ON THE NONEXISTENCE OF PURELY STEPANOV ALMOST-PERIODIC SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. It is shown that, in uniformly convex Banach spaces, Stepanov almost-periodic functions with Stepanov almost-periodic derivatives are uniformly almost-periodic in the sense of Bohr. This in natural situations yields, jointly with the derived properties of the associated Nemytskii operators, the nonexistence of purely (i.e. nonuniformly continuous) Stepanov almost-periodic solutions of ordinary differential equations. In particular, the existence problem of such solutions, considered in a series of papers [Hu1], [Hu2], [HM1], [HM2], [HM3] of Z. Hu and A. B. Mingarelli, is so answered in a negative way.

1. Introduction

Some results about Stepanov almost-periodic solutions of ordinary differential equations indicate that these solutions are in fact Bohr almost-periodic (see e.g. [AB], [AP], [C2], [C3], [Ha], [Pa], [Rd], [Ro]). This especially concerns the *Bohr–Neugebauer-type theorems* and the *Favard-type results*, because solutions are assumed there to be bounded.

On the other hand, in a series of papers [Hu1], [Hu2], [HM1], [HM2], [HM3], Z. Hu and A. B. Mingarelli posed and elaborated the question *"whether boundedness of solutions can imply their pure (i.e. nonuniformly continuous) Stepanov almost-periodicity (?)"*. A Stepanov-like extension of their Favard-type results was, nevertheless, shown by M. Tarallo [Ta] just apparent.

In [AP], we have presented several examples of scalar ordinary differential equations which admit bounded or unbounded purely Stepanov almost-periodic solutions, but in none of them the right-hand sides were Stepanov almost-periodic in a time variable. Moreover, we observed in [AP] that in order to have, under natural assumptions imposed on the right-hand sides (like the Lipschitzianity in a space variable and the Stepanov almost-periodicity in a time variable), a purely Stepanov almost-periodic solution, no its discretization can be curiously (Stepanov) almost-periodic. At least in Euclidean spaces, such solutions must be, under the same conditions, still unbounded.

In the present note, we will show that also in (not necessarily finite-dimensional) uniformly convex Banach spaces (e.g. in Hilbert spaces) such solutions must be oppositely bounded, and subsequently uniformly almost-periodic.

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2. Stepanov space $\mathrm{S}_{\mathrm{ap}}^{\infty}$ and some further preliminaries

Let E be a Banach space endowed with the norm $|\cdot|_E$. Let us recall (cf. [ABG], [Be], [C1], [Le], [LZ]) that a function $f \in \mathrm{L}_{\mathrm{loc}}^{p}(\mathbb{R}, E)$, $p \geq 1$, is said to be *almost-periodic* (a.p.) *in the sense of Stepanov* ($\mathrm{S}_{\mathrm{ap}}^{p}$) if, for every $\varepsilon > 0$, there corresponds a relatively dense set $\{\tau\}_\varepsilon$ of ε-Stepanov almost-periods such that

$$\sup_{x\in\mathbb{R}} \left[\int_x^{x+1} \left|f(t+\tau) - f(t)\right|_E^p \, dt\right]^{\frac{1}{p}} < \varepsilon, \quad \text{for all } \tau \in \{\tau\}_\varepsilon,$$

where the integral is understood in a Bochner sense (see e.g. [PK]).

The Banach space of Stepanov almost-periodic (shortly $\mathrm{S}_{\mathrm{ap}}^{p}$-) functions $f \in \mathrm{L}_{\mathrm{loc}}^{p}(\mathbb{R}, E)$, $p \geq 1$, endowed with the norm

$$\|f\|_{\mathrm{S}^p} := \sup_{x\in\mathbb{R}} \left[\int_x^{x+1} \left|f(t)\right|_E^p \, dt\right]^{\frac{1}{p}},$$

will be denoted by $\mathrm{S}_{\mathrm{ap}}^{p}(\mathbb{R}, E)$. For $p = 1$, we shall simply write $\mathrm{S}_{\mathrm{ap}}(\mathbb{R}, E)$ and speak about S_{ap}*-functions.*

Since *uniformly (Bohr) almost-periodic functions* can be characterized, according to the Bochner theorem (see e.g. [Be], [C1], [LZ]), as entirely uniformly continuous $\mathrm{S}_{\mathrm{ap}}^{p}$-functions, $p \geq 1$, we can use here this characterization as their definition. By *purely* $\mathrm{S}_{\mathrm{ap}}^{p}$*-functions*, we shall therefore mean those which are not uniformly continuous.

