Nonautonomous Dynamical Systems
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Existence of different kind of solutions for discrete time equations

Abstract

The aim of this paper is to extend the classical linear condition concerning diagonal dominant bloc matrix to fully nonlinear equations. Even if assumptions are strong, we obtain an explicit condition which exactly extend the one known in linear case, and the setting allows also to consider bicontinuous operator instead of the schift and as particular case, we receive periodic or almost periodic solutions for discrete time equations.

Denis PENNEQUIN[1]*

1 Université Paris 1 Panthéon-Sorbonne,
Laboratoire SAMM, Centre PMF, 90 rue de Tolbiac, 75634
PARIS Cedex 13, FRANCE

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The first aim of the paper is to obtain some existence results of solutions $\underline{x} = (x_t)_t$ for difference equations:

$$A(t, x_t, x_{t+1}, \dots, x_{t+n}) = 0,$$

where A is fully nonlinear, which extend the diagonal dominant condition of the linear setting. We look for standart solutions $\underline{x} = (x_t)_t$ defined on $\mathbf{Z}$, but also for periodic and almost periodic solutions. Closed techniques are used in a continuous setting in [12], and give there different results.

We are interested in giving a slighly more general result. We consider a continuous bijective operator Θ on an abstract space whose generic element will be denoted $\underline{x}$ (even if we have mainly in mind spaces of sequences). Θ will be first assumed to be an isometry. We will prove an existence result for an equation:

$$A(., \underline{x}, \Theta(\underline{x}), \dots, \Theta^n(\underline{x})) = 0,$$

which permits to take into account various situations. Assumptions on A are more or less boundedness of partial differentials w.r.t. the last variables and the fact that one variable is growing more than all the other ones together (precise assumptions will be given in the text).

The idea is to adapt basic technics used in linear elliptic partial differential equations. We will write our problem in a kind of *variational form*. Usually, the Hilbert space used (a Sobolev one) is identified to its dual space, and here we won't do this. Since we are in a nonlinear setting, we will use Newton's method to approximate the solution of the equation. In fact we obtain local solvability, but with a "uniformity" on the local aspect, so it is possible to go to global.

We have in mind in this paper:

- to give an extension of a kind of diagonal dominant condition in a nonlinear setting, even with strong assumptions;
- to give it in a more general setting than the one of difference equations. The proof are completely the same;
- to show a new technic to solve this. In order that this could be improved by some readers, I decided to keep the proof as closed as basics ideas. For instance, the considered operator is Fréchet-differentiable, but in fact we don't need this for the proof and from the theorem of composition we just receive Gâteaux-differentiability. Fréchet differentiability comes from assumption **(H4)**, a reader who could relax this could follow our proof, even with an operator which could only have a Gâteaux differential.

Now we describe the context. We consider a measured space $(G, \mathcal{G}, \mu_G)$ and an Hilbert space $(H, \langle ., . \rangle_H)$ (norm $||.||_H$) equipped with its Lebesgue measure (the Borel completed one). Since we have mainly the idea to work with sequences, the spaces $L^p(G, H)$ will be denoted by $\ell^p(G)$ (norm $._{\ell^p}$. We assume that $\ell^2(G, H) \subset \ell^\infty(G, H)$ with continuous injection, i.e. there exists $c > 0$ s.t.:

$$\forall \underline{x} \in \ell^2(G, H), \quad ||\underline{x}||_{\ell^\infty} \leq c||\underline{x}||_{\ell^2},$$

which is equivalent to:

$$\forall G \in \mathcal{G}, \qquad (\mu_G(A) > 0) \Rightarrow (\mu_G(A) \geq 1/c^2).$$

For sets G, we are interested in particular cases as $G = \mathbf{Z}$ (discrete case), $G = \mathbf{Z}/\varpi\mathbf{Z}$ (discrete $\varpi-$periodic case, with $\varpi \in \mathbf{N}$) or $G = b\mathbf{Z}$ (Here, $b\mathbf{Z}$ is the Bohr compactification of $\mathbf{Z}$ and this permits us to consider the discrete a.p.(=almost periodic) case). For such examples, where G is a topological group, we will assume that μ_G is its Haar measure. In all these cases, we have the former continuous injection. This is for instance not the case when $G = \mathbf{R}$, but our techniques are been presented in a continuous setting (but in a very less general context) in [12].

