guérékata and Pankov ([14]). Recently, J. Liang et al. have suggested the notion of pseudo almost automorphic functions, i.e. functions that can be written uniquely as a sum of an almost automorphic function and an ergodic term, i.e. a function with vanishing mean (cf [6], and [7], [8]). This latter turns out to be more general than asymptotic almost automorphy. However it seems to be more complicated.

There has been a considerable interest in the existence of (these various types of) almost automorphic solutions of evolution equations. Semigroups theory and fixed point techniques have been frequently used for semilinear evolution equations. In [3] the authors studied the existence and uniqueness of an almost automorphic mild solution to the equation

$$\frac{d}{dt}\left[u(t)+f(t,u(t))\right]=Au(t)+g(t,u(t)) \;\; t\in\mathbb{R}, \tag{1.1}$$

where the functions $f(t,u)$ and $g(t,u)$ are almost automorphic in t, for each u. This latter motivated our recent paper [15], where we study the existence and uniqueness of a pseudo almost automorphic mild solution the semilinear evolution equations of the form

$$\frac{du}{dt}=Au(t)+g(t,u(t)), \qquad t\in\mathbb{R}, \tag{1.2}$$

where A is an unbounded sectorial operator with not necessarily dense domain in a Banach space $\mathbb{X}$ and $g:\mathbb{R}\times\mathbb{X}_\alpha\to\mathbb{X}$, where $\mathbb{X}_\alpha$, $\alpha\in(0,1)$, is any intermediate Banach space between $D(A)$ and $\mathbb{X}$.

In this paper, we study pseudo almost automorphic solutions to perturbations to Equations:

$$\frac{du}{dt}=Au(t) \;\; t\in\mathbb{R}, \tag{1.3}$$

consisting of the class of abstract partial evolution equations of the form

$$\frac{d}{dt}\left[u(t)+f(t,u(t))\right]=Au(t)\quad t\in\mathbb{R}, \tag{1.4}$$

where A is the infinitesimal generator of an exponentially stable C_0-semigroup acting on $\mathbb{X}$, B, C are two densely defined closed linear operators on $\mathbb{X}$, and f is continuous functions.

Under some appropriate assumptions, we establish the existence and uniqueness of an almost automorphic (mild) solution to Eq. (1.4) using the Banach fixed-point principle.

We start this work by presenting some properties of pseuso almost automorphic functions in Section 2 including an application to a Volterra-like integral equation. Our main result (Theoreme 3.3) is presented in Section 3.

2 Preliminaries

In this work, $(\mathbb{X}, \|\cdot\|)$ will stand for a Banach space. The collection of all bounded linear operators from $\mathbb{X}$ is denoted by $B(\mathbb{X})$ — this is a Banach space when it is equipped with its natural norm $\|A\|_{B(\mathbb{X})} := \sup\limits_{x\in\mathbb{X}, x\neq 0}\dfrac{\|Ax\|_{\mathbb{X}}}{\|x\|_X}$.

The fields of real and complex numbers, are respectively denoted by $\mathbb{C}$ and $\mathbb{R}$. We let $BC(\mathbb{R},\mathbb{X})$ denote the space of all $\mathbb{X}$-valued bounded continuous functions $\mathbb{R}\to\mathbb{X}$– it is a Banach space when equipped with the sup norm $\|u\|_\infty := \sup\limits_{t\in\mathbb{R}}\|u(t)\|_{\mathbb{X}}$ for each $u\in B(\mathbb{R},\mathbb{X})$.

We will use the following well-known concepts in the sequel.

Definition 2.1. A continuous function $f:\mathbb{R}\mapsto\mathbb{X}$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}}\subset(s'_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\mapsto\infty} f(t+s_n)$$

is well-defined for each $t\in\mathbb{R}$, and

$$f(t) = \lim_{n\mapsto\infty} g(t-s_n)$$

for each $t\in\mathbb{R}$.

