QUASI-PERIODIC FUNCTION SPACES AND OSCILLATIONS IN DIFFERENTIAL EQUATIONS

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Abstract

We build spaces of q.p. (quasi-periodic) functions and we establish some of their properties. They are motivated by the Percival approach to q.p. solutions of Hamiltonian systems. We use this approach to obtain some regularization theorems of weak q.p. solutions of Differential Equations. For a large class of Differential Equations, the first theorem gives a result of density: a particular form of perturbated equations have strong solutions. The second theorem gives a condition which insures that any essentially bounded solution is a strong one.

Key Words: quasi-periodic functions, Ordinary Differential Equations.

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Introduction

The question of the existence of a.p. (almost periodic) solutions of ODE (Ordinary Differential Equations) is a hard problem. On the a.p. solutions of dissipative systems, a large literature exists [26], but on the a.p. solutions of Lagrangian Systems (or Hamiltonian systems), the situation is different.
During the last decennies, the methods of Nonlinear Functional Analysis have provided powerful results to treat the existence of periodic solutions of Hamiltonian systems [30]. After that, methods of Nonlinear Analysis have arose to treat the existence of a.p. solutions of Hamiltonian Systems.
In this viewpoint, the question is to define spaces of the a.p. functions, functionals or operators on these functions spaces such that the critical points of functionals, or the fixed points of operators, coincide with the a.p. solutions of the considered ODE.
About the spaces of a.p. functions, there exist several constructions.
Denoting by $AP^1(\mathbb{R}^N)$ the (Banach) space of the Bohr-a.p. functions which possess a Bohr-a.p. derivative, we can consider functionals of the form: $J(x) := \mathcal{M}\{L(\cdot, x, \dot{x})\}$, where $\mathcal{M}\{\varphi\} = \lim\limits_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \varphi(t)dt$ and $L : \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}$ such that their critical points in $AP^1(\mathbb{R}^N)$ are exactly the a.p. solutions of the Euler-Lagrange equation: $L_x(t, x(t), \dot{x}(t)) = \frac{d}{dt}L_{\dot{x}}(t, x(t), \dot{x}(t))$. This approach, called Calculus of Variations in Mean Time, permits to obtain results, essentially when $L(t, \cdot, \cdot)$ is convex (or concave), [9].
But $AP^1(\mathbb{R}^N)$ does not possess good properties, notably it is not reflexive. And so, it becomes natural to build Hilbertian spaces of a.p. functions. $B^{1,2}(\mathbb{R}^N)$ is the Hilbert space of the (classes of) Besicovitch-a.p. functions which possess a Besicovitch-a.p. generalized derivative [10, 13]. Associated to this space, we have a notion of weak a.p. solution: $L_x(\cdot, x, \dot{x}) \sim_2 \nabla L_{\dot{x}}(\cdot, x, \dot{x})$, where ∇x denotes the generalized derivative of x, and $\phi \sim_2 \psi$ means $\mathcal{M}\{|\phi-\psi|^2\} = 0$. This notion of weak a.p. solution means also that the Fourier-Bohr series of $L_x(\cdot, x, \dot{x})$ is equal to the formal derivative of the Fourier-Bohr series of $L_{\dot{x}}(\cdot, x, \dot{x})$.
This Hilbertian approach have permitted to obtain existence results of weak a.p. solutions (in $B^{1,2}$) [12,13] and also resuls of density of strong a.p. solutions [11] in the following sense: when $b \in AP^0(\mathbb{R}^N)$, for each $\varepsilon > 0$, we can find $b_\varepsilon \in AP^0(\mathbb{R}^N)$ such that $d(b, b_\varepsilon) < \varepsilon$ and such that there exists a strong a.p. solution to $L_x(t, x(t), \dot{x}(t)) - \frac{d}{dt}L_{\dot{x}}(t, x(t), \dot{x}(t)) = b_\varepsilon(t)$.

The use of the space $B^{1,2}$ can be considered like a step towards the strong a.p. solutions. Naturally, it motivates the question of the regularization of weak a.p. solutions into strong a.p. solutions. One of the difficulties in the use of the space $B^{1,2}$ is the absence of results like theorem of Sobolev imbeddings. This is due to the presence of small divisors.

