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Notion of Weak variational solutions for almost periodic or more general problems

Denis Pennequin*
Université Paris 1 Panthéon-Sorbonne
Centre MF, Laboratoire SAMM
90, Rue de Tolbiac
75 634 Paris Cedex 13, France

Abstract

The aim of this paper is to introduce of notion of weak variational solution in an abstract setting, although we are mainly interested in almost periodic type solutions. We give two existence and uniqueness theorems. Even if assumptions are strong, we obtain two theorems with explict bounds.

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In this paper, we will introduce a notion of variational solution in an abstract setting. This has been already used in the particular case in a quasiperiodic with fixed modulus of frequencies by M.S. Berger and L. Zhang ([3] and [4]), but here we are interested in a more general setting which allows different cases of almost-periodicity and more general situations too. In the papers by Berger and Zhang were considered a case of convexity with standart techniques of Calculus of Variations (direct methods). Here, as example of our theorems, we can obtain a perturbation of linear situations with strong assumptions but with explicit bounds.

As examples of situations, we quote the cases of periodic, quasi-periodic with prescribed modulus of frequencies and almost periodic case, but some more general situations could be explored, even in some almost-periodic settings. A good classification of a.p. function spaces has been made in J. Andres, A.M. Bersani and R.F. Grande [1]. Probably for instance to study the Stepanov case should be interesting, but this can be seen as a standart a.p. case through the Bohr transform, and we know that in many standart situations any Stepanov a.p. solution is in fact Bohr a.p. (see M. Tarallo [22] for the linear case and J. Andres and D. Pennequin [2] for a more general one).

*E-mail address: pennequi@univ-paris1.fr

1 Introduction.

1.1 The setting.

We will consider general spaces as Sobolev spaces based on a linear operator T on $L^2_G = L^2(G, H)$ is endowed with its classical $||.||_2$ norm, where $(H, \langle ., . \rangle_H)$ is a Hilbert space and G is a set G equipped with a measure μ_G. In all what follows, G will be a locally compact group and μ_G its Haar measure, but this is not necessary.

Now consider the domain of the operator T:

$$H^1_G = \{u \in L^2(G, \mathbb{R}), \quad T u \in L^2(G, \mathbb{R})\}.$$

Analogy with Sobolev's notations is due to the fact that we have the derivative in mind, but this could be a different operator. H^1_G is an Hilbert space with the norm:

$$||u||_{H^1_G} = \sqrt{||u||_2^2 + ||T u||_2^2}.$$

In this case, we obtain that $T : H^1_G \to L^2_G$ is a linear continuous operator. We also assume that

$$\forall (u, v) \in H^1_G \times L^2_G, \qquad \int_G \langle T u, v \rangle_H d\mu_G = -\int_G \langle u, T v \rangle_H d\mu_G.$$

Here, we assume that we have a subspace of H^1_G, called $H^1_{G,0}$, where a Poincaré-Wirtinger inequality holds:

$$\exists \alpha_{PW} > 0, \quad \forall u \in H^1_{G,0}, \quad ||T u||_2 \geq \alpha_{PW} ||u||_2.$$

In this case, $H^1_{G,0}$ is an Hilbert space with the following norm, equivalent to the one of H^1_G:

$$||u||_0 = ||T u||_2.$$

1.2 Examples of considered spaces.

Taking first $G = \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, L^2_G is the set of $L^2_{loc}(\mathbb{R}, H)$ functions which are $(2\pi-)$periodic in each variable. For T we take the standart (distributional) derivative and H^1_G is the same space of standart $H^1_{loc}(\mathbb{R}, H)$ functions which are $2\pi-$periodic. If we introduce the mean of function $f \in L^2_G$:

$$\mathcal{M}\{f\} = \frac{1}{2\pi} \int_0^{2\pi} f(t) dt = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} f(t) dt = \int_{\mathbb{T}} f d\mu_{\mathbb{T}},$$

we have a Poincaré-Wirtinger inequality in:

$$H^1_G = \left\{ f \in H^1_G, \quad \mathcal{M}\{f\} = 0 \right\},$$

since Fourier's series theory give $||f'||_2 \geq ||f||_2$ in $L^2(\mathbb{T})$.