Similarly,

Definition 2.2. A continuous function $f:\mathbb{R}\times\mathbb{X}\mapsto\mathbb{X}$ is said to be almost automorphic in $t\in\mathbb{R}$ for each $u\in\mathbb{X}$ if every sequence of real numbers $(\sigma_n)_{n\in\mathbb{N}}$ contains a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$g(t,u) := \lim_{n\mapsto\infty} f(t+s_n,u)$$

is well defined for each $t\in\mathbb{R}$ and each $u\in\mathbb{X}$ and,

$$f(t,u) = \lim_{n\mapsto\infty} g(t-s_n,u)$$

exists for each $t\in\mathbb{R}$ and $u\in\mathbb{X}$.

The following natural properties hold: If $f, h : \mathbb{R} \mapsto \mathbb{X}$ are almost automorphic functions and if $\lambda \in \mathbb{R}$, then $f + h$, λf, and f_λ are almost automorphic, where $f_\lambda(t) := f(t + \lambda)$. Moreover, $R(f) := \{f(t), t \in \mathbb{R}\}$ is relatively compact.

Since the range of an almost automorphic function f is relatively compact on $\mathbb{X}$, then it is bounded. Almost automorphic functions constitute a Banach space $AA(\mathbb{X})$ when it is endowed with the sup norm. This naturally generalizes the concept of (Bochner) almost periodic functions.

Definition 2.3. Let $\mathbb{X}$ be a Banach space.

1. A bounded continuous function with vanishing mean value can be defined as

$$AA_0(\mathbb{R}, \mathbb{X}) = \left\{ \phi \in BC(\mathbb{R}, \mathbb{X}) : \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \|\phi(\sigma)\| d\sigma = 0 \right\}.$$

2. Similarly we define $AA_0(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ to be the collection of all functions $f : t \mapsto f(t, x) \in BC(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ satisfying

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \|f(\sigma, x)\| d\sigma = 0$$

uniformly for x in any bounded subset of $\mathbb{X}$.

Now we describe the sets $PAA(\mathbb{R}, \mathbb{X})$ and $PAA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ of pseudo almost automorphic functions:

$$PAA(\mathbb{R}, \mathbb{X}) = \left\{ \begin{array}{l} f = g + \phi \in BC(\mathbb{R}, \mathbb{X}), \\ g \in AA(\mathbb{R}, \mathbb{X}) \quad and \quad \phi \in AA_0(\mathbb{R}, \mathbb{X}) \end{array} \right\};$$

$$PAA(\mathbb{R} \times \mathbb{X}, \mathbb{X}) = \left\{ \begin{array}{l} f = g + \phi \in BC(\mathbb{R} \times \mathbb{X}, \mathbb{X}), \\ g \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{X}) \quad and \quad \phi \in AA_0(\mathbb{R} \times \mathbb{X}, \mathbb{X}) \end{array} \right\}.$$

In both cases above, g and ϕ are called respectively the principal and the ergodic terms of f.

We have the following elementary properties of pseudo almost automorphic functions.

Theorem 2.4. *([8] Theorem 2.2).*

$PAA(\mathbb{R}, \mathbb{X})$ is a Banach space under the supremum norm.

Let now $f, h : \mathbb{R} \to \mathbb{R}$ and consider the convolution

$$(f \star h)(t) := \int_{\mathbb{R}} f(s) h(t - s) ds, \quad t \in \mathbb{R},$$

if the integral exists.

Remark 2.5. The operator $J : PAA(\mathbb{R}, \mathbb{X}) \to PAA(\mathbb{R}, \mathbb{X})$ such that $(Jx)(t) := x(-t)$ is well-defined and linear. Moreover it is an isometry and $J^2 = I$.

Remark 2.6. The operator T_a defined by $(T_a x)(t) := x(t+a)$ for a fixed $a \in \mathbb{R}$ leaves $PAA(\mathbb{R}, \mathbb{X})$ invariant.

Let us now discuss conditions which do ensure the pseudo almost automorphy of the convolution function $f \star h$ of f with h where f is pseudo almost automorphic and h is a Lebesgue mesurable function satisfying additional assumptions.

Let $f : \mathbb{R} \to X$ and $h : \mathbb{R} \to \mathbb{R}$; the convolution function (if it does exist) of f with h denoted $f \star h$ is defined by:

$$(f \star h)(t) := \int_{\mathbb{R}} f(\sigma)h(t-\sigma)d\sigma = \int_{\mathbb{R}} f(t-\sigma)h(\sigma)d\sigma = (h \star f)(t), \quad \text{for all } t \in \mathbb{R}.$$

Hence, if $f \star h$ is well-defined, then $f \star h = h \star f$.