Let us give more details concerning the above exemples, for instance, when $H = \mathbf{R}$:

- if $G = \mathbf{Z}$ we obtain standart $\ell^2(\mathbf{Z})$ space, and:

$$\int_{\mathbf{Z}} \underline{x} d\mu_{\mathbf{Z}} = \sum_{t \in \mathbf{Z}} x_t,$$

- if $G = b\mathbf{Z}$ we obtain the discrete case of Besicovitch a.p. sequences (see [5]), and:

$$\int_{\mathbf{Z}} \underline{x} d\mu_{b\mathbf{Z}} = \mathcal{M}\{\underline{x}\} = \lim_{T \to \infty} \frac{1}{T} \sum_{t=-T}^{T} x_t,$$

- if $G = \mathbf{Z}/\varpi\mathbf{Z}$ we simply obtain the $\varpi-$periodic sequences, and:

$$\int_{\mathbf{Z}/\varpi\mathbf{Z}} \underline{x} d\mu_{\mathbf{Z}/\varpi\mathbf{Z}} = \frac{1}{\varpi} \sum_{t=0}^{\varpi-1} x_t.$$

Some other kind of oscillating solutions could also be introduced. But some are not necessary to do. For instance, we know that in the discrete case, Stepanov a.p. is equivalent to Bohr's one (made in [1] for the sup norm, but follows for Besicovitch's one), and even in the continuous case in standart situations every Stepanov solution is in fact Bohr a.p. (see [2]).

1. Linear autonomous case

Let us recall some basic facts, when $H = \mathbf{R}$ for sake of simplicity. Here, we look for solutions $\underline{x} = (x_t)_t$ for linear equations:

$$a_n x_{t+n} + \ldots + a_0 x_t = y_t,$$

where $(y_t)_t$ is assumed to be in a suitable space of sequences E and $(a_0, \ldots, a_n) \in \mathbf{R}^{n+1}$. Some basic calculations show that under an Hadamard's like condition:

$$\exists j_0 \in \{0, \ldots, n\}, \qquad a_{j_0} > \sum_{j \neq j_0} |a_j|,$$

given $\underline{y} = (y_t)_t$, there exists a unique solution $\underline{x} \in E$, for instance in the following cases $E = \ell^p(\mathbf{Z}, \mathbf{R})$, $E = \ell^p(b\mathbf{Z}, \mathbf{R})$ (almost periodic case), $E = \mathbf{R}^{\mathbf{Z}/(\varpi \mathbf{Z})}$ ($\varpi-$periodic case). This condition (or matrix equivalent one when $H = \mathbf{R}^N$) is usual in discrete systems, and is applied for instance in Economics problems by Blot and Crettez (see [3], [4]).

One way to see that is to introduce the characteristic polynomial, $P = \sum_{k=0}^{n} a_k X^k$ and the shift operator $S : (x_t)_t \mapsto (x_{t+1})_t$. Under Hadamard's condition, the polynomial has not root of modulus one, and so, for each of its root α, $S - \alpha Id$ is invertible, so is $P(S)$.
In the periodic case, the problem can also be seen as a linear algebra problem. First of all, we reduce modulo ϖ the problem, by introducing:

$$J_k = \{j \in \{0, \ldots, n-1\}, j \equiv k\ [\varpi]\} = (k + \varpi \mathbf{Z}) \cap \{\mathbf{0}, \ldots, \mathbf{n-1}\},$$

and: $\hat{P} = \sum_{k=0}^{\varpi - 1} \left(\sum_{j \in J_k} a_j \right) X^k$. It is easy to see that if P satisfies an Hadamard condition, this is also the case for $\hat{P}$. Moreover, if M is the matrix of the shift operator, we have $M^{\varpi} = I_{\varpi}$, from which we deduce the eigenvalues of M and the fact that $P(M) = \hat{P}(M)$ is invertible.

In what follows, we shall extend this in a fully nonlinear and implicit case, but with strong boundedness assumptions. Moreover, we will consider not only the shift, but also continuous bijective operators. In order to make the proof simpler to read, we will first give a theorem for a bijective isometry, but we will after explain some improvements.

2. A first theorem

2.1. Assumptions

We also consider $A : G \times H^{n+1} \to H$, $A : (t, s_0, \ldots, s_n) \mapsto A(t, s_0, \ldots, s_n)$ such that the following assumptions hold:

(H1) $A(., s)$ is measurable from $(G, \mathcal{G})$ to H.