There exists an another viewpoint on the spaces of a.p. functions described into the works of [4] which consider hilbertian spaces with an additional restrictive condition on the Fourier-Bohr exponents in order to ensure a theorem of Sobolev imbedding. The counterpart of this gain is the unstability of these spaces for nonlinear operators, that induces a notion of weak a.p. solution with a correcting term in the ODE.

Some deep contributions to the study of of this type of functions spaces are due to Avantaggiati and alii [2,3, 25].

In this short list of functions spaces useful to the study of the a.p. solutions of ODE, we must talk about the spaces $BP^k(I\!\!R^N)$, the spaces of the Bohr-a.p. functions which possess bounded primitives until order k, introduced by J. Mawhin [28], [29]. These spaces permit to obtain strong a.p. solutions. Among the a.p. solutions, it is classical to distinguish the classes of q.p. (quasi-periodic) functions. The q.p. functions are related to the famous problem of the invariant tori and to the famous KAM method. There exists perturbative methods to study the q.p. solutions of Lagrangian systems [15]. In a radically different spirit, there exists a variational approach due to Percival [33].

In the present work, we adapt to the q.p. functions the spaces $B^{1,2}$ and more generally $B^{p,2}$. We give several equivalent constructions to these spaces, and we establish a process of regularization (of the weak a.p. solutions) special to the q.p. solutions.

Let $\omega = (\omega_1, \omega_2, ..., \omega_m)$ be a list of m $Z\!\!\!Z$-linearly independent real numbers. Let $F : I\!\!R^{1+Np} \longrightarrow I\!\!R^N$ a Bohr q.p. (quasi-periodic) function depending uniformly on parameters such that ω is a $Z\!\!\!Z$-basis of its module of frequencies. We assume that F satisfies the following Lipschitz condition:

$$\exists c \in (0, \infty), \forall t \in I\!\!R, \forall (\xi_i)_i \in (I\!\!R^N)^p, \forall (\zeta_i)_i \in (I\!\!R^N)^p,$$

$$|F(t,\xi_1,...,\xi_p) - F(t,\zeta_1,...,\zeta_p)| \leq c\sum_{i=1}^{p} |\xi_i - \zeta_i|. \tag{1}$$

From it, we formulate the following forced ordinary differential equation:

$$q^{(p)}(t) = F(t, q(t), ..., q^{(p-1)}(t)). \tag{2}$$

We seek q.p. solutions of (2) with a module of frequencies generated by ω.

To study this problem, we associate to F a function $\Phi : I\!R^{m+Np} \longrightarrow I\!R^N$ periodic with respect to its m first variables such that $\Phi(t\omega, \alpha_1, ..., \alpha_p) = F(t, \alpha_1, ..., \alpha_p)$, and we seek periodic solutions $u : I\!R^m \longrightarrow I\!R^N$ of the following partial differential equation:

$$\partial_\omega^p u(x) = \Phi(x, u(x), ..., \partial_\omega^{p-1} u(x)), \tag{3}$$

where $\partial_\omega u := \sum\limits_{j=1}^{m} \omega_j \frac{\partial u}{\partial x_j}$ and $\partial_\omega^p u := \partial_\omega^{p-1}(\partial_\omega u)$.

We shall use a notion of weak q.p. solution of (2), i.e. a solution of the following equation:

$$\nabla^p q \sim_2 F(., q, ..., \nabla^{p-1} q). \tag{4}$$

in the sense defined in [10] and [13] that we recall in Section 1.

We shall also use a notion of weak periodic solution of (3), i.e. a solution of the following equation

$$\partial_\omega^p u(x) = \Phi(x, u(x), ..., \partial_\omega^{p-1} u(x)), \tag{5}$$

which is an equality of distributions whose sense will be precised in Section 2.

The idea to use (3) to study (2) was expressed in [33], but, in this paper, Percival does not build any existence result by using this idea. In [6, 7], Berger and Zhang constructed adequate functions spaces and provided some existence results of weak and strong q.p. solutions of forced second-order Lagrangian systems (with coercive potentials) by using this way.

