WEIGHTED PSEUDO ALMOST AUTOMORPHIC FUNCTIONS AND APPLICATIONS TO ABSTRACT DIFFERENTIAL EQUATIONS

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Abstract. This paper is concerned with weighted pseudo almost automorphic functions, which are more general and complicated than weighted pseudo almost periodic functions. Using these properties, we establish an existence and uniqueness theorem for pseudo almost automorphic mild solutions to semilinear differential equations in a Banach spaces which extends known results.

1. Introduction

In this paper, we introduce the notion of weighted pseudo almost automorphic functions with values in a Banach space. This generalizes the concept of weighted pseudo almost periodic functions introduced by T. Diagana [2]. We firstly extend some basic properties of weighted pseudo almost automorphic functions in a further sight. We present a result on the composition of weighted pseudo almost automorphic functions under conditions which are weaker than the Lipschtiz condition. Furthermore, under these conditions, we deal with the existence and uniqueness of weighted pseudo almost automorphic mild solutions to the abstract differential equations in Banach space $\mathbb{X}$ of the form

$$x'(t) = Ax(t) + f(t, x(t)), \quad \text{t} \in \mathbb{R}, \tag{1}$$

where A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space $\mathbb{X}$, and $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ a weighted pseudo almost automorphic function.

Throughout this paper, $\mathbb{X}$ will be Banach spaces, and $BC(\mathbb{R}, \mathbb{X})$ the space of all bounded continuous functions under the sup norm. To begin this paper, we recall some primary definitions of almost automorphic.

The paper is presented as follows. In Section 2, we first study basic properties of weighted pseudo almost automorphic functions and then use these results to study the existence and uniqueness of weighted almost automorphic mild solution of a

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semilinear abstract differential equation. The most important result here is the composition Theorem 2.10 which generalizes Theorem 2.3 [13].

2. Weighted pseudo almost automorphic functions

Let's first recall some properties of almost automorphic functions. Detailed presentations can be found in [10, 11].

Definition 2.1. (S. Bochner)

(i) Let $f : \mathbb{R} \to \mathbb{X}$ be a bounded continuous function. We say that f is almost automorphic if for every sequence of real numbers $\{s_n\}_{n=1}^{\infty}$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that:

$$g(t) = \lim_{n\to\infty} f(t+\tau_n)$$

is well-defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t-\tau_n) = f(t)$$

for each $t \in \mathbb{R}$. Denote by $AA(\mathbb{R}, \mathbb{X})$ the set of all such functions.

(ii) A continuous function $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is said to be almost automorphic if $f(t, x)$ is almost automorphic in $t \in \mathbb{R}$ uniformly for all $x \in B$, where B is any bounded subset of $\mathbb{X}$.

Clearly when the convergence above is uniform in $t \in \mathbb{R}$, f is almost periodic. The function g is measurable, but not continuous in general. Denote by $AA(\mathbb{X})$ (resp. $AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$), the set of all almost automorphic function $f : \mathbb{R} \to \mathbb{X}$, (resp. $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$). With the sup norm $\sup_{t\in\mathbb{R}} \|f(t)\|$ (resp. $\sup_{(t,x)\in\mathbb{R}\times\mathbb{X}} \|f(t, x)\|$), these spaces turn out to be Banach spaces.

Now like in [2], let $\mathcal{U}$ be the set of all functions $\rho : \mathbb{R} \to (0, \infty)$ which are positive and locally integrable over $\mathbb{R}$.

For a given $r > 0$, set

$$m(r, \rho) := \int_{-r}^{r} \rho(x)dx$$

for each $\rho \in \mathcal{U}$.

Define

$$\mathcal{U}_\infty := \{\rho \in \mathcal{U} : \lim_{r\to\infty} m(r, \rho) = \infty\}$$

and

$$\mathcal{U}_b := \{\rho \in \mathcal{U}_\infty : \rho \textit{ is bounded and } \inf_{x\in\mathbb{R}} \rho(x) > 0\}.$$

It is clear that $\mathcal{U}_b \subset \mathcal{U}_\infty \subset \mathcal{U}$.