(H2) $(A(t, 0))_t \in \ell^2(G)$;

(H3) all partial Fréchet-differentials $\frac{\partial A}{\partial s_j} : G \times H^{n+1} \to \mathcal{L}(H)$ (written $\partial_{j+2} A$),$j = 0, \ldots, n$, exist everywhere;

(H4) all partial Fréchet-differentials $\partial_{j+2} A$,$j = 0, \ldots, n$, are uniformly continuous, i.e.:

$$\lim_{\delta \to 0} \left[\sup_{|t-t'| + ||s-s'||_{H^{n+1}} \leq \delta} ||\partial_{j+2} A(t, s) - \partial_{j+2} A(t', s')||_{\mathcal{L}(H)} \right] = 0.$$

(H5) all partial Fréchet-differentials $\partial_{j+2} A$,$j = 0, \ldots, n$, are uniformly bounded:

$$\forall j \in \{0, \ldots, n\}, \quad M_j := \sup_{(t,s)} ||\partial_{j+2} A(t, s)||_{\mathcal{L}(H)} < \infty.$$

(H6) there exists j_0 such that:

$$m_{j_0} := \inf_{(t,s,v)\in \mathbf{R}\times \mathbf{H}^{n+1}\times (\mathbf{H}\setminus\{0\})} \frac{\langle \partial_{j_0+2} A(t,s)v, v\rangle_H}{||v||_H^2} > \sum_{j\neq j_0} M_j.$$

Remark 1.

Assumptions **(H1)**, **(H3)** and **(H4)** imply that A is a Caratheodory function.

We will prove the following theorem:

Theorem 1.

Under the assumptions **(H1)-(H6)**, given a bijective isometry $\Theta : \ell^2(G) \to \ell^2(G)$, there exists $\underline{x} \in \ell^2(G)$ s.t.:

$$A\left(., \underline{x}, \Theta(\underline{x}), \dots, \Theta^n(\underline{x})\right) = 0. \tag{1}$$

2.2. Outline of the proof.

Using an idea close to the variational technics we see in linear P.D.E., we replace our problem by the equivalent one, to find a zero of the operator:

$$\phi_{\underline{b}} : \ell^2(G) \to (\ell^2(G))'$$

$$\phi_{\underline{b}}(\underline{x}) = \left[\underline{v} \mapsto \int_G \langle A(., \underline{x}, \dots, \Theta^n(\underline{x}) - \underline{b}, \Theta^{k_0}\underline{v}\rangle_H d\mu_G\right].$$

Using Newton's method, we are able to prove that if for some $\underline{b}'$, $\phi_{\underline{b}'}$ has a zero, this is the case for all $\phi_{b''}$ with $\underline{b}''$ sufficently close to $\underline{b}'$. This criterium of *sufficently* close does not depends on $\underline{b}'$, so $\phi_{\underline{b}}$ has a zero. One clue to may the Newton's method possible is the Lax-Milgram theorem, which is of standart application in the case of linear P.D.E.

We have in mind to separate all ideas of the proof, in case a reader would like to improve our theorem. For instance, with the strong assumptions on the partial differentials, our operator admits a Fréchet-derivative. But from the chain rule, we only first obtain Gâteaux derivative, since as it is well known, the Nemytskii operator (with good growth assumptions) is only Gâteaux-differentiable, the Fréchet is obtained only in trivial cases. By the composition, we only obtain Gâteaux, which is sufficent to apply Newton's method. Here in fact since the operator is Fréchet, a proof could be more straightforward, but we prefer to give the different tools in order to make it improvable.

2.3. A first result concerning Newton's method

Let us first give a version concerning Newton's convergence method we will use after. Even if for our theorem the function will be Fréchet differentiable, we will give a Gâteaux version for an improvement of our theorem. The proposition and the proof are adapted form Ciarlet's, [6], Theorem 7.5-1.

Proposition 1.

Consider a continuous and Gâteaux-differentiable function $f : \Omega \subset X \to Y$, where X and Y are linear normed spaces, and $r > 0$ s.t. $\overline{B}(x_0, r) \subset \Omega$. If we can find $M > 0$ and $\alpha \in (0, 1)$ s.t.:

- $\sup_{x\in\overline{B}(x_0,r)} ||D_G f(x)^{-1}||_{\mathcal{L}(Y,X)} \leq M$;
- $\sup_{(x,x')\in\overline{B}(x_0,r)^2} ||D_G f(x) - D_G f(x')||_{\mathcal{L}(X,Y)} \leq \alpha/M$;
- $||f(x_0)||_Y \leq r(1-\alpha)/M$

Then $f(x) = 0$ as a unique solution in $\overline{B}(x_0, r)$.