Taking now $G = \mathbb{T}^m$, we obtain for L^2_G the set of $L^2_{loc}(\mathbb{R}^m, H)$ functions which are $2\pi-$periodic in each variable. Here the mean of a function f is:

$$\mathcal{M}\{f\} = \frac{1}{(2\pi)^m} \int_{[0;2\pi]^m} f = \int_{\mathbb{T}^m} f d\mu_{\mathbb{T}^m}.$$

If $\omega = (\omega_1, \ldots, \omega_m)$ is a set of $\mathbb{Z}-$linearly independants numbers, L^2_G is isomorphic and isometrical to the space of quasi-periodic functions whose modulus of frequencies has ω as $\mathbb{Z}-$basis. This has been used by Percival [20], [21] which introduce the derivative:

$$\partial_\omega u(x) = \lim_{s \to 0} \frac{u(x + s\omega) - u(x)}{s}.$$

The set H^1_G that we obtain when $T = \partial_\omega$ has been used by Berger and Zhang [3], [4] and Blot and Pennequin [12], [13]. We will call it $H^1_\omega(\mathbb{T}^m, H)$. As Berger and Zhang proved in [4], we have a Poincaré-Wirtinger inequality in $H^1_{G,0}$, here written $H^1_{\omega,0}(\mathbb{T}^m, H)$, the closure in H^1_G of the set of functions whose restriction to $[0; 2\pi]^m$ has compact support in $(0; 2\pi)^m$. As done in [13], we could also directly work in $H^1(\mathbb{T}^m)$ and $H^1_0(\mathbb{T}^m)$, this could give some different results.

Taking now for G the Bohr compactification of $\mathbb{R}$, $b\mathbb{R}$, $L^2(G)$ is isometric and isometrical to Besicovitch space $B^2(\mathbb{R})$. Here the mean is:

$$\mathcal{M}\{f\} = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau f(t) dt = \int_{b\mathbb{R}} f d\mu_{b\mathbb{R}},$$

and, if we take for T the derivative (infinitesimal generator of the set of translations):

$$T f = \nabla f = \lim_{s \to 0} \frac{f(. + s) - f(.)}{s},$$

then H^1_G is isomorphic and isometrical to the Blot space $B^{1,2}(\mathbb{R})$ (see all papers by J. Blot in the references).

1.3 Notion of weak variational solution.

We would like to solve the equation:

$$-T^2 u(x) = X(x, u(x), T u(x)) \tag{1.1}$$

with growth assumptions on X. We will see that adapting some ideas used for standart elliptic linear equations, we can obtain a result of existence is a nonlinear setting, which could be seen as a quantitative perturbation result for the linear case. We can look for strong or weak solutions. By a *weak* solution, we mean that the left and right hand side are in L^2_G and that the equality is true on L^2_G (so, $T u \in H^1_G$).

The idea is to replace (1.1) by the problem to find $u \in \mathcal{H}$ ($\mathcal{H} = H^1_G$ or $H^1_{G,0}$) s.t.:

$$\forall v \in \mathcal{H}, \quad \int_G (\langle T u, T v \rangle_H + \langle X(., u, T u), v \rangle_H) d\mu_G = 0. \tag{1.2}$$

This problem will be called *the variational form* of the first equation.

It is clear that if u satisfies the first equation, even in a weak sense, it also satisfies the variational form. Indeed, for each v in L^2_G, since $T^2u = T(Tu)$ with $Tu \in L^2_G$, we have:

$$\int_G \langle -T^2u, v\rangle_H d\mu_G = \int_G \langle Tu, Tv\rangle_H d\mu_G.$$