Let $\varphi \in L^1$ and $\lambda \in \mathbb{C}$. It is well-known that the operator $A_{\varphi,\lambda}$ defined by

$$A_{\varphi,\lambda} u = \lambda u + \varphi \star u \tag{2.1}$$

acts continuously in $BC(\mathbb{R}, \mathbb{X})$ i.e., there exists $K > 0$ such that

$$\|A_{\varphi,\lambda} u\|_{BC(\mathbb{R},\mathbb{X})} \leq K \|u\|_{BC(\mathbb{R},\mathbb{X})}, \forall u \in BC(\mathbb{R}, \mathbb{X}) \tag{2.2}$$

Moreover $A_{\varphi,\lambda}$ leaves $BC(\mathbb{R}, \mathbb{X})$ invariant.

Now denote $\mathcal{M} := \{PAP(\mathbb{R}, \mathbb{X}), PAA(\mathbb{R}, \mathbb{X})\}$ where $PAP(\mathbb{R}, X)$ is the Banach space of all pseudo almost periodic functions $f : \mathbb{R} \to \mathbb{X}$. Then we have.

Theorem 2.7. *For* $\Omega \in \mathcal{M}$,

$$A_{\varphi,\lambda}(\Omega) \subset \Omega.$$

Proof. . It is an immediate consequence of the remarks above. □

Application: A Volterra-like equation

Consider the equation

$$x(t) = g(t) + \int_{-\infty}^{+\infty} a(t-\sigma)x(\sigma)d\sigma, \ t \in \mathbb{R}, \tag{2.3}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function and $a \in L^1(\mathbb{R})$.

Theorem 2.8. *Suppose* $g \in PAA(\mathbb{R}, \mathbb{X})$ *and* $\|a\|_{L^1} < 1$. *Then* (2.3) *above has a unique pseudo almost automorphic solution.*

Proof. It is clear that the operator

$$x \in PAA(\mathbb{R}, \mathbb{X}) \to \int_{-\infty}^{+\infty} a(t-\sigma)x(\sigma)d\sigma \in PAA(\mathbb{R}, \mathbb{X})$$

is well-defined. Now consider $\Gamma : PAA(\mathbb{R},\mathbb{X}) \to PAA(\mathbb{R},\mathbb{X})$ such that

$$(\Gamma x)(t) = g(t) + \int_{-\infty}^{+\infty} a(t-\sigma)x(\sigma)d\sigma, \quad t \in \mathbb{R}.$$

We can easily show that

$$\|(\Gamma x) - (\Gamma y)\| \leq \|a\|_{L^1}\|x-y\|.$$

The conclusion is immediate by the principle of contraction. □

3 Main results

This section is devoted to the proof of the main result of the paper, that is, the existence and uniqueness of an almost automorphic (mild) solution to Eq. (1.4). For that we need to establish a few preliminary results.

Definition 3.1. A function $u \in BC(\mathbb{R},\mathbb{X})$ is said to be a mild solution to Eq. (1.4) if the function $s \to AT(t-s)f(s,u(s))$ is integrable on $(-\infty,t)$ for each $t \in \mathbb{R}$ and

$$u(t) = -f(t,u(t)) - \int_{-\infty}^{t} AT(t-s)f(s,u(s))ds$$

for each $t \in \mathbb{R}$.

We now make the following assumptions.

(H.1) The operator A is the infinitesimal generator of an exponentially stable semigroup $(T(t))_{t\geq 0}$ such that there exist constants $M > 0$ and $\delta > 0$ with

$$\|T(t)\|_{B(\mathbb{X})} \leq Me^{-\delta t}, \quad \forall t \geq 0.$$

Furthermore, the function $\sigma \to AT(\sigma)$ defined from $(0,\infty)$ into $B(\mathbb{X})$ is strongly (Lebesgue) measurable and there exist a function $\gamma : (0,\infty) \to [0,\infty)$ such that $\sup_{s\geq s_0}\gamma(s) < \infty$ for any $s_0 > 0$, and a constant $\omega > 0$ with $\rho := \int_0^{\infty} e^{-\omega s}\gamma(s)ds < \infty$ such that