Proof. We introduce the sequence defined by Newton's method:

$$x_{k+1} = x_k - D_G f(x_k)^{-1} f(x_k)$$

and prove it has a limit, which is the unique solution in $B = \overline{B(x_0, r)}$ of $f(x) = 0$.
The first step is to prove by induction that for all k:

- $||x_{k+1} - x_k||_X \leq M||f(x_k)||_Y$;
- $||x_{k+1} - x_k||_X \leq \alpha^k ||x_1 - x_0||_X$;
- $x_{k+1} \in B$;
- $||f(x_{k+1})||_Y \leq \frac{\alpha}{M}||x_{k+1} - x_k||_X$.

For $k = 0$, we write for the first one:

$$||x_1 - x_0||_X = ||D_G f(x_0)^{-1} f(x_0)||_X \leq M||f(x_0)||_Y.$$

From this and the fact that $||f(x_0)||_Y \leq r(1-\alpha)/M$, we receive $x_1 \in B$. The last comes from:

$$||f(x_1)||_Y = ||f(x_1) - f(x_0) - D_G f(x_0)(x_1 - x_0)||_Y \leq$$

$$\sup_{\zeta \in [x_0, x_1]} ||D_G f(\zeta) - D_G f(x_0)||_{\mathcal{L}(X,Y)} ||x_1 - x_0||_X \leq \frac{\alpha}{M}.$$

Assuming the properties are true for $k-1$, the first and fourth assumptions are proved by the same way. For the second:

$$||x_{k+1} - x_k||_X \leq \alpha ||x_k - x_{k-1}||_X \leq \alpha^k ||x_1 - x_0||_X,$$

from which we obtain the third:

$$||x_{k+1} - x_0||_X \leq \sum_{j=0}^{k} \alpha^j ||x_1 - x_0||_X \leq \frac{1}{1-\alpha} r(1-\alpha) = r.$$

From these property, we obtain that:

$$||x_{k+\ell} - x_k||_X \leq \sum_{j=k}^{\ell-1} \alpha^j ||x_1 - x_0||_X \leq \alpha^k r,$$

which proves that $(x_k)_k$ is a Cauchy sequence so has a limit $x \in B$. But since:

$$f(x_k) = Df(x_k)(x_{k+1} - x_k)$$

by continuity of f and boundedness in B of $s \mapsto D_G f(s)$ (by $||D_G f(x_0)|| + \alpha/M$), we obtain that $f(x) = 0$. Finally, if $y \in B$ satisfies $f(y) = 0$, we have:

$$y - x = -D_G f(x)^{-1}(f(y) - f(x) - D_G f(x)(y - x)),$$

so: we obtain:

$$||y - x||_X \leq ||D_G f(x)^{-1}||_{\mathcal{L}(Y,X)} \sup_{\zeta \in [x,y]} ||D_G f(\zeta) - D_G f(x)||_{\mathcal{L}(X,Y)} ||y - x||_X,$$

and finaly

$$||y - x||_X \leq ||D_G f(x)^{-1}||_{\mathcal{L}(Y,X)} \frac{\alpha}{M} ||y - x||_X \leq \alpha ||y - x||_X,$$

from which we receive $y = x$. □

2.4. Proof of the theorem

In fact, we will prove that there exists an absolute constant C, depending only on the M_j and m_{j_0}, s.t., if

$$A(., \underline{x}, \Theta(\underline{x}), \dots, \Theta^n(\underline{x})) = \underline{b}$$

has a solution (given $(b_t)_t \in \ell^2(G)$), then for any $\underline{b'} \in \ell^2(G)$ s.t. $||\underline{b} - \underline{b'}||_{\ell^2} \leq C$, then the equation admits a unique solution closed to those of the first equation. Starting with $\underline{b} = A(., 0) \in \ell^2(G)$ and take N a positive integer s.t. $\frac{||\underline{b}||_{\ell^2}}{N} \leq C$, by induction on n we prove that the equation:

$$A(., \underline{x}, \Theta(\underline{x}), \dots, \Theta^n(\underline{x})) = \frac{N-n}{N}\underline{b}$$

has a solution for each n. Our result is reached by taking $N = n$.