Now we describe the contents of the present paper.

In Section 1, we recall some notations and some notions about the Bohr a.p. (almost periodic), the Besicovitch a.p. functions, the generalized derivatives of the Besicovitch a.p. functions, and about various spaces of periodic functions defined on $\mathbb{R}^m$.

In Section 2, we define a notion of generalized Gâteaux variation, denoted by ∇_ω or ∂_ω, that we introduce as the infinitesimal generator of a group of transformations. We prove that this notion of derivative coincides with a distributional notion and with a notion of generalized derivative like a Sobolev derivative. And so we can define a space of periodic functions on $\mathbb{R}^m$ which is like a Sobolev space that we denote by $H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$. This space coincides with a space introduced by Berger and Zhang.

In Section 3, we study the relations between various classes of q.p. functions defined on $\mathbb{R}$ and various classes of periodic functions defined on $\mathbb{R}^m$.

In Section 4, we extend the relation shown in Section 3 to the case of quasi-periodic functions depending uniformly on parameters.

In Section 5, given $u \in H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$, we study the absolute continuity of the associated functions $t \longmapsto u(t\omega + \xi)$.

In Section 6, we study the relations between the q.p. solutions of (2) and (3) and the periodic solutions of (4) and (5).

1 Notations and usual functions spaces

We denote by $BC^0(\mathbb{R}, \mathbb{R}^N)$ the space of the bounded continuous functions from $\mathbb{R}$ in $\mathbb{R}^N$, and by $\|.\|_\infty$ the norm of the supremum on this space. When $r \in \mathbb{N} \cup \{\infty\}$, $BC^r(\mathbb{R}, \mathbb{R}^N)$ denotes the space of the functions $f \in C^r(\mathbb{R}, \mathbb{R}^N)$ such that f and all its derivatives, until order r, belong to $BC^0(\mathbb{R}, \mathbb{R}^N)$. $H^r_{loc}(\mathbb{R}, \mathbb{R}^N) = W^{r,2}_{loc}(\mathbb{R}, \mathbb{R}^N)$ denotes the usual Sobolev space.

Let $f \in L^1_{loc}(\mathbb{R}, \mathbb{R}^N)$. The mean value of f (when it exists) is the following vector of $\mathbb{R}^N$:

$$\mathcal{M}\{f\} = \mathcal{M}\{f(t)\}_t := \lim_{T\to\infty} (2T)^{-1} \int_{-T}^{T} f(t)dt,$$

and, when $N = 1$, the upper mean value of f is the following scalar quantity:

$$\overline{\mathcal{M}}\{f\} = \overline{\mathcal{M}}\{f(t)\}_t := \limsup_{T\to\infty} (2T)^{-1} \int_{-T}^{T} f(t)dt.$$

When $f : \mathbb{R}^m \longrightarrow \mathbb{R}^N$ and when $p \in \mathbb{R}^m$, the translation of f by p is the following function:

$$\tau_p f : \mathbb{R}^m \longrightarrow \mathbb{R}^N, \quad \tau_p f(x) := f(x + p).$$

We denote by $TP(\mathbb{C}^N)$ the space of the trigonometric polynomials with coefficients in $\mathbb{C}^N$, i.e. $f \in TP(\mathbb{C}^N)$ means that $f : \mathbb{R} \longrightarrow \mathbb{C}^N$, $f(t) = \sum_{\ell=0}^{n} c_\ell e^{i\lambda_\ell t}$, with $c_\ell \in \mathbb{C}^N$, and $\lambda_\ell \in \mathbb{R}$.

The space of the Bohr a.p. functions [8, Chapter I] and [18, Chapter VI] from $\mathbb{R}$ in $\mathbb{R}^N$ is denoted by $AP^0(\mathbb{R}^N)$. When $r \in \mathbb{N} \cup \{+\infty\}$, $AP^r(\mathbb{R}^N)$ is the space of the functions $f \in C^r(\mathbb{R}, \mathbb{R}^N)$ such that f and all its derivatives, until order r, belong to $AP^0(\mathbb{R}^N)$.