Now for $\rho \in \mathcal{U}_\infty$ define

$$PAA_0(\mathbb{X}, \rho) := \{f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{r\to\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \|f(s)\|\rho(s)ds = 0\}$$

Similarly we define $PAA_0(\mathbb{R}\times\mathbb{X},\rho)$ as the collection of all functions $F:\mathbb{R}\times\mathbb{X}\to\mathbb{X}$ is jointly continuous and $F(\cdot,y)$ is bounded for each $y\in\mathbb{X}$, and

$$\lim_{r\to\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}\|F(s,y)\|\rho(s)ds=0$$

uniformly in $y\in\mathbb{X}$.

We are now ready to introduce the sets $WPAA(\mathbb{R},\mathbb{X})$ and $WPAA(\mathbb{R}\times\mathbb{X},\mathbb{X})$ of weighted pseudo almost automorphic functions:

$$WPAA(\mathbb{X},\rho)=\left\{\begin{array}{l} f=g+\phi\in BC(\mathbb{R},\mathbb{X}): \\ g\in AA(\mathbb{R},\mathbb{X}) \quad and \quad \phi\in PAA_0(\mathbb{X},\rho)\end{array}\right\};$$

$$WPAA(\mathbb{R}\times\mathbb{X},\mathbb{X})=\left\{\begin{array}{l} f=g+\phi\in BC(\mathbb{R}\times\mathbb{X},\mathbb{X}): \\ g\in AA(\mathbb{R}\times\mathbb{X},\mathbb{X}) \quad and \quad \phi\in PAA_0(\mathbb{R}\times\mathbb{X},\mathbb{X},\rho)\end{array}\right\}.$$

Remark 2.2. When $\rho=1$, we obtain the standard spaces $PAA(\mathbb{R},\mathbb{X})$ and $PAA(\mathbb{R}\times\mathbb{X},\mathbb{X})$.

Lemma 2.3. *If $f=g+\phi$ with $g\in AA(\mathbb{R},\mathbb{X})$, and $\phi\in PAA_0(\mathbb{X},\rho)$ where $\rho\in\mathcal{U}_b$, then $g(\mathbb{R})\subset\overline{f(\mathbb{R})}$.*

Proof. The proof is an adaptation of [14] Theorem 2.2. □

Theorem 2.4. *The decomposition of a weighted pseudo almost automorphic function is unique for any $\rho\in\mathcal{U}_b$.*

Proof. Assume that $f=g_1+\phi_1$ and $f=g_2+\phi_2$. Then $0=(g_1-g_2)+(\phi_1-\phi_2)$. Since $g_1-g_2\in AA(\mathbb{R},X)$, and $\phi_1-\phi_2\in PAA_0(X,\rho)$, in view of Lemma 2.3, we deduce that $g_1-g_2=0$. Consequently $\phi_1-\phi_2=0$, that is $\phi_1=\phi_2$, which proves the uniqueness of the decomposition of f. □

From the above it is clear that $AA(\mathbb{R},\mathbb{X})\bigcap PAA_0(\mathbb{X},\rho)=\{0\}$.

Theorem 2.5. *If $\rho\in\mathcal{U}_b$, then $(WPAA(\mathbb{X},\rho),\|.\|_\infty)$ is a Banach space.*

Proof. Assume that (f_n) is a Cauchy sequence in $WPAA(\mathbb{X},\rho)$. We can write uniquely $f_n=g_n+\varphi_n$. Using Lemma 2.3, we see that : $\|g_p-g_q\|_\infty\le\|f_p-f_q\|_\infty$, from which we deduce that (g_n) is a Cauchy sequence in the Banach space $AA(\mathbb{R},\mathbb{X})$. So, $\phi_n=f_n-g_n$ is also a Cauchy sequence in the Banach space $PAA_0(\mathbb{X},\rho)$. We can deduce that $g_n\to g\in AA(\mathbb{R},\mathbb{X})$, $\phi_n\to\phi\in PAA_0(\mathbb{R},\rho)$, and finally $f_n\to g+\phi\in WPAA(\mathbb{X},\rho)$. □

Proposition 2.6. Let $\rho\in\mathcal{U}_b$, $f\in WPAA(\mathbb{X},\rho)$ and $g\in L^1(\mathbb{R})$. Then the operator $J_g:WPAA(\mathbb{X},\rho)\to WPAA(\mathbb{X},\rho)$ defined by

$$(J_gf)(t):=(f\star g)(t)$$

is well defined.