The reverse way can be true or not. For instance, in our examples of H^1_G spaces, the reverse is true, provided $\varphi := X(.,u,Tu)$ is in L^2_G, since in these examples,

$$\forall v \in H^1_G, \quad \int_G \langle Tu, Tv\rangle_H d\mu_G = \int_G \langle \varphi, v\rangle_H d\mu_G,$$

implies the fact that Tu is in H^1_G and that:

$$\forall v \in H^1_G, \quad \int_G \langle Tu, Tv\rangle_H d\mu_G = \int_G \langle -T^2u, v\rangle_H d\mu_G.$$

But in $H^1_{G,0}$ this can not be true. Let us take our simplest example, the one of periodic case. Consider the problem:

$$-\ddot{q} + \theta q = \varphi$$

with $\theta \in \mathbb{R}$. In $H^1(\mathbb{T})$, we receive the existence (and also uniqueness) of a variational solution provided $\theta > 0$. This can be directly seen using standart linear elliptic PDE arguments we will extend there. With the Fourier expansion, if $\varphi \sim \sum_{n\in\mathbb{Z}} \varphi_n e_n$, where $e_n(t) = e^{int}$, we see that there exist a unique solution $q \sim \sum_{n\in\mathbb{Z}} q_n e_n$ in $H^1(\mathbb{T})$, whose Fourier coefficients are $q_n = \frac{\varphi_n}{\theta+n^2}$ (we see that $\sum_n (1+n^2)|q_n|^2 < \infty$ which proves that $q \in H^1(\mathbb{T})$). But in $H^1_{G,0}$, we obtain existence of a solution provided $\theta > -\alpha^2_{PW} = -1$. If the solution is $H^1_0(\mathbb{T})$, its coefficients should also satisfy $(\theta + n^2)q_n = \varphi_n$, which is impossible for instance with $\theta = 0$ for some functions φ as $\varphi : t \mapsto 1$. Thus, $-\ddot{q} = 1$ admits a $H^1_0(\mathbb{T})$ variational solution, but no solution, even in a $L^2(\mathbb{T})$ weak sense. In our example, to come back, we need that the variational form is also true for the constant function $h = 1$, since $H^1(\mathbb{T}) = H^1_0(\mathbb{T}) \oplus \mathrm{span}(1)$. When $\theta = 0$, this means that the mean of φ should be 0. In this case, the Fourier analysis show us that we can find the solution, since here $\varphi_0 = 0$.

2 Existence and uniqueness theorems.

In this section, we will give an existence theorem of a variational solution in H^1_G and in $H^1_{G,0}$. Let us firstly introduce the assumptions on X, common to the two theorems.

In all what follows, we assume that X is a Caratheodory function s.t. $X(.,0) \in L^2_G$ and that the partial derivatives $\partial_2 X$ and $\partial_3 X$ exist and are bounded. We note, for $j \in \{2,3\}$, $M_j = \sup_{(t,u,v)\in G\times H\times H} \|\partial_j X(t,u,v)\|_H$. Moreover, introduce m_2 s.t.:

$$\forall (t,u,v,w) \in \mathbb{R}\times H\times H\times H, \quad \langle \partial_2 X(t,u,v).w, w\rangle_H \geq m_2\|w\|^2_H.$$

Let us introduce for $i = 2,3$:

$$\delta_i = \sup_{(t,u_2,v_2,u_1,v_1)\in G\times H^4} \|\partial_i X(t,u_2,v_2) - \partial_i X(t,u_1,v_1)\|_H \in [0;\infty].$$

2.1 An existence theorem in H^1_G.

Theorem 2.1. *Assume that X is a Caratheodory function s.t.* $X(.,0) \in L^2_G$ *and that the partial derivatives* $\partial_2 X$ *and* $\partial_3 X$ *exist and are bounded. If:*

- $m_2 > \frac{M_3^2}{4}$*;*
- $\delta_2^2 + \delta_3^2 < \frac{1-m_2+\sqrt{(1+m_2)^2+M_3^2}}{2}$.