When $\alpha \in [1, \infty)$, $B^\alpha(\mathbb{R}^N)$ denotes the space of the Besicovitch a.p. functions from $\mathbb{R}$ in $\mathbb{R}^N$ [8, Chapter II]. We recall that $B^\alpha(\mathbb{R}^N)$ is a quotient space, and when f and g are representants of the same element of $B^\alpha(\mathbb{R}^N)$, we set $f \sim_\alpha g$, that means: $\mathcal{M}\{\|f(t) - g(t)\|^\alpha\}_t = 0$.
Following [10, 13], when $f \in B^2(\mathbb{R}^N)$, we denote by ∇f (when it exists) the following limit in $B^2(\mathbb{R}^N)$:

$$\nabla f = \lim_{s\to 0} s^{-1}(\tau_s f - f),$$

and $B^{1,2}(\mathbb{R}^N) := \{f \in B^2(\mathbb{R}^N) : \nabla f \text{ exists in } B^2(\mathbb{R}^N)\}$. We can iterate this process to define

$$B^{r+1,2}(\mathbb{R}^N) := \{f \in B^r(\mathbb{R}^N) : \nabla(\nabla^r f) \text{ exists in } B^2(\mathbb{R}^N)\},$$

and $\nabla^{r+1} f := \nabla(\nabla^r f)$.

When f belongs to $AP^r(\mathbb{R}^n)$ or to $B^2(\mathbb{R}^N)$, we associate to f its Fourier-Bohr series [8]:

$$f(t) \sim_2 \sum_{\lambda\in\mathbb{R}} \mathsf{a}(f;\lambda)e^{i\lambda t},$$

where $\mathsf{a}(f;\lambda) := \mathcal{M}\{f(t)e^{-i\lambda t}\}_t \in \mathbb{C}^N$, and we define $\Lambda(f) := \{\lambda \in \mathbb{R} : \mathsf{a}(f;\lambda) \neq 0\}$, and $\mathrm{Mod}(f) := \mathbb{Z}\langle\Lambda(f)\rangle$ the $\mathbb{Z}$-module generated by $\Lambda(f)$ in $\mathbb{R}$.

When M is a $\mathbb{Z}$-module in $\mathbb{R}$, we define the following spaces:

$$AP^r(\mathbb{R}^N, M) := \{f \in AP^r(\mathbb{R}^N) : \mathrm{Mod}(f) \subset M\}$$

$$B^2(\mathbb{R}^N, M) := \{f \in B^2(\mathbb{R}^N) : \mathrm{Mod}(f) \subset M\}$$

$$B^{r,2}(\mathbb{R}^N, M) := \{f \in B^{r,2}(\mathbb{R}^N) : \mathrm{Mod}(f) \subset M\},$$

and, denoting by $\mathbb{Z}\langle\omega\rangle$ the $\mathbb{Z}$-module generated by $\{\omega_j : j = 1, ..., m\}$,

$$AP^r_\omega(\mathbb{R}^N) := AP^r(\mathbb{R}^N, \mathbb{Z}\langle\omega\rangle)$$

$$B^2_\omega(\mathbb{R}^N) := B^2(\mathbb{R}^N, \mathbb{Z}\langle\omega\rangle)$$

$$B^{r,2}_\omega(\mathbb{R}^N) := B^{r,2}(\mathbb{R}^N, \mathbb{Z}\langle\omega\rangle).$$

Now, we consider some spaces of functions defined on $\mathbb{R}^m$.
When $r \in \mathbb{N} \cup \{\infty\}$, $C^r_c(\mathbb{R}^m, \mathbb{R})$ denotes the space of the functions of $C^r(\mathbb{R}^m, \mathbb{R})$ whose support is compact. $\mathcal{D}(\mathbb{R}^m) := C^\infty_c(\mathbb{R}^m, \mathbb{R})$, and the topological space $\mathcal{D}(\mathbb{R}^m)^*$ is the space of the distributions of L. Schwartz on $\mathbb{R}^m$ [36].