Proof. The proof is immediate using [7] Lemma 3.5 and Proposition 2.8 [3] □

On $\mathcal{U}_\infty$, we use the following $\sim$ introduced (and denoted by $\prec$ by Diagana ([3]) :

$$(\rho_1 \sim \rho_2) \Leftrightarrow \left(\frac{\rho_2}{\rho_1} \in \mathcal{U}_b\right).$$

Theorem 2.7. *Let $\rho_1, \rho_2 \in \mathcal{U}_\infty$. If $\rho_1 \sim \rho_2$, then $WPAA(\mathbb{X}, \rho_1) = WPAA(\mathbb{X}, \rho_2)$.*

Proof. Assume that $\rho_1 \sim \rho_2$. There exists $(a, b) \in \mathbb{R}^2$ s.t.: $a\rho_1 \le \rho_2 \le b\rho_1$. So,

$$am(r, \rho_1) \le m(r, \rho_2) \le bm(r, \rho_1),$$

and:

$$\begin{aligned}\frac{a}{b}\frac{1}{m(r,\rho_1)}\int_{-r}^{r}\|\phi(s)\|\rho_1(s)ds &\le \frac{1}{m(r,\rho_2)}\int_{-r}^{r}\|\phi(s)\|\rho_2(s)ds\\ &\le \frac{b}{a}\frac{1}{m(r,\rho_1)}\int_{-r}^{r}\|\phi(s)\|\rho_1(s)ds,\end{aligned}$$

which completes the proof. □

We make the following assumptions

- H1. $f(t, x)$ is uniformly continuous in any bounded subset $K \subset X$ uniformly in $t \in \mathbb{R}$
- H2. $g(t, x)$ is uniformly continuous in any bounded subset $K \subset X$ uniformly in $\in \mathbb{R}$.

Let's recall the result (see [13] Theorem 2.4).

Theorem 2.8. *Let $f = g + \phi \in PAA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ where $g(t, x) \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, $\phi(t, x) \in AA_0(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ such that H1 and H2 are satisfied.*

If $x(t) \in PAA(\mathbb{R}, \mathbb{X})$, then $f(\cdot, x(\cdot)) \in PAA(\mathbb{R}, \mathbb{X})$

Lemma 2.9. *Let $f \in BC(\mathbb{R}, \mathbb{X})$. Then $f \in PAA_0(\mathbb{X}, \rho)$ where $\rho \in \mathcal{U}_b$ if and only if for every $\epsilon > 0$,*

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} meas(M_{r,\epsilon}(f)) = 0$$

where $M_{r,\epsilon}(f) := \{t \in [-r, r] \ / \ \|f(t)\| \ge \epsilon\}$.

Proof. a) Necessity. We follow the proof of Lemma 2.1 [16]. By contradiction suppose that there exists $\epsilon_0 > 0$ such that

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} mes(M_{r,\epsilon_0}(f)) \neq 0.$$

Then there exists $\delta > 0$ such that for every $n \in \mathbb{N}$, $\frac{1}{m(r_n,\rho)} mes(M_{r_n,\epsilon_0}(f)) \ge \delta$ for some $r_n > n$.