Then there exists a unique H^1_G*−variational solution to (1.2).*

Proof. **Step 1: introducing an operator.** To find a solution of:

$$\forall v \in H^1_G, \quad \int_G (\langle Tu, Tv\rangle_H + \langle X(.,u,Tu), v\rangle)\, d\mu_G = 0$$

is equivalent to find a zero of the following operator $\Phi : H^1_G \to (H^1_G)'$

$$\Phi(u) = \left[v \mapsto \int_G (\langle Tu, Tv\rangle_H + \langle X(.,u,Tu), v\rangle)\, d\mu_G \right].$$

Step 2 : Φ is continuous and Gâteaux differentiable. We prove here that Φ admits everywhere a Gâteaux derivative, which is:

$$D_G\Phi(u).h = \left[v \mapsto \int_G (\langle Th, Tv\rangle_H + \langle (\partial_2 X(.,u,Tu)h + \partial_3 X(.,u,Tu)Th), v\rangle_H)\, d\mu_G \right].$$

For, we write : $\Phi = \Phi_1 + \Phi_2$ with:

$$\Phi_1(u) = \left[v \mapsto \int_G \langle Tu, Tv\rangle_H d\mu_G \right]$$

and:

$$\Phi_2(u) = \left[v \mapsto \int_G \langle X(.,u,Tu), v\rangle_H d\mu_G \right].$$

Let us concentrate on the second one, the first is easier. We can write: $\Phi_2 = L \circ \mathcal{N}_X \circ S$, with:

$$S : H^1_G \to L^2_G \times L^2_G, \quad S(u) = (u, Tu)$$

is linear continuous so Fréchet-differentiable;

$$\mathcal{N}_X : L^2_G \times L^2_G \to L^2, \quad \mathcal{N}_X(u,v) = X(.,u,v)$$

is continuous and Gâteaux-differentiable (having a look at [18] Th. 2.3. and proof of Theorem 2.7) and:

$$L : L^2_G \times (H^1_G)', \quad L(\varphi) = \left[v \mapsto \int_G \langle \varphi, v\rangle_H d\mu_G \right]$$

is linear continuous so Fréchet-differentiable. By the chain rule, and since S is linear, we receive the result.

Step 3 : invertibility of the Gâteaux derivative. We see that

$$D_G\Phi(u).h = [v \mapsto \beta(h,v)],$$

where $\beta : H^1_G \times H^1_G$ is the continuous bilinear form:

$$\beta(h,v) = \int_G (\langle Th, Tv\rangle_H + \langle(\partial_2 X(.,u,Tu)h + \partial_3 X(.,u,Tu)Th), v\rangle_H)\, d\mu_G.$$

β is clearly continuous, since:

$$|\beta(h,v)| \leq ||Th||_2||Tv||_2 + M_2||h||_2||v||_2 + M_3||Th||_2||v||_2 \leq (1+M_2+M_3)||h||_{H^1_G}||v||_{H^1_G}.$$

Moreover, when $m_2 > \frac{M_3^2}{4}$, β is elliptic. Indeed, we would like to find $\varepsilon > 0$ s.t. for all $h \in H^1_G$:

$$\beta(h,h) \geq \varepsilon||h||^2_{H^1_G}.$$

But:

$$\beta(h,h) \geq ||Th||_2^2 + m_2||h||_2^2 - M_3||h||_2||Th||_2.$$

So, dividing par $||h||_2^2$, it is sufficient that for all $U > 0$, we have:

$$(1-\varepsilon)U^2 - M_3U + (m_2 - \varepsilon) \geq 0.$$

It is sufficient to have this that the discriminant is negative. But its value is: $M_3^2 - 4(1-\varepsilon)(m_2-\varepsilon)$. Since the value is negative when $\varepsilon = 0$, we can find positive ε s.t. the value is again negative. We need to choose $\varepsilon \in (0, \varepsilon_0)$, with:

$$\varepsilon_0 = \frac{1 - m_2 + \sqrt{(1+m_2)^2 + M_3^2}}{2}.$$

By choosing $\varepsilon = \varepsilon_0$, the large inequality remains true.