An important role will be played by the *Bohl–Bohr–Amerio theorem* (see e.g. [LZ, p. 80]), which we state here in the form of the following lemma.

Lemma 1. *In uniformly convex spaces, the boundedness of an indefinite integral of a Stepanov almost-periodic function implies its Bohr almost-periodicity.*

It is well-known that a linear combination of $\mathrm{S}_{\mathrm{ap}}^{p}$-functions is, for any $p \geq 1$, an $\mathrm{S}_{\mathrm{ap}}^{p}$-function. On the other hand, even for $p = 1$, the product of two $\mathrm{S}_{\mathrm{ap}}^{p}$-functions need not be $\mathrm{S}_{\mathrm{ap}}^{p}$. Therefore, the properties of products will be firstly derived in the next section. In order the product to be an $\mathrm{S}_{\mathrm{ap}}^{p}$-function, we need to recall the pair class ($p = 1$) of $\mathrm{S}_{\mathrm{ap}}^{\infty}$-functions before.

For more properties and details, we recommend the monographs ([ABG], [Be], [C1], [Le], [LZ]).

Hence, let us denote by χ_A the indicator of $A \subset E$:

$$\chi_A(x) = \begin{cases} 1, & \text{for } x \in A, \\ 0, & \text{for } x \notin A, \end{cases}$$

and let us consider the space $\mathrm{L}_{\mathrm{loc}}^{\infty}(\mathbb{R}, E)$ of functions f s.t.:

$$\sup_{x\in\mathbb{R}} \|f\chi_{[x,x+1]}\|_\infty < +\infty$$

(this means to take $p = \infty$ in standard Stepanov spaces). In fact, this space becomes $\mathrm{L}^{\infty}(\mathbb{R}, E)$, because

$$\sup_{x\in\mathbb{R}} \|f\chi_{[x,x+1]}\|_\infty = \|f\|_\infty.$$

Indeed, the inclusion related to the inequality $\leq$ is obvious, For the reverse inequality, for any $\varepsilon > 0$, we can find a set A with a positive measure s.t. $|f|_E \geq \|f\|_\infty - \varepsilon$, on A. Thus, there exists $x \in \mathbb{R}$ s.t. $A \cap [x, x+1]$ has a positive measure and $|f|_E \geq \|f\|_\infty - \varepsilon$, on $A \cap [x, x+1]$, by which $\|f\chi_{[x,x+1]}\|_\infty \geq \|f\|_\infty - \varepsilon$.

Let us still denote by $\mathrm{S}^\infty_{\mathrm{ap}}(\mathbb{R}, E)$ the space of functions $f \in \mathrm{L}^\infty(\mathbb{R}, E)$ s.t., for any $\varepsilon > 0$, there exists a relatively dense (shortly, r.d.) set $\{\tau_\varepsilon\}_\varepsilon$ s.t., for any $\tau \in \{\tau_\varepsilon\}_\varepsilon$:

$$\|f(.+\tau) - f(.)\|_\infty \leq \varepsilon.$$

This space contains, in particular, the space of Bohr a.p. functions, but also some more general functions. For instance, any measurable essentially bounded periodic function belongs to $\mathrm{S}^\infty_{\mathrm{ap}}(\mathbb{R}, E)$.

Remark 1. We can easily check that, for any measurable function f, we have:

$$\lim_{p\to\infty} \|f\|_{\mathrm{S}^p_{\mathrm{ap}}} = \|f\|_\infty.$$

Let us prove this, provided $\|f\|_\infty$ is finite. The proof of the infinite case is almost the same. We know that $\|f\|_{\mathrm{S}^p_{\mathrm{ap}}} \leq \|f\|_\infty$. For the reverse inequality, take $\varepsilon > 0$. We know that there exists $x \in \mathbb{R}$ s.t.: $\|f\chi_{[x,x+1]}\|_\infty \geq \|f\|_\infty - \varepsilon$. We also know that, for a sufficiently large p, $\|f\|_{\mathrm{L}^p([x,x+1],E)} \geq \|f\|_{\mathrm{L}^\infty([x,x+1],E)} - \varepsilon$. Thus, for a sufficiently large p, we obtain:

$$\|f\|_{\mathrm{S}^p_{\mathrm{ap}}} \geq \|f\|_{\mathrm{L}^p([x,x+1],E)} \geq \|f\|_\infty - 2\varepsilon.$$