Step 1: introducing an operator. Given $\underline{b} \in \ell^2(G)$, let us consider the mapping:

$$\phi_{\underline{b}} : \ell^2(G) \to (\ell^2(G))'$$

$$\phi_{\underline{b}}(\underline{x}) = \left[\underline{v} \mapsto \int_G \langle A(., \underline{x}, \dots, \Theta^n(\underline{x})) - \underline{b}, \Theta^{k_0}\underline{v}\rangle_H d\mu_G\right].$$

Step 2: existence and Lipschitzianity of our operator. We remark that $\phi_{\underline{b}}(0)$ is well defined since $A(., 0)$ is assumed to be in $\ell^2(G)$ and by Lipschitzianity of $A(t, .)$ with a uniform constant $L(A)$, we obtain that $\phi_{\underline{b}}$ is well defined and Lipschitzian:

$$||\phi_{\underline{b}}(\underline{u}) - \phi_{\underline{b}}(\underline{v})||_{(\ell^2)'} \leq L(A)||\underline{u} - \underline{v}||_{\ell^2}.$$

So, $\phi_{\underline{b}}$ is continuous.

Step 3: Gâteaux differentiability of the operator. Now, let us see that $\phi_{\underline{b}}$ is Gâteaux differentiable with as Gâteaux derivative:

$$D_G\phi_{\underline{b}}(\underline{x}).\underline{h} = \left[\underline{v} \mapsto \int_G \sum_{j=0}^{n} \langle \partial_{j+2}A(., \underline{x}, \dots, \Theta^n(\underline{x}))\Theta^j h, \Theta^{k_0} v\rangle_H d\mu_G\right].$$

Set $T : \ell^2(G) \to (\ell^2(G))^{n+1}$ as:

$$T(\underline{x}) = (\underline{x}, \Theta(\underline{x}), \dots, \Theta^n(\underline{x}))$$

and $J : \ell^2(G) \to (\ell^2(G))'$ as:

$$J(\underline{u}) = \left[\underline{v} \mapsto \int_G \langle u_t, v_t\rangle_H d\mu_G(t)\right].$$

T and J are continuous linear operators, so Lipschitzian and Fréchet differentiable.
For $\phi_{\underline{b}}$, we see that: $\phi_{\underline{b}} = J \circ \mathcal{N}_{A-b} \circ T$. The Nemytskii operator is Gâteaux differentiable, since with boundedness of the derivative, there exists an absolute constant C s.t.:

$$||A(t, s) - b_t||_H \leq C||s||_{H^{n+1}} + ||A(t, 0) - b_t||_H.$$

Since $(A(t, 0))_t \in \ell^2(G)$ we can have a look at [9] Theorem 2.3 and proof of Theorem 2.7 to see that the Nemytskii operator is well defined and Gâteaux-differentiable. Since T is linear, we receive that $\mathcal{N}_{A-b} \circ T$ is Gâteaux differentiable[1] and since J is Fréchet, we can conclude.

[1] *in fact with next step we will see that $\phi_{\underline{b}}$ is Fréchet-differentiable.*

Step 4: uniform continuity of the Gâteaux derivative. Let us fix a positive ε. By uniform continuity of all $\partial_{j+2}A$, we can find a common $\delta > 0$ s.t.:

$$\forall (t, s_1, s_2, j) \in G \times H^{n+1} \times H^{n+1} \times \{0, \dots, n+1\},$$

$$(||s_1 - s_2||_{H^{n+1}} \leq \delta) \Rightarrow \left(||\partial_{j+2}A(t, s_1) - \partial_{j+2}A(t, s_2)||_H \leq \frac{\varepsilon}{n+1} \right).$$

Now by assumption on norms, we have:

$$||T(\underline{x}) - T(\underline{x}')||^2_{(\ell^\infty)^{n+1}} \leq c^2 ||T(\underline{x}) - T(\underline{x}')||^2_{(\ell^2)^{n+1}} =$$

$$c^2 \sum_{k=0}^{n} ||(\Theta^k)(\underline{x}) - (\Theta^k)(\underline{x}')||^2_{\ell^2} = (n+1)c^2 ||\underline{x} - \underline{x}'||^2_{\ell^2}.$$

This means that if we assume $||\underline{x} - \underline{x}'||_{\ell^2} \leq \delta_0 := \frac{\delta}{c\sqrt{n+1}}$, we receive:

$$||\partial_{j+2}A(t, T(\underline{x})_t) - \partial_{j+2}A(t, T(\underline{x}')_t)||_{\mathcal{L}(H)} \leq \frac{\varepsilon}{n+1}.$$

But:

$$||D_G\phi_{\underline{b}}(\underline{x}) - D_G\phi_{\underline{b}}(\underline{x}')||_{\mathcal{L}(\ell^2, (\ell^2)')} =$$

$$\sup_{||\underline{h}||_{\ell^2}=1, ||\underline{v}||_{\ell^2}=1} \left| \int_G \sum_{j=0}^{n} \langle ((\partial_{j+2}A(., T(\underline{x})) - (\partial_{j+2}A(., T(\underline{x}'))).(\Theta^j \underline{h}), (\Theta^{k_0}\underline{v})\rangle_H d\mu_G \right|$$

$$\leq \int_G \sum_{j=0}^{n} ||(\partial_{j+2}A(., T(\underline{x})) - (\partial_{j+2}A(., T(\underline{x}'))||_{\mathcal{L}(H)} d\mu_G \leq \varepsilon.$$