From this and from Lax Milgram's theorem, we obtain invertibility of $D_G\Phi(u)$ and that, when $h_L = (D_G\Phi(u))^{-1}(L)$:

$$||h_L||^2_{H^1_G} \leq \frac{\beta(h_L,h_L)}{\varepsilon_0} = \frac{L(h_L)}{\varepsilon_0} \leq \frac{||L||_{(H^1_G)'}||h_L||_{H^1_G}}{\varepsilon_0},$$

so:

$$\left\|(D_G\Phi(u))^{-1}\right\|_{\mathcal{L}((H^1_G)',H^1_G)} \leq \frac{1}{\varepsilon_0}.$$

Step 4 : using Newton's Method. Now, let us apply Newton's theorem, following Ciarlet's Theorem 7.5-1 in [15] with $A_k = D_G\Phi$. Having a look at the proof, we see that the result is longer true with a continuous and Gâteaux differentiable function. Let us recall it here to fix the notations.

Proposition 2.2. *Consider a continuous and Gâteaux-differentiable function* $f : \Omega \subset X \to Y$*, where* X *and* Y *are linear normed spaces, and* $r > 0$ *s.t.* $\overline{B}(x_0, r) \subset \Omega$*. If we can find* $M > 0$ *and* $\beta \in (0,1)$ *s.t.:*

- $\sup_{x \in \overline{B}(x_0,r)} \|D_G f(x)^{-1}\|_{\mathcal{L}(Y,X)} \leq M$*;*
- $\sup_{(x,x') \in \overline{B}(x_0,r)^2} \|D_G f(x) - D_G f(x')\|_{\mathcal{L}(X,Y)} \leq \beta/M$*;*
- $\|f(x_0)\|_Y \leq r(1-\beta)/M$

Then $f(x) = 0$ *as a unique solution in* $\overline{B}(x_0, r)$*.*

We wish to apply this with $f = \Phi$. We have to take $M = \varepsilon_0^{-1}$ for the first condition, avaliable for any $r > 0$. To find a β for the second condition, it is necessary that:

$$\sup \|D_G\Phi(u) - D_G\Phi(u')\|_{\mathcal{L}(H^1_G,(H^1_G)')} < \varepsilon_0$$

and when this is true, by noting σ the sup, we could choose $\beta = \sigma/\beta_1$. But:

$$\|D_G\Phi(u) - D_G\Phi(u')\|_{\mathcal{L}(H^1_G,(H^1_G)')} \leq$$

$$\sup_{\|v\|_{H^1_G} = \|w\|_{H^1_G} = 1} \int_G \left|\langle(\partial_2 X(.,S\,u) - \partial_2 X(.,S\,u')).v + (\partial_3 X(.,S\,u) - \partial_3 X(.,S\,u')).Tv, w\rangle_H\right| d\mu_G.$$

Moreover:

$$\int_G \left|\langle(\partial_2 X(.,S\,u) - \partial_2 X(.,S\,u')).v + (\partial_3 X(.,S\,u) - \partial_3 X(.,S\,u')).Tv, w\rangle_H\right| d\mu_G$$

$$\leq \delta_2\|v\|_2\|w\|_2 + \delta_3\|Tv\|_2\|w\|_2 \leq \sqrt{\delta_2^2 + \delta_3^2}\|v\|_{H^1_G}\|w\|_{H^1_G},$$

so:

$$\sup \|D_G\Phi(u) - D_G\Phi(u')\|_{\mathcal{L}(H^1_G,(H^1_G)')} \leq \sqrt{\delta_2^2 + \delta_3^2}.$$

Thus, when $\delta_2^2 + \delta_3^2 < \varepsilon_0^2$, we can also obtain the second property for any $r > 0$. We choose $\beta = \frac{\sqrt{\delta_2^2+\delta_3^2}}{\varepsilon_0}$ such that the first and second property are true. Now let us take $x_0 = 0$ for instance. By Newton's theorem, there exists a unique solution in the ball $\overline{B}(0, r)$. Since this is true for any r, there exists a unique solution. □