So we get

$$\begin{aligned}
\frac{1}{m(r_n,\rho)}\int_{-r_n}^{r_n}\|f(s)\|\rho(s)ds &= \frac{1}{m(r_n,\rho)}\int_{M_{r_n,\epsilon_0}(f)}\|f(s)\|\rho(s)ds \\
&+ \frac{1}{m(r_n,\rho)}\int_{[-r_n,r_n]-M_{r_n,\epsilon_0}(f)}\|f(s)\|\rho(s)ds \\
&\geq \frac{1}{m(r_n,\rho)}\int_{M_{r_n,\epsilon_0}(f)}\|f(s)\|\rho(s)ds \\
&\geq \frac{\epsilon_0}{m(r_n,\rho)}\int_{M_{r_n,\epsilon_0}(f)}\rho(s)ds \\
&\geq \epsilon_0\delta\gamma,
\end{aligned}$$

where $\gamma = \inf_{s\in\mathbb{R}}\rho(s)$. This contradicts the assumption .

b) Sufficiency. Assume that $\lim_{r\to\infty}\frac{1}{m(r,\rho)}mes(M_{r,\epsilon}(f)) = 0$. Then for every $\epsilon > 0$, there exists $r_0 > 0$ such that for every $r > r_0$,

$$\frac{1}{m(r,\rho)}mes(M_{r,\epsilon}(f)) < \frac{\epsilon}{KM}$$

where $M := \sup_{t\in\mathbb{R}}\|f(t)\| < \infty$ and $K := \sup_{t\in\mathbb{R}}\rho(t) < \infty$.

Now we have

$$\begin{aligned}
\frac{1}{m(r,\rho)}\int_{-r}^{r}\|f(s)\|\rho(s)ds &= \frac{1}{m(r,\rho)}\left(\int_{M_{r,\epsilon}(f)}\|f(s)\|\rho(s)ds + \int_{[-r,r]-M_{r,\epsilon}(f)}\|f(s)\|\rho(s)ds\right) \\
&\leq \frac{MK}{m(r,\rho)}meas(M_{r,\epsilon}(f)) + \frac{\epsilon}{m(r,\rho)}\int_{[-r,r]-M_{r,\epsilon}(f)}\rho(s)ds \\
&\leq 2\epsilon.
\end{aligned}$$

Which shows that $\lim_{r\to\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}\|f(s)\|\rho(s)ds = 0$, that is $f \in PAA_0(X,\rho)$. The proof is now complete. □

The following result generalizes Theorem 2.8 above.

Theorem 2.10. *Let $f = g + \phi \in WPAA(\mathbb{X},\rho)$ where $\rho \in \mathcal{U}_\infty$ and assume that H1 and H2 are satisfied.*

Then $L(\cdot) := f(\cdot, h(\cdot)) \in WPAA(\mathbb{X},\rho)$ if $h \in WPAA(\mathbb{X},\rho)$.

Proof. We have $f = g + \phi$ where $g \in AA(\mathbb{R},\mathbb{X})$ and $\phi \in PAA_0(\mathbb{X},\rho)$ and $h = \mu + \nu$ where $\mu \in AA(\mathbb{R},\mathbb{X})$ and $\nu \in PAA_0(\mathbb{X},\rho)$.

Now let's write

$L(\cdot) = g(\cdot,\mu(\cdot)) + f(\cdot,h(\cdot)) - g(\cdot,\mu(\cdot)) = g(\cdot,\mu(\cdot)) + f(\cdot,h(\cdot)) - f(\cdot,\mu(\cdot)) + \phi(\cdot,\mu(\cdot))$.

By [13] Lemma 2.2, $g(\cdot,\mu(\cdot)) \in AA(\mathbb{R},\mathbb{X})$. Consider now the function

$$\Phi(\cdot) := f(\cdot,h(\cdot)) - f(\cdot,\mu(\cdot)).$$

Clearly $\Phi(t) \in BC(\mathbb{R},\mathbb{X})$. For Φ to be in PAA_0, it is enough to show that $\lim_{r\to\infty}\frac{1}{m(r,\rho)}mes(M_{r,\epsilon}(\Phi)) = 0$.