Note that since $D_G\phi_{\underline{b}}$ is uniformly continuous, in fact $\phi_{\underline{b}}$ is Fréchet-C^1. We will now write $\phi'_{\underline{b}}$ instead of $D_G\phi_{\underline{b}}$.

Step 5: invertibility of the derivative. Let us now remark that $(\underline{h}, \underline{v}) \mapsto (\phi'_{\underline{b}}(\underline{x}).(\underline{h}))(\underline{v})$ is a continuous bilinear form (let us call it β), with $||\beta|| \leq \sum_{j=1}^{n} M_j$. Moreover, this form is elliptic:

$$\beta(\underline{h}, \underline{h}) \geq \left(m_{j_0} - \sum_{j \neq j_0} M_j \right) ||\underline{h}||^2.$$

Indeed:

$$\beta(\underline{h}, \underline{h}) = \int_G \sum_{j=0}^{n} \langle \partial_{j+2}A(., T(\underline{x}))(\Theta^j \underline{h}), (\Theta^{j_0}\underline{h})\rangle_H d\mu_G =$$

$$\int_G \left[\langle \partial_{j_0+2}A(., T(\underline{x}))(\Theta^{j_0} \underline{h}), (\Theta^{j_0}\underline{h})\rangle_H + \sum_{j \neq j_0} \langle \partial_{j+2}A(., T(\underline{x}))(\Theta^j \underline{h}), (\Theta^{j_0}\underline{h})\rangle_H \right] d\mu_G.$$

But:

$$\int_G \langle \partial_{j_0+2}A(., T(\underline{x}))(\Theta^{j_0} \underline{h}), (\Theta^{j_0}\underline{h})\rangle_H d\mu_G \geq m_{j_0} ||\Theta^{j_0}\underline{h}||^2_{\ell^2} = m_{j_0} ||\underline{h}||^2_{\ell^2}$$

and by Cauchy-Schwarz inequality:

$$\int_G |\langle \partial_{j+2}A(., T(\underline{x}))(\Theta^j \underline{h}), (\Theta^{j_0}\underline{h})\rangle_H| d\mu_G \leq M_j ||\Theta^j \underline{h}||_{\ell^2} ||\Theta^{j_0}\underline{h}||_{\ell^2} = M_j ||\underline{h}||^2_{\ell^2},$$

which proves our assumption.

Let us now set $\beta_1 = m_{j_0} - \sum_{j\neq j_0} M_j$. It follows from Lax-Milgram's theorem that the linear form $\phi'_{\underline{b}}(\underline{x})$ is invertible, and by writing $\underline{h}' = (\phi'_{\underline{b}}(\underline{x}))^{-1}(L)$, we have:

$$\beta_1 ||\underline{h}'||^2_{\ell^2} \leq \beta(\underline{h}', \underline{h}') = L(\underline{h}') \leq ||L||_{(\ell^2)'} ||\underline{h}'||_{\ell^2},$$

so:

$$||(\phi'_{\underline{b}}(\underline{x}))^{-1}||_{\mathcal{L}((\ell^2)', \ell^2)} \leq \beta_1^{-1}.$$

The last constant does not depends on $\underline{x}$ or $\underline{b}$, but only on A.

Step 6: applying Newton's method. We wish to apply this with $f = \phi_{\underline{b}}$. We have to take $M = \beta_1^{-1}$ for the first condition. To find a β for the second condition, it is necessary that:

$$\sup_{(x,x')\in \overline{B}(x_0,r)^2} ||\phi'_{\underline{b}}(x) - \phi'_{\underline{b}}(x')||_{\mathcal{L}(\ell^2, (\ell^2)')} < \beta_1$$

and when this is true, by noting σ the sup, we could choose $\alpha = \sigma/\beta_1$. But this is possible by using the uniform continuity of $\phi'_{\underline{b}}$: for sufficiently small $r > 0$ the ball $\overline{B}(x_0, r)$ will satisfy this property.