3. Nemytskii's operators in Stepanov spaces

Let us define the Nemytskii operator $\mathcal{N}_f$ built on $f\colon \mathbb{R}\times E \to E$:

$$\mathcal{N}_f(u) := [t \mapsto f(t, u(t))].$$

We begin with the particular case of products, but the product "$\cdot$" can have different meanings. The following lemma is an extension of the property which can be found in [Le, pp. 204–205].

Lemma 2. *Assume that $p^{-1} + q^{-1} = r^{-1}$ with $p, q, r \in [1, \infty]$. Then:*

(i) *If $A \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R}, \mathcal{L}(E))$ and $\beta \in \mathrm{S}^q_{\mathrm{ap}}(\mathbb{R}, E)$, then $A \cdot \beta \in \mathrm{S}^r_{\mathrm{ap}}(\mathbb{R}, E)$, where $\mathcal{L}(E)$ stands for the Banach space of all linear bounded transformations $L\colon E \to E$ endowed with the sup-norm.*

(ii) *If $\alpha \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R}, \mathbb{R})$ and $\beta \in \mathrm{S}^q_{\mathrm{ap}}(\mathbb{R}, E)$, then $\alpha \cdot \beta \in \mathrm{S}^r_{\mathrm{ap}}(\mathbb{R}, E)$.*

(iii) *If $A \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R}, \mathcal{L}(E)) \cap \mathrm{L}^\infty(\mathbb{R}, \mathcal{L}(E))$ and $\beta \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R}, E) \cap \mathrm{L}^\infty(\mathbb{R}, E)$, then $A \cdot \beta \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R}, E)$.*

(iv) *If $\alpha \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R}, \mathbb{R}) \cap \mathrm{L}^\infty(\mathbb{R}, \mathbb{R})$ and $\beta \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R}, E) \cap \mathrm{L}^\infty(\mathbb{R}, E)$, then $\alpha \cdot \beta \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R}, E)$.*

Proof. Let us show the first case. It is well-known (see e.g. [Be], [C1], [Le]) that we can find a set of ε-almost periods common to A and β. Let τ be such an ε-almost period. We can easily check that:

$$\begin{aligned} &|A(t+\tau)\beta(t+\tau) - A(t)\beta(t)|_E \\ &\leq \|A(t+\tau)\|_{\mathcal{L}(E)}|\beta(t+\tau) - \beta(t)|_E + \|A(t+\tau) - A(t)\|_{\mathcal{L}(E)}|\beta(t)|_E. \end{aligned}$$

Let us recall that, for any positive a, b:

$$(a+b)^r \leq 2^{r-1}(a^r + b^r).$$

We obtain:

$$\begin{aligned} &|A(t+\tau)\beta(t+\tau) - A(t)\beta(t)|_E^r \\ &\leq 2^{r-1}\left(\|A(t+\tau)\|^r_{\mathcal{L}(E)}|\beta(t+\tau) - \beta(t)|^r_E + \|A(t+\tau) - A(t)\|^r_{\mathcal{L}(E)}|\beta(t)|^r_E\right). \end{aligned}$$

Applying the well-known Hölder inequality with p/r and q/r as exponents, we obtain, for any $s \in \mathbb{R}$:

$$\int_s^{s+1} |A(t+\tau)\beta(t+\tau) - A(t)\beta(t)|_E^r dt$$

$$\leq 2^{r-1} \left(\left(\int_s^{s+1} \|A(t+\tau)\|_{\mathcal{L}(E)}^p dt \right)^{r/p} \left(\int_s^{s+1} |\beta(t+\tau) - \beta(t)|_E^q dt \right)^{r/q} \right.$$

$$\left. + \left(\int_s^{s+1} \|A(t+\tau) - A(t)\|_{\mathcal{L}(E)}^p dt \right)^{r/p} \left(\int_s^{s+1} |\beta(t)|_E^q dt \right)^{r/q} \right),$$

from which:

$$\|(A \cdot \beta)(. + \tau) - (A \cdot \beta)(.)\|_{\mathrm{S}^r_{\mathrm{ap}}}$$

$$\leq 2^{r-1} \left(\|A\|_{\mathrm{S}^p_{\mathrm{ap}}} \|\beta(. + \tau) - \beta(.)\|_{\mathrm{S}^q_{\mathrm{ap}}} + \|A(. + \tau) - A(.)\|_{\mathrm{S}^p_{\mathrm{ap}}} \|\beta(.)\|_{\mathrm{S}^q_{\mathrm{ap}}} \right)$$

$$\leq 2^{r-1} \left(\|A\|_{\mathrm{S}^p_{\mathrm{ap}}} + \|\beta(.)\|_{\mathrm{S}^q_{\mathrm{ap}}} \right) \varepsilon.$$

The second property can be proved by the same arguments, or can be seen as a particular case, when taking $A := [x \mapsto \alpha(t)x]$ as a linear mapping.

The two remaining cases are even a bit simpler to prove. For instance, for the last one, this begins with:

$$|\alpha(t+\tau)\beta(t+\tau) - \alpha(t)\beta(t)|_E^p$$

$$\leq 2^{p-1} \left(\|\beta\|_\infty^p |\alpha(t+\tau) - \alpha(t)|_E^p + \|\alpha\|_\infty^p |\beta(t+\tau) - \beta(t)|_E^p \right),$$

etc. □

Now, consider the situation of the autonomous Nemytskii operator $\mathcal{N}_f$ built on $f(t,x) \equiv f(x)$. Let us denote by $\mathcal{S}_D(\mathbb{R}, E)$ the space which was originally denoted by $\mathcal{S}(\mathbb{R}, E)$ in [D1]; this means that $\mathcal{S}_D(\mathbb{R}, E)$ is the set of measurable functions $f \colon \mathbb{R} \to E$ s.t., for any $\varepsilon, \delta > 0$, the set of numbers τ satisfying:

$$\sup_{\xi \in \mathbb{R}} \mu(\{t \in [\xi, \xi+1], |f(t+\tau) - f(t)|_E \geq \varepsilon\}) < \delta$$

is relatively dense, where μ stands for the Lebesgue measure. Lemma 3 and its corollary in [D1] give some conditions ensuring that the Nemytskii operator maps $\mathcal{S}_D(\mathbb{R}, E)$ into itself. For instance, when f is continuous, then the Nemytskii operator on f has its property. This notion of Stepanov almost-periodicity must be compared with the usual one, used in our paper.

For this, using Danilov's notations in [D1], consider $M'_p(\mathbb{R}, E)$ as the set of measurable functions $f \colon \mathbb{R} \to E$ s.t. $\|f\|_{\mathrm{S}^p} < \infty$ and:

$$\lim_{\delta \to 0^+} \sup_{\xi \in \mathbb{R}} \sup_{T \subset [\xi, \xi+1], \mu(T) \leq \delta} \int_T |f(t)|_E^p \, dt = 0.$$

The space S_p used in [D1] is our $\mathrm{S}^p_{\mathrm{ap}}$ and equation (1) on p. 1420 in [D1] says that:

$$\mathrm{S}^p_{\mathrm{ap}}(\mathbb{R}, E) = \mathcal{S}_D(\mathbb{R}, E) \cap M'_p(\mathbb{R}, E).$$

Let us now give the following consequence of Danilov's corollary:

Lemma 3. *If $f \in \mathrm{C}^0(E,E)$, $a,b>0$ and $p,q \geq 1$ satisfy:*

$$\forall x \in E,\ |f(x)|_E \leq a|x|_E^{p/q} + b,$$

then, for any $g \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R},E)$, we have: $f \circ g \in \mathrm{S}^q_{\mathrm{ap}}(\mathbb{R},E)$.

Proof. Taking $z \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R},E)$, we know that $z \in \mathcal{S}_D(\mathbb{R},E)$. Thus, $f \circ z \in \mathcal{S}_D(\mathbb{R},E)$, by the corollary in [D1].