By Lemma 2.3, $\mu(\mathbb{R}) \subset \overline{h(\mathbb{R})}$ which is a bounded set. Using assumption H1 with $K = \overline{h(\mathbb{R})}$, we say that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in K, \ \|x - y\| < \delta \Longrightarrow \|f(t,x) - f(t,y)\| < \epsilon, \ \ \forall t \in \mathbb{R}.$$

Thus we obtain

$$\begin{aligned} \frac{1}{m(r,\rho)} mes(M_{r,\epsilon}(\Phi(t))) &= \frac{1}{m(r,\rho)} mes(M_{r,\epsilon}(f(t,h(t)) - f(t,\mu(t)))) \\ &\leq \frac{1}{m(r,\rho)} mes(M_{r,\delta}(h(t) - \mu(t))) \\ &= \frac{1}{m(r,\rho)} mes(M_{r,\delta}(\nu(t))). \end{aligned}$$

Now since $\nu \in PAA_0(\mathbb{X}, \rho)$, then by Lemma 2.9, $\lim_{r\to\infty} \frac{1}{m(r,\rho)} mes(M_{r,\epsilon}(\nu(t))) = 0$. Consequently $\lim_{r\to\infty} \frac{1}{m(r,\rho)} mes(M_{r,\epsilon}(\Phi)(t)) = 0$. Thus $\Phi \in PAA_0(\mathbb{X}, \rho)$.

Finally we need to show that $\phi(t, \mu(t)) \in PAA_0(X, \rho)$. Note that $\phi(t, \mu(t))$ is uniformly continuous on $[-r, r]$, and that $\mu([-r, r])$ is compact since μ is continuous on $\mathbb{R}$ as an almost automorphic function. Thus given $\epsilon > 0$, there exists $\delta > 0$ such that $\mu([-r, r]) \subset \cup_{k=1}^m B_k$ where $B_k = \{x \in X \ / \ \|x - x_k\| < \delta\}$ for some $x_k \in \mu([-r, r])\}$, and

$$\|\phi(t, \mu(t)) - \phi(t, x_k)\| < \frac{\epsilon}{2}, \ \mu(t) \in B_k, \ t \in [-r, r]. \tag{2}$$

It is easy to see that the set $U_k := \{t \in [-r, r] \ / \ \mu(t) \in B_k\}$ is open in $[-r, r]$ and that $[-r, r] = \cup_{k=1}^m U_k$. Define V_k by

$$V_1 = U_1, \ \ V_k = U_k - \cup_{i=1}^{k-1} U_i, \ 2 \leq k \leq m.$$

Then it is clear that $V_i \cap V_j = \emptyset$, if $i \neq j$, $1 \leq i, j \leq m$. So we get

$$\Lambda := \{t \in [-r, r] \ / \ \|\phi(t, \mu(t))\| \geq \frac{\epsilon}{2}\|\}$$

$$\subset \cup_{k=1}^m \{t \in V_k \ / \ \|\phi(t, \mu(t)) - \phi(t, x_k)\| + \|\phi(t, x_k)\| \geq \epsilon\}$$

$$\subset \cup_{k=1}^m (\{t \in V_k \ / \ \|\phi(t, \mu(t)) - \phi(t, x_k)\| \geq \frac{\epsilon}{2}\} \cup \{t \in V_k \ / \ \|\phi(t, x_k)\| \geq \frac{\epsilon}{2}\}).$$

In view of (2), it follows that

$$\{t \in V_k \ / \ \|\phi(t, \mu(t)) - \phi(t, x_k)\| \geq \frac{\epsilon}{2}\} = \emptyset, \ k = 1, 2, ..., m.$$

Thus we get

$$\frac{1}{m(r,\rho)} mes(M_{r,\epsilon}(\phi(t, \alpha(t))) \leq \sum_{k=1}^m \frac{1}{m(r,\rho)} mes(M_{r,\epsilon}(\phi(t, x_k)).$$

And since $\phi(t, x) \in PAA_0(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ and $\lim_{r\to\infty} \frac{1}{m(r,\rho)} mes(M_{r,\frac{\epsilon}{2}}(\phi(t, x_k))) = 0$, it follows that $\lim_{r\to\infty} \frac{1}{m(r,\rho)} mes(M_{r,\frac{\epsilon}{2}}(\phi(t, \mu(t)))) = 0$, i.e. $\phi(t, \mu(t)) \in PAA_0(\mathbb{X}, \rho)$. This completes the proof. □

We can establish the following consequence.

Corollary 2.11. $f = g + \phi \in WPAA(\mathbb{X}, \rho)$ where $\rho \in \mathcal{U}_\infty$ assume both f and g are Lipschitzian in $x \in \mathbb{X}$ uniformly in $t \in \mathbb{R}$.