$$\|AT(s)\|_{B(\mathbb{X})} \leq e^{-\omega s} \,.\, \gamma(s), \quad s > 0.$$

(H.2) The function $f : \mathbb{R}\times\mathbb{X} \mapsto \mathbb{X}$, $(t,u) \mapsto f(t,u)$ is jointly continuous and

$$\|f(t,u) - f(t,v)\|_{\mathbb{X}} \leq k(t) \,.\, \|u-v\|, \quad \text{and}$$

for all $t \in \mathbb{R}$, and $\forall u,v \in \mathbb{X}$. Here $k \in L^1(\mathbb{R},\mathbb{R}^+)$.

(H.3) $f = g + \psi \in PAA(\mathbb{R}\times\mathbb{X},\mathbb{X})$, where g and ψ are the principal and the ergodic terms of f respectively and $f(t,u)$ and $g(t,u)$ are uniformly continuous on every bounded subset $K \subset \mathbb{X}$ uniformly in $t \in \mathbb{R}$.

Lemma 3.2. *Suppose that assumptions (H.1)-(H.2)-(H.3) hold. Define the nonlinear operator* Λ_1 *by: For each* $\xi \in PAA(\mathbb{X})$,

$$(\Lambda_1\xi)(t) = \int_{-\infty}^{t} AT(t-s)f(s,\xi(s))ds$$

Then Λ_1 *maps* $PAA(\mathbb{X})$ *into itself.*

Proof. Set h defined by: $h(.) = f(.,\xi(.))$. Since $h \in PAA(\mathbb{R},\mathbb{X})$ using [6, Theorem 2.4] with assumption (H.3), we can write $h = \beta + \phi$ where β is the principal part and ϕ the ergodic term of h. Using the same argument as in [11], we can prove that $t \mapsto \int_{-\infty}^{t} AT(t-s)\beta(s)ds$ is in $AA(\mathbb{X})$. Now, set:

$$\nu(t) = -\int_{-\infty}^{t} AT(t-s)\phi(s)ds.$$

We have:

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T}\|\nu(t)\|_X dt &\leq \frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{t}\|A(t-s)\phi(s)\|_{\mathbb{X}}dsdt\\ &\leq \frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{t}e^{-\omega(t-s)}\gamma(t-s)\|\phi(s)\|_{\mathbb{X}}dsdt.\end{aligned}$$

Let's write:

$$\frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{t}e^{-\omega(t-s)}\gamma(t-s)\|\phi(s)\|_{\mathbb{X}}dsdt = I_1 + I_2,$$

where:

$$I_1 = \frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{-T}e^{-\omega(t-s)}\gamma(t-s)\|\phi(s)\|_{\mathbb{X}}dsdt$$

and

$$I_2 = \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{t}e^{-\omega(t-s)}\gamma(t-s)\|\phi(s)\|_{\mathbb{X}}dsdt.$$

We prove know that $I_1 \to 0$ and $I_2 \to 0$ as $T \to \infty$.

Indeed, for I_1, let $s_0 > 0$ and set $M(s_0) = \sup_{s\geq s_0}\gamma(s)$, and $K = \sup_{t\in\mathbb{R}}\|\phi(t)\|_{\mathbb{X}}$. We have:

$$I_1 \leq K\frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{-T}e^{-\omega(t-s)}\gamma(t-s)dsdt = \frac{K}{2T}\int\int_{D}e^{-\omega(t-s)}\gamma(t-s)dsdt,$$

where $D = \{(s,t) \in \mathbb{R}^2, |t| \leq T, \quad s \leq -T\}$. We introduce also:

$$D_1 = \{(s,t) \in D, t-s \geq s_0\}, \quad D_2 = D \setminus D_1.$$

We have:

$$\begin{aligned}&\int\int_{D_1}e^{-\omega(t-s)}\gamma(t-s)dsdt \leq M(s_0)\int\int_{D_1}e^{-\omega(t-s)}dsdt\\ &\leq M(s_0)\int\int_{D}e^{-\omega(t-s)}dsdt = \frac{M(s_0)e^{-\omega T}}{\omega}\int_{-T}^{T}e^{-\omega t}dt \leq 2T\frac{M(s_0)e^{-\omega T}}{\omega}.\end{aligned}$$