Now, let us show that $f \circ z \in M'_q(\mathbb{R},E)$. For any bounded interval T, we have:

$$\int_T |f(z(t))|_E^q\,dt \leq 2^{q-1}\int_T (a^q|z(t)|_E^p + b^q)\,dt \leq 2^{q-1}\left(a^q\int_T |z(t)|_E^p\,dt + b^q m\right),$$

where m is the Lebesgue measure of T. Since $z \in L^p_{loc}(\mathbb{R},E)$, we deduce from this inequality that $f\circ z \in L^q_{loc}(\mathbb{R},E)$. Now given any $\delta>0$, any $\xi \in \mathbb{R}$ and $T \subset [\xi,\xi+1]$ whose measure is less than δ, we get:

$$\int_T |f(z(t))|_E^q\,dt \leq 2^{q-1}\left(a^q\int_T |z(t)|_E^p\,dt + b^q \delta\right).$$

Since $z \in M'_p(\mathbb{R},E)$, we obtain from the last inequality that $f\circ z \in M'_q(\mathbb{R},E)$. So, $f \circ z \in M'_q(\mathbb{R},E) \cap \mathcal{S}_D(\mathbb{R},E) = \mathrm{S}^q_{\mathrm{ap}}(\mathbb{R},E)$. □

Remark 2. It is not true that if f is continuous and $g \in \mathrm{S}_{\mathrm{ap}}$, then $f\circ g \in \mathrm{S}_{\mathrm{ap}}$. Indeed, take for instance $g = \sum_{n=2}^{+\infty} g_n$, with g_n $4n$-periodic, and:

$$g_n|_{[-2n,2n]} := \beta_n\left(1 - \frac{2}{\alpha_n}|x-n|\right)\chi_{[n-\alpha_n/2,n+\alpha_n/2]},$$

where $\alpha_n \in (0,1/2)$, $\beta_n>0$ will be specified later. Since g_n is periodic, continuous and bounded, then $g_n \in \mathrm{S}_{\mathrm{ap}}$. Moreover, $\|g_n\|_{\mathrm{S}_{\mathrm{ap}}} = 2\alpha_n\beta_n$. So, if we assume that $\sum_n \alpha_n\beta_n < \infty$, then $g \in \mathrm{S}_{\mathrm{ap}}$. Now, as a continuous function, let us choose $f(.) := \exp(.)$. Then:

$$\exp(g) = \exp\left(\sum_n g_n\right) \geq \exp(g_N),$$

for all N. Moreover:

$$\exp(g_N) = \frac{\alpha_N}{\beta_N}\left(e^{\beta_N}-1\right) \geq \frac{\alpha_N\beta_N^2}{6}.$$

So, if we have $\alpha_n\beta_n^2 \to \infty$ when $n\to\infty$, then $\|\exp(g)\|_{\mathrm{S}_{\mathrm{ap}}} \geq \frac{\alpha_n\beta_n^2}{6} \to \infty$, which verifies that $\exp(g) \notin \mathrm{S}_{\mathrm{ap}}$, when taking, for instance, $\alpha_n = 1/n^5$ and $\beta_n = n^3$.

Furthermore, let us consider the nonautonomous Nemytskii operator $\mathcal{N}_f$.

Proposition 1. *Assume that:*

(i) *for all $t\in\mathbb{R}$, $f(t,.) \in L^\infty(E,E)$;*
(ii) *$[t \mapsto f(t,.)] \in \mathrm{S}^q_{\mathrm{ap}}(\mathbb{R}, L^\infty(E,E))$.*

Then $\mathcal{N}_f := f(t,\cdot)$ maps $\mathrm{S}^p_{\mathrm{ap}}(\mathbb{R},E)$ into $\mathrm{S}^q_{\mathrm{ap}}(\mathbb{R},E)$, where $p,q\geq 1$.

Proof. From Danilov's Lemma 3 in [D1], we know that if $z \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R},E)$, then $\mathcal{N}_f(z) \in \mathcal{S}_D(\mathbb{R},E)$. Now, taking

$$b(t) := \sup_{x\in E}|f(t,x)|_E,$$

we obtain

$$|b(t+\tau)-b(t)|_E \le \sup_{x\in E} |f(t+\tau,x)-f(t,x)|_E \le \|f(t+\tau,.)-f(t,.)\|_\infty.$$

Thus, if τ is an ε-almost-period for $F=[t\mapsto f(t,.)]$, we have:

$$\|b(.+\tau)-b(.)\|_{\mathrm{S}^q_{\mathrm{ap}}} \le \|F(.+\tau)-F(.)\|_{\mathrm{S}^q_{\mathrm{ap}}} \le \varepsilon,$$

which verifies that $b\in \mathrm{S}^q_{\mathrm{ap}}(\mathbb{R},E)$. This implies that $\mathcal{N}_f(z)\in M'_q(\mathbb{R},E)$, because:

$$\int_T |f(t,z(t))|^q_E\,dt \le \int_T |b(t)|^q_E\,dt.$$

□

It is possible to combine these results to obtain more general functions mapping Stepanov spaces into themselves. For instance, let us combine Lemma 2(i) and Lemma 3 to obtain immediately the following proposition.