Then $L(\cdot) := f(\cdot, h(\cdot)) \in WPAA(\mathbb{X}, \rho)$ if $h \in WPAA(\mathbb{X}, \rho)$.

3. Applications

Consider the abstract differential equation

$$x'(t) = Ax(t) + f(t, x(t)), \quad \text{t} \in \mathbb{R}, \tag{3}$$

with the following assumptions

- (H3) A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space $\mathbb{X}$ such that

$$\|T(t)\| \leq Ne^{-\omega t},\ t \geq 0$$

- (H4) $f = g + \phi \in WPAA(\mathbb{X}, \rho)$ where $\rho \in \mathcal{U}_\infty$
- (H5) $\|f(t, x) - f(t, y)\| \leq L_f \|x - y\|,\ \forall x, y \in \mathbb{X}$
- (H6) $\|g(t, x) - g(t, x)\| \leq L_g \|x - y\|,\ \forall x, y \in \mathbb{X}$

Lemma 3.1. *Let $f = g + \phi \in WPAA(\mathbb{X}, \rho)$ where $\rho \in \mathcal{U}_\infty$ and $(T(t))_{t\geq 0}$ is an exponentially stable semigroup. Then $F(t) := \int_{-\infty}^{t} T(t-s)f(s)ds \in WAAP(\mathbb{X}, \rho)$.*

Proof. Let $F(t) = G(T) + \Phi(t)$ where $G(t) := \int_{-\infty}^{t} T(t-s)g(s)ds$ and $\Phi(t) := \int_{-\infty}^{t} T(t-s)\phi(s)ds$.

Then by [9], $G(t) \in AA(\mathbb{X})$. Now let's show that $\Phi(t) \in PAA_0(\mathbb{X}, \rho)$.

We have

$$\frac{1}{m(r,\rho)} \int_{-r}^{r} \|\Phi(s)\|\rho(s)ds = \frac{1}{m(r,\rho)} \int_{-r}^{r} \| \int_{-\infty}^{s} T(s-\sigma)\phi(\sigma)d\sigma\|\rho(s)ds = I_1 + I_2,$$

where

$$I_1 := \frac{1}{m(r,\rho)} \int_{-r}^{r} \| \int_{-\infty}^{-r} T(s-\sigma)\phi(\sigma)d\sigma\|\rho(s)ds$$

and

$$I_2 := \frac{1}{m(r,\rho)} \int_{-r}^{r} \| \int_{-r}^{s} T(s-\sigma)\phi(\sigma)d\sigma\|\rho(s)ds.$$

We have

$$
\begin{aligned}
I_1 &:= \frac{1}{m(r,\rho)}\int_{-r}^{r}\left\|\int_{-\infty}^{-r} T(s-\sigma)\phi(\sigma)d\sigma\right\|\rho(s)ds\\
&\leq \frac{1}{m(r,\rho)}\int_{-r}^{r}\left(\int_{-\infty}^{-r}\|T(s-\sigma)\|\,\|\phi(\sigma)\|\,d\sigma\right)\rho(s)ds\\
&\leq \frac{N}{m(r,\rho)}\int_{-r}^{r}\left(\int_{-\infty}^{-r} e^{-\omega(s-\sigma)}\,\|\phi(\sigma)\|\,d\sigma\right)\rho(s)ds\\
&\leq \frac{N}{m(r,\rho)}\left(\int_{-r}^{r} e^{-\omega s}\rho(s)ds\right)\left(\int_{-\infty}^{-r} e^{\omega\sigma}\,\|\phi(\sigma)\|\,d\sigma\right)\\
&\leq \frac{N}{m(r,\rho)}\|\rho\|_{L^1_{Loc}(\mathbb{R})}\left(\int_{-r}^{r} e^{-\omega s}ds\right)\left(\int_{-\infty}^{-r} e^{\omega\sigma}\,\|\phi(\sigma)\|\,d\sigma\right)\\
&\leq \frac{N}{\omega\, m(r,\rho)}\|\rho\|_{L^1_{Loc}(\mathbb{R})}\left(e^{-\omega\, r}-e^{\omega\, r}\right)\sup_{t\in\mathbb{R}}\|\phi(t)\|\left(\int_{-\infty}^{-r} e^{\omega\sigma}d\sigma\right)\\
&\leq \frac{N}{\omega\, m(r,\rho)}\|\rho\|_{L^1_{Loc}(\mathbb{R})}e^{-\omega\, r}\sup_{t\in\mathbb{R}}\|\phi(t)\|\left(\int_{-\infty}^{-r} e^{\omega\sigma}d\sigma\right)\\
&\leq \frac{N}{\omega^2\, m(r,\rho)}\|\rho\|_{L^1_{Loc}(\mathbb{R})}e^{-2\omega\, r}\sup_{t\in\mathbb{R}}\|\phi(t)\|\\
&\leq \frac{N}{\omega^2\, m(r,\rho)}\|\rho\|_{L^1_{Loc}(\mathbb{R})}\sup_{t\in\mathbb{R}}\|\phi(t)\|
\end{aligned}
$$