2.2 A version when a Poincaré-Wirtinger's inequality holds.

All what has been done before is again true, but we can relax the condition for ellipticity. But now, we just want to find a positive ε s.t. for all $U > \alpha_{PW}$:

$$(1-\varepsilon)U^2 - M_3 U + m_2 \geq 0.$$

Let us call $f_0 : U \mapsto U^2 - M_3 U + m_2$. Before we have considered the case where f_0 has no real root. Now, if $f_0(\alpha_{PW}) > 0$ and $f_0'(\alpha_{PW}) > 0$, the result is longer true. This mean that we need:

$$\begin{cases} \alpha_{PW}^2 - M_3\alpha_{PW} + m_2 > 0 \\ 2\alpha_{PW} - M_3 > 0 \end{cases}.$$

This means that if $M_3 < 2\alpha_{PW}$ and $(\alpha_{PW} - M_3/2)^2 + (m_2 - (M_3/2)^2) > 0$, then we have ellipticity. The new ε_0 is the greatest s.t. we have simultaneously:

$$\begin{cases} \alpha_{PW}^2 - M_3\alpha_{PW} + m_2 \geq \varepsilon_0\alpha_{PW}^2 \\ 2\alpha_{PW} - M_3 > 2\alpha_{PW}\varepsilon_0 \end{cases}$$

i.e.

$$\varepsilon_0 = \min\left\{\frac{\alpha_{PW}^2 - M_3\alpha_{PW} + m_2}{\alpha_{PW}^2}, \frac{2\alpha_{PW} - M_3}{2\alpha_{PW}}\right\}.$$

And in these conditions, when $\delta_1^2 + \delta_2^2 < \varepsilon_0^2$ we obtain the result. So finaly we have proved:

Theorem 2.3. *Assume that X is a Caratheodory function s.t. $X(.,0) \in L_G^2$ and that the partial derivatives $\partial_2 X$ and $\partial_3 X$ exist and are bounded. Let α_{PW} be a Poincaré-Wirtinger inequality in $H_{G,0}^1$. If:*

- $\min\{\alpha_{PW}^2 - M_3\alpha_{PW} + m_2, 2\alpha_{PW} - M_3\} > 0$*;*
- $\delta_2^2 + \delta_3^2 < \left(\min\left\{\frac{\alpha_{PW}^2 - M_3\alpha_{PW} + m_2}{\alpha_{PW}^2}, \frac{2\alpha_{PW} - M_3}{2\alpha_{PW}}\right\}\right)^2.$

Then there exists a unique $H_{G,0}^1$−variational solution to (1.2).

3 Comments.

3.1 Our theorems as perturbative results.

Our two theorems can be seen as extensions in a nonlinear setting of what occurs in the linear setting. They can be seen as perturbation theorem with known constants. To explain this, let us consider the first theorem in the following particular case, with for instance $H = \mathbb{R}$ (just for simplicity):

$$X(x, u, v) = (a_f(x)u + \epsilon_f(x, u)) + (a_g(x)v + \epsilon_g(x, v)).$$

When $\epsilon_f = \epsilon_g = 0$, we have $\delta_i = 0$, and our assumption is that a_f and a_g are bounded with:

$$\inf a_f > \frac{(\sup|a_g|)^2}{4}.$$

Our theorem shows that if ϵ_f and ϵ_g are sufficently small with sufficently small variations on the partial second derivatives (with explicit bounds of variations), we have existence and uniqueness of the solution. Assumptions are strong but we obtain explicit bounds.

3.2 Possible extensions.

This may be probably interesting to have a look on the particular case of Stepanov. Indeed, as mentionned in the introduction, in standart situations the Stepanov a.p. are in fact Bohr, as we know through Andres and Pennequin [2], but situation could be different for variational solutions, or we could obtain weaker conditions in a Stepanov-weak setting than in Besicovitch setting.

Another thing is that we have strong assumptions since control of $||u||_{H^1_G}$ does not in general implies control of $|u(x)|$ for (almost) all x. This is true for $G = \mathbb{T}$ but not in our other examples. When we have this property, assumptions with the δ_i are weaker. This remark has been used in a discrete setting by the author [19].

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