Proposition 2. *Set $F(t,X):=A(t)\cdot F_1(X)$, $p,q\ge 1$, $r\in(0,p/q]$, where:*

(i) $A\in \mathrm{S}^{\frac{pq}{p-qr}}(\mathbb{R},\mathcal{L}(E))$;

(ii) $F_1\in C^0(E,E)$ *and satisfies:*

$$\forall X\in E,\quad |F_1(X)|_E\le a|X|^r_E+b,$$

with $a,b>0$.

Then $\mathcal{N}_F\colon\ \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R},E)\to \mathrm{S}^q_{\mathrm{ap}}(\mathbb{R},E)$.

Proof. From Lemma 3, it follows that $\mathcal{N}_{F_1}\colon\ \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R},E)\to \mathrm{S}^{p/r}(\mathbb{R},E)$. Thus, taking $z\in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R},E)$, we obtain $F_1\circ z\in \mathrm{S}^{p/r}(\mathbb{R},E)$. Applying still Lemma 2(i), we arrive at $\mathcal{N}_F\colon\ \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R},E)\to \mathrm{S}^q_{\mathrm{ap}}(\mathbb{R},E)$. □

4. Main proposition

Let us recall that, according to the Milman–Pettis theorem (see e.g. [Br]), uniformly convex Banach spaces are reflexive. Moreover, reflexivity implies the Radon–Nikodym property (see e.g. [PK]). Thus, the following proposition holds, for instance, in Hilbert spaces.

Proposition 3. *Let $f\colon\mathbb{R}\to E$ be a locally absolutely continuous function, i.e. $f\in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R},E)$, where E is a Banach space satisfying the Radon–Nikodym property. If f and f' are bounded, in the Stepanov norm, then f is bounded in the essential sup-norm. In particular, if $f\in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R},E)$, where E is a uniformly convex Banach space, is s.t. f and f' are Stepanov almost-periodic, i.e. $f,f'\in \mathrm{S}^1_{\mathrm{ap}}(\mathbb{R},E)$, then f is Bohr (uniformly) almost-periodic.*

Proof. For the first part, since E has the *Radon–Nikodym property* (see e.g. [PK, p. 694]) and since $f\in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R},E)$, the Fréchet derivative f' exists a.e. and the fundamental calculus formula holds, where the integral should be unterstood in Bochner's sense (see e.g. [PK, p. 695]):

$$f(t)-f(s)=\int_s^t f'(u)\,du.$$

Firstly, by denoting by $[t]$ the integer part of t, one can easily check that:

$$|f(t)-f([t])|_E\le \|f'\|_{\mathrm{S}^1_{\mathrm{ap}}}. \tag{1}$$

Indeed:

$$|f(t)-f([t])|_E = \left|\int_{[t]}^{t} f'(s)\,ds\right|_E \le \int_{[t]}^{t} |f'(s)|_E\,ds \le \int_{[t]}^{[t]+1} |f'(s)|_E\,ds \le \|f'\|_{\mathrm{S}^1_{\mathrm{ap}}}.$$

This, in particular, yields that:

$$\sup_{t\in\mathbb{R}} |f(t)|_E \le \|f'\|_{\mathrm{S}^1_{\mathrm{ap}}} + \sup_{n\in\mathbb{Z}} |f(n)|_E.$$

At the same time, for any $n \in \mathbb{Z}$, we have:

$$f(n) = \int_n^{n+1} f(n)\,dt = \int_n^{n+1} (f(n)-f(t))\,dt + \int_n^{n+1} f(t)\,dt.$$