Last of all, let us note that if we take x_0 a solution for $\phi_{\underline{b}'}(x_0) = 0$, then:

$$||\phi_{\underline{b}}(x_0)||_{(\ell^2)'} = ||\underline{b} - \underline{b}'||_{\ell^2}.$$

Thus, if $||\underline{b} - \underline{b}'||_{\ell^2} \leq r(1-\alpha)/M$, we will receive a $\underline{x} \in \overline{B}(x_0, r)$ s.t. $\phi_{\underline{b}}(\underline{x}) = 0$. Note that the constant $C := r(1-\alpha)/M$ depends only on A.

Remark 2.
The uniform continuity and the continuous injection are used only in the step 4. We could remark that instead of uniform continuity of the partial differential, we use is fact:

$$\lim_{\delta\to 0} \left[\sup_t \sup_{||s-s'||_{H^{n+1}} \leq \delta} ||\partial_{j+2} A(t,s) - \partial_{j+2} A(t,s')||_{\mathcal{L}(H)} \right] = 0,$$

which is more general, but we made for assumption **(H4)** which is simpler to write.

2.5. Examples

As a first example, let us come back to the linear case in $\ell^2(G)$. Let us take for instance:

$$A(t,x) = -y_t + \sum_{k=0}^{n} \langle a_k(t), s_k \rangle_H;$$

here $\partial_{j+2} A(t,x) = a_j(t)$. If we assume that all a_j are bounded and uniformly continuous, we obtain the usual condition, given before when $H = \mathbf{R}$ and when the system is autonomous (i.e. the a_j does not depends on t). Here, assumption **(H2)** means that $\underline{b}$ is assumed to be in ℓ^2. So, our result can be seen as an extension of the linear case.

Let us take a quasilinear case. For sake of simplicity, let us assume $H = \mathbf{R}$, although this is not necessary:

$$A(t,s) = A_1(t,s) - y_t + \sum_{k=0}^{n} a_k(t) s_k.$$

Assume that all assumptions **(H1)-(H5)** are true on A_1 and that there exist j_0 s.t.:

$$\inf_t a_{j_0}(t) > \sum_{j \neq j_0} |a_j(t)|.$$

Then if:

$$-\inf \partial_{j_0+2} A_1(t,s) + \sum_{j \neq j_0} \sup |\partial_j A_1(t,s)| < \inf_t a_{j_0}(t) - \sum_{j \neq j_0} \sup_t |a_j(t)|,$$

then:

$$a_p x_{t+p} + \ldots + a_0 x_t + A_1(t, x_t, \ldots, x_{t+p}) = y_t$$

has a solution. Indeed, we shall have the condition:

$$\inf_{(t,s)} (a_{j_0}(t) + \partial_{j_0+2} A_1(t,s)) > \sum_{j \neq j_0} \sup_{(t,s)} |a_j(t) + \partial_{j+2} A_1(t,s)|.$$

But since $\inf_{(t,s)}(a_{j_0}(t)+\partial_{j_0+2}A_1(t,s)) \geq \inf_{(t,s)}(a_{j_0}(t))+\inf_{(t,s)}(\partial_{j_0+2}A_1(t,s))$ and $\sup_{(t,s)} |a_j(t)+\partial_{j+2}A_1(t,s)| \leq \sup_{(t,s)} |a_j(t)|+ \sup_{(t,s)} |\partial_{j+2}A_1(t,s)|$, we see that the given condition is sufficient.

Remark 3.
The nonlinear case can be seen as a **quantitative** result of perturbation of the linear one.

3. Some extensions

3.1. Non isometric case

Instead assuming that Θ is a bijective isometry, we may assume that Θ is continuous and bijective. In this case, by Banach's Theorem, Θ^{-1} is also continuous. Let us introduce $(\alpha, \beta) \in (\mathbf{R}_*^+)^2$ s.t.:

(H7) : for each $\underline{x} \in E$, $\quad \alpha_\Theta ||\underline{x}|| \leq ||\Theta(\underline{x})|| \leq \beta_\Theta ||\underline{x}||$.

Here, assumption **(H6)** should be replaced by **(H8)**:
(H8) : $\alpha_\Theta^{2j_0} m_{j_0} > \sum_{j \neq j_0} \beta_\Theta^{j+j_0} M_j$.