Moreover,

$$\int\int_{D_2} e^{-\omega(t-s)}\gamma(t-s)dsdt \leq \int_{-T-s_0}^{-T}\int_{-T}^{s+s_0} e^{-\omega(t-s)}\gamma(t-s)dtds$$
$$\leq \int_{-T-s_0}^{-T}\int_{-T-s}^{s_0} e^{-\omega\sigma}\gamma(\sigma)d\sigma ds$$
$$\leq \int_{-T-s_0}^{-T}\int_{0}^{s_0} e^{-\omega\sigma}\gamma(\sigma)d\sigma ds$$
$$\leq s_0\int_{0}^{s_0} e^{-\omega\sigma}\gamma(\sigma)d\sigma.$$

So, for any $T \geq 1$, we have:

$$I_1 \leq \frac{K}{2T}\left(2T\frac{M(s_0)e^{-\omega T}}{\omega} + s_0\int_0^{s_0} e^{-\omega\sigma}\gamma(\sigma)d\sigma\right)$$
$$\leq K\left(e^{-\omega T}\frac{M(s_0)}{\omega} + s_0\int_0^{s_0} e^{-\omega\sigma}\gamma(\sigma)d\sigma\right).$$

Let $\epsilon > 0$. We can find $s_0 > 0$ such that $Ks_0\int_0^{s_0} e^{-\omega\sigma}\gamma(\sigma)d\sigma < \varepsilon/2$. Let us take such an s_0. After, for T sufficiently large, $Ke^{-\omega T}\frac{M(s_0)}{\omega} < \epsilon/2$, and so, for sufficiently large T, $I_1 \leq \epsilon$.
Now, we consider I_2. We have:

$$I_2 = \frac{1}{2T}\int_{-T}^{T}\|\phi(s)\|_{\mathbb{X}}ds\int_s^T e^{-\omega(t-s)}\gamma(t-s)dt$$
$$\leq \frac{1}{2T}\int_{-T}^{T}\|\phi(s)\|_{\mathbb{X}}ds\int_0^{T-s} e^{-\omega\sigma}\gamma(\sigma)d\sigma$$
$$\leq \rho\frac{1}{2T}\int_{-T}^{T}\|\phi(s)\|_{\mathbb{X}}ds \to 0 \qquad \text{as} \qquad T\to\infty.$$

□

Now we are ready to state and prove the following.

Theorem 3.3. *Suppose that assumptions (H.1)-(H.2) hold. Then Eq. (1.4) has a unique pseudo almost automorphic (mild) solution if*

Proof. Define the nonlinear operator $\Gamma : AA(\mathbb{X}) \mapsto AA(\mathbb{X})$ by:

$$\Gamma(u) : t \mapsto -f(t,u(t)) - \int_{-\infty}^{t} AT(t-s)f(s,u(s))ds.$$

We have:

$$\|\Gamma(u)(t)-\Gamma(v)(t)\|_{\mathbb{X}} \leq \|f(t,u(t))-f(t,v(t))\|_{\mathbb{X}} + \int_{-\infty}^{t}\|AT(t-s)(f(s,u(s))-f(s,v(s)))\|_{\mathbb{X}}ds$$

$$\leq k(t)\|u(t)-v(t)\|_{\mathbb{X}} + \int_{-\infty}^{t} e^{-\omega(t-s)}\gamma(t-s)k(s)\|u(x)-v(s)\|_{\mathbb{X}}ds$$
$$\leq \left[k(t) + \int_{-\infty}^{t} e^{-\omega(t-s)}\gamma(t-s)k(s)ds\right]\|u-v\|_{\infty}$$
$$\leq (1+\rho)\|k\|_{\infty}\|u-v\|_{\infty}.$$

So, we obtain:

$$\|\Gamma(u)-\Gamma(v)\|_{\infty} \leq (1+\rho)\|k\|_{\infty}\|u-v\|_{\infty},$$

and we can conclude using the Banach's fixed point principle.

□

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