Using again (1), this implies that, for any $n \in \mathbb{Z}$:

$$|f(n)|_E \le \int_n^{n+1} |f(n)-f(t)|_E\,dt + \int_n^{n+1} |f(t)|_E\,dt \le \|f'\|_{\mathrm{S}^1_{\mathrm{ap}}} + \|f\|_{\mathrm{S}^1_{\mathrm{ap}}}.$$

Thus, we obtain:

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} |f(t)|_E \le 2\|f'\|_{\mathrm{S}^1_{\mathrm{ap}}} + \|f\|_{\mathrm{S}^1_{\mathrm{ap}}}.$$

For the second part of the proposition, let us consider $f \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}, E)$ s.t. f and f' are both Stepanov almost-periodic. We know from the first part that f is essentially bounded. Therefore, as a direct consequence of Lemma 1, we get that f is uniformly almost-periodic, as claimed. □

Remark 3. Since we obviously have (see e.g. [ABG]):

$$\forall p \ge 1, \quad \|f\|_{\mathrm{S}_{\mathrm{ap}}} \le \|f\|_{\mathrm{S}^p_{\mathrm{ap}}},$$

the foregoing Proposition 3 is also valid for $f \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R}, E)$ and $f' \in \mathrm{S}^q_{\mathrm{ap}}(\mathbb{R}, E)$, with any $p, q \ge 1$.

5. Main results

In order to apply Proposition 3 to the nonexistence of purely Stepanov almost-periodic solutions, let us start this section with an intuitively clear proposition.

Proposition 4. *Assume that $\mathcal{N}_F \colon \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R}, E) \to \mathrm{S}^q_{\mathrm{ap}}(\mathbb{R}, E)$, where $p, q \ge 1$. Then every $\mathrm{S}^p_{\mathrm{ap}}$-solution of the differential equation $X' = F(t, X)$, in a uniformly convex Banach space E, is uniformly almost-periodic.*

Proof. In such cases, the derivative of any solution $x(.) \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R}, E)$ will be $\mathrm{S}^q_{\mathrm{ap}}$, and, in view of Remark 4, $x(.)$ as well as $x'(.)$ are S_{ap}. Applying the second part of Proposition 3, the assertion follows. □

On the basis of Proposition 4 and the results from foregoing sections, we can consider many cases where purely Stepanov solutions do not exist. For instance, let us formulate the main result for the differential equation:

$$X' = A(t) \cdot F_1(X) + F_2(t, X). \tag{2}$$

Theorem 1. *Let E be a uniformly convex Banach space. Assume that there exist constants p, q, r with $p, q \ge 1$, $r^{-1} = p^{-1} + q^{-1}$ and $r \in (0, \frac{p}{q}]$, such that the following conditions are satisfied:*

(i) *$A \in \mathrm{S}_{\mathrm{ap}}^{\frac{pq}{p-qr}}(\mathbb{R}, \mathcal{L}(E))$;*

(ii) $\forall X \in E,\ \ |F_1(X)|_E \le C_1 |X|_E^r + C_2$,
with $C_1, C_2 \ge 0$, *holds for* $F_1 \in \mathrm{C}^0(E,E)$;
(iii) *for all* $t \in \mathbb{R}$, $\ F_2(t,.) \in \mathrm{L}^\infty(E,E)$;
(iv) $[t \mapsto F_2(t,.)] \in \mathrm{S}^q_{\mathrm{ap}}(\mathbb{R}, \mathrm{L}^\infty(E,E))$.

Then the equation (2) has no purely $\mathrm{S}^p_{\mathrm{ap}}$*-solution.*

Proof. Setting $F(t,X) := A(t)\cdot F_1(X) + F_2(t,X)$, let us take $z \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R},E)$. According to Proposition 1, the Nemytskii operator $\mathcal{N}_{F_2}$ maps $\mathrm{S}^p_{\mathrm{ap}}(\mathbb{R},E)$ into $\mathrm{S}^q_{\mathrm{ap}}(\mathbb{R},E)$, i.e. $[t \mapsto F_2(t,z(t))] \in \mathrm{S}^q_{\mathrm{ap}}(\mathbb{R},E)$. Applying Proposition 2, we get $[t \mapsto A(t)F_1(z(t))] \in \mathrm{S}^q_{\mathrm{ap}}(\mathbb{R},E)$. Thus, the Nemytskii operator $\mathcal{N}_F$ built on F maps $\mathrm{S}^p_{\mathrm{ap}}(\mathbb{R},E)$ into $\mathrm{S}^q_{\mathrm{ap}}(\mathbb{R},E)$ which, in view of Proposition 4, completes the proof. $\square$