Theorem 2.
Under the assumptions **(H1)-(H5), (H7), (H8)**, there exists $\underline{x} \in \ell^2(G)$ s.t.:

$$A(., \underline{x}, \Theta(\underline{x}), \ldots, \Theta^n(\underline{x})) = 0. \tag{2}$$

Proof. The proof is similar. The only change concerns ellipticity of β. We see by induction that:

$$||\Theta^{j_0}(\underline{x})|| \geq \alpha_\Theta^{j_0} ||\underline{x}||$$

and:

$$\forall j, \quad ||\Theta^j(\underline{x})|| \leq \beta_\Theta^j ||\underline{x}||.$$

So,

$$m_{j_0} ||\Theta^{j_0}(\underline{h})||^2 \geq m_{j_0} \alpha_\Theta^{2j_0} ||\underline{h}||^2,$$

$$M_j ||\Theta^j \underline{h}||.||\Theta^{j_0} \underline{h}|| \leq M_j \beta_\Theta^{j+j_0} ||\underline{h}||^2,$$

and **(H8)** gives ellipticity. □

As an example for a nonisometric case, let us know assume that we are with a weighted ℓ^2: given $p : G \to \mathbf{R}_*^+$ we are in:

$$\ell_p^2 = \left\{ \underline{x} \in H^G, \quad \int_G |x_t|^2 p_t d\mu_G(t) < +\infty \right\}.$$

Note that in a weighted ℓ^2, the corresponding measure with density $\underline{p}$ w.r.t. μ_G does not necessary satisfy an assertion as:

$$\exists c > 0, \quad \forall A, \quad (\nu(A) > 0) \Rightarrow (\nu(A) \geq 1/c^2).$$

Moreover, assume **(H9)**:

$$\exists (c_1, c_2) \in (\mathbf{R}_*^+)^2, \forall \mathbf{t} \in \mathbf{G}, \quad \mathbf{c_1} \leq \frac{\mathbf{p_{t-1}}}{\mathbf{p_t}} \leq \mathbf{c_2}.$$

Then, it is easy to see that the shift operator $S : \ell_p^2 \to \ell_p^2$, no longer isometric, is still bijective and that:

$$\forall \underline{x} \in \ell_p^2, \quad c_1 ||\underline{x}|| \leq ||S(\underline{x})|| \leq c_2 ||\underline{x}||.$$

Here $\alpha_\Theta = c_1$ and $\beta_\Theta = c_2$.

A more illustrative example is a case of discrete Sobolev space, where $p_t = 1 + t^2$. In this case, simple calculations show that we can take:

$$c_1^{-1} = c_2 = \frac{5}{2}.$$

Corollary 1.

Under the assumptions **(H1)-(H5)**, and $p_t = 1 + t^2$, if:

$$m_{j_0} > \sum_{j \neq j_0} \left(\frac{5}{2} \right)^{j + 3j_0} M_j,$$

then there exists $\underline{x} \in \ell_p^2(G)$ s.t.:

$$A(t, x_t, x_{t+1}, \ldots, x_{t+n}) = 0. \tag{3}$$

Proof. Here $\beta_\Theta = \alpha_\Theta^{-1} = \frac{5}{2}$. Assertion **(H8)** can be written as:

$$m_{j_0} \beta_\Theta^{-2j_0} > \sum_{j \neq j_0} \beta_\Theta^{j + j_0} M_j,$$

i.e.:

$$m_{j_0} > \sum_{j \neq j_0} \left(\frac{5}{2} \right)^{j + 3j_0} M_j.$$

□

4. A case of uniqueness

We come back to Theorem 3 (a similar remark could be made for Theorem 2). Assume for instance instead of **(H4)** the assumption:

$$\sum_{j=0}^{n} \sup_{(t,s,s')} ||\partial_{j+2}A(t,s) - \partial_{j+2}A(t,s')||_{\mathcal{L}(H)} < \left(m_{j_0} - \sum_{j \neq j_0} M_j \right)^{-1}.$$

In this case, the first two properties for Newton's method are true with any r. So, we can take r as huge as we want. We would start for instance with $\underline{x}_0 = 0$. Taking $\underline{b} = 0$, uniqueness in Newton's method for any r gives uniqueness of the solution of our problem. So we obtain the following:

Theorem 3.

Under the assumptions **(H1)-(H3), (H5), (H6)** and:

$$\sum_{j=0}^{n} \sup_{(t,s,s')} ||\partial_{j+2}A(t,s) - \partial_{j+2}A(t,s')||_{\mathcal{L}(H)} < \left(m_{j_0} - \sum_{j \neq j_0} M_j \right)^{-1}.$$

Given a bijective isometry $\Theta : \ell^2(G) \to \ell^2(G)$, there exists a unique $\underline{x} \in \ell^2(G)$ s.t.:

$$A\left(., \underline{x}, \Theta(\underline{x}), \ldots, \Theta^n(\underline{x})\right) = 0. \tag{4}$$

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