Since $\sup_{t\in\mathbb{R}}\|\phi(t)\| < \infty$ and $\lim_{r\to\infty} m(r,\rho) = \infty$, then $\lim\limits_{r\to\infty} I_1 = 0$.

$$
\begin{aligned}
I_2 &:= \frac{1}{m(r,\rho)}\int_{-r}^{r}\left\|\int_{-r}^{s} T(s-\sigma)\phi(\sigma)d\sigma\right\|\rho(s)ds\\
&\leq \frac{1}{m(r,\rho)}\int_{-r}^{r}\left(\int_{-r}^{s}\|T(s-\sigma)\|\,\|\phi(\sigma)\|\,d\sigma\right)\rho(s)ds\\
&\leq \frac{N}{m(r,\rho)}\int_{-r}^{r}\left(\int_{-r}^{s} e^{-\omega(s-\sigma)}\,\|\phi(\sigma)\|\,d\sigma\right)\rho(s)ds\\
&\leq \frac{N}{\omega\, m(r,\rho)}\int_{-r}^{r}\left(1-e^{-\omega(s+r)}\right)\|\phi(s)\|\,\rho(s)ds\\
&\leq \frac{N}{\omega\, m(r,\rho)}\int_{-r}^{r}\|\phi(s)\|\,\rho(s)ds
\end{aligned}
$$

Since $\phi \in PAA_0(\mathbb{X},\rho)$, then $\lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \|\phi(s)\| \rho(s)ds = 0$.
Thus $\lim_{r\to\infty} I_2 = 0$

□

Theorem 3.2. *Under assumptions (H3-H6) above, Eq (2) has a unique mild solution in $WPAA(\mathbb{X},\rho)$ provided $\frac{NL_f}{\omega} < 1$.*

Proof. In view of Corollary 1.11 above, the operator $\Gamma : WPAA(\mathbb{X},\rho) \to WPAA(\mathbb{X},\rho)$ such that

$$(\Gamma x)(t) := \int_{-\infty}^{t} T(t-\sigma)f(\sigma, x(\sigma))d\sigma,\ t \in \mathbb{R}$$

is well-defined.

Now if $x, y \in WPAA(\mathbb{X},\rho)$, we have

$$\begin{aligned}\|(\Gamma x)(t) - (\Gamma y)(t)\| &= \|\int_{-\infty}^{t} T(t-\sigma)(f(\sigma,x(\sigma)) - f(\sigma,y(\sigma))d\sigma\| \\ &\leq NL_f \int_{-\infty}^{t} e^{-\omega(t-\sigma)}\|x(\sigma)-y(\sigma)\|d\sigma \\ &\leq \frac{NL_f}{\omega}\|x-y\|_\infty, \quad \forall t \in \mathbb{R}.\end{aligned}$$

Thus

$$\|\Gamma x - \Gamma y\|_\infty \leq \frac{NL_f}{\omega}\|x-y\|_\infty.$$

The conclusion follows by the principle of contraction.

□

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