Remark 4. One can readily check that, for $F_2(t,X) := P(t)$, $P \in \mathrm{S}^q_{\mathrm{ap}}(\mathbb{R},E)$ need not be essentially bounded for the same conclusion. Putting still $F_1(X) := X$ and taking, this time, $A \in \mathrm{S}^\infty_{\mathrm{ap}}(\mathbb{R},\mathcal{L}(E))$, $P \in \mathrm{S}^1_{\mathrm{ap}}(\mathbb{R},E)$, we have by means of Lemma 2(i) that $[t \mapsto A(t)z(t) + P(t)] \in \mathrm{S}^1_{\mathrm{ap}}(\mathbb{R},E)$, for $z \in \mathrm{S}^1_{\mathrm{ap}}(\mathbb{R},E)$. Therefore, by virtue of Proposition 4, the linear differential equation $X' = A(t)X + P(t)$, where $A \in \mathrm{S}^\infty_{\mathrm{ap}}$ and $P \in \mathrm{S}^1_{\mathrm{ap}}$, has no purely S_{ap}-solution. This slightly generalizes the result of M. Tarallo [Ta].

For particular forms of the right-hand sides in (2), Theorem 1 can be still modified as follows.

Theorem 2. *Let E be a uniformly convex Banach space. Assume that there exist constants $\alpha > 1$, $r > 0$, such that:*

(i) $A \in \mathrm{S}^\alpha_{\mathrm{ap}}(\mathbb{R},\mathcal{L}(E))$;
(ii) $F_1 \in \mathrm{C}^0(E,E)$ *and satisfies:*

$$\forall X \in E,\ \ |F_1(X)|_E \le C_1 |X|^r + C_2,$$

with $C_1, C_2 \ge 0$.

Then, for any $p > \frac{\alpha r}{\alpha - 1}$, the equation (2), where $F_2(t,X) \equiv 0$, has no purely $\mathrm{S}^p_{\mathrm{ap}}$-solution. In particular, for $r < 1 - \alpha^{-1}$, there are no purely S_{ap}-solutions.

Proof. For $p > \frac{\alpha r}{\alpha-1}$, we have $\frac{1}{\alpha} + \frac{r}{p} \le 1$, and there certainly exists $q \ge 1$ such that $q^{-1} = \alpha^{-1} + (p/r)^{-1}$.

Taking $z \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R},E)$, the Nemytskii operator $\mathcal{N}_{F_1}$ maps, according to Lemma 3, $\mathrm{S}^p_{\mathrm{ap}}(\mathbb{R},E)$ into $\mathrm{S}^{p/r}_{\mathrm{ap}}(\mathbb{R},E)$. Hence, applying still Lemma 2(i), we obtain that $[t \mapsto A(t)F_1(z(t))] \in \mathrm{S}^q_{\mathrm{ap}}(\mathbb{R},E)$ which, in view of Proposition 4, completes the general part of the proof.

For the particular part of the proof, it is enough to realize that, for $p = 1$, the inequality $1 > \frac{\alpha r}{\alpha-1}$ can be equivalently expressed as $r < 1 - \alpha^{-1}$. $\square$

Remark 5. Under the assumptions (i) and (ii), imposed on A and F_1 in Theorem 2, Proposition 2 immediately implies that, for $z \in \mathrm{S}^p_{\mathrm{ap}}(\mathbb{R},E)$, $[t \mapsto A(t)F_1(z_1(t))] \in \mathrm{S}^{\frac{\alpha p}{p+\alpha r}}_{\mathrm{ap}}(\mathbb{R},E)$, provided the constants α, p, r satisfy suitable conditions, as in Proposition 2. The inequality $p > \frac{\alpha r}{\alpha-1}$ then simply means that $\frac{\alpha p}{p+\alpha r} > 1$ by which $[t \mapsto A(t)F_1(z(t))] \in \mathrm{S}^1_{\mathrm{ap}}(\mathbb{R},E)$. Therefore, assuming still (iii) and (iv) with $q \ge 1$ in Theorem 1, the same conclusion holds also for the equation (2), where $F_2(t,X) \not\equiv 0$.

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