$P(\mathbb{T}^m, \mathbb{C}^N)$ denotes the space of the functions $u : \mathbb{R}^m \longrightarrow \mathbb{C}^N$ in the following form:

$$u(x_1, ..., x_m) = \sum_{k_1=1}^{n_1} ... \sum_{k_m=1}^{n_m} c_{k_1...k_m} e^{ik_1x_1} ... e^{ik_mx_m},$$

and $P(\mathbb{T}^m, \mathbb{R}^N) := \{u \in P(\mathbb{T}^m, \mathbb{C}^N) : u(\mathbb{R}^m) \subset \mathbb{R}^N\}$.

A vector $p \in \mathbb{R}^m$ is called a period of a function $u : \mathbb{R}^m \longrightarrow \mathbb{R}^N$ when we have $\tau_p u = u$. A function $u : \mathbb{R}^m \longrightarrow \mathbb{R}^N$ is called a periodic function when it possesses a non zero period. We denote by $\mathrm{per}(u)$ the set of the periods of u. It is well-known that $\mathrm{per}(u)$ is a $\mathbb{Z}$-module in $\mathbb{R}^m$ [14, TG VII 10].
When $r \in \mathbb{N} \cup \{\infty\}$, we set

$$C^r(\mathbb{T}^m, \mathbb{R}^N) := \{u \in C^r(\mathbb{R}^m, \mathbb{R}^N) : \mathrm{per}(u) \supset 2\pi\mathbb{Z}^m\}.$$

Let $\alpha \in [1, \infty)$. When $u \in L^{\alpha}_{loc}(I\!\!R^m, I\!\!R^N)$, we say that $p \in I\!\!R^m$ is a period of u when $\tau_p u = u$ in $L^{\alpha}_{loc}(I\!\!R^m, I\!\!R^N)$, and $\text{per}(u)$ denotes the set of the periods of u. We set

$$L^{\alpha}(I\!\!T^m, I\!\!R^N) := \{u \in L^{\alpha}_{loc}(I\!\!R^m, I\!\!R^N) : \text{per}(u) \supset 2\pi Z\!\!\!Z^m\}.$$

When $u \in L^2(I\!\!T^m, I\!\!R^N)$, we associate to u its Fourier series [37, Chapter VII]:

$$u(x) \sim \sum_{k \in Z\!\!\!Z^m} \hat{u}(k) e^{ik \cdot x},$$

where $k \cdot x := \sum_{j=1}^{m} k_j x_j$ and $\hat{u}(k) := (2\pi)^{-m} \int_{Q^m} u(x) e^{ik \cdot x} dx$, with $Q^m := [-\pi, \pi]^m \subset I\!\!R^m$.
We recall that, when $u \in L^1(I\!\!T^m, I\!\!R^N)$, we have

$$\int_{I\!\!T^m} u(x) dx := (2\pi)^{-m} \int_{Q^m} u(x) dx.$$

We use the abbreviation "L.a.e." to say: "Lebesgue almost everywhere" or " Lebesgue almost every".

About the periodic distributions we refer to [16, Chapter I]. And so, by taking $\mathcal{D}(I\!\!T^m) := C^{\infty}(I\!\!T^m, I\!\!R)$, its topological dual space $\mathcal{D}(I\!\!T^m)^*$ is the space of the periodic distributions. It is also possible to define the periodic distributions as special distributions on $I\!\!R^m$ like it is made in [38, pp.64-65]. These two constructions are equivalent [38, Chapter CC, Section III]. The distributional derivative with respect the j-th variable is denoted by ∂_j in $\mathcal{D}(I\!\!T^m)^*$ and by D_j in $\mathcal{D}(I\!\!R^m)^*$.
Since the notation $I\!\!T^m$ in the above-mentioned sense, we denote the m-dimensional geometric torus as follows:

$$\mathbf{U}^m := \{(z_1, ..., z_m) \in \mathbb{C}^N : \forall j = 1, ..., m, \ \ |z_j| = 1\}.$$

And we can assimilate the periodic functions (respectively distributions) defined on $I\!\!R^m$ and the functions (respectively distributions) defined on $\mathbf{U}^m$ [37, p.245] (respectively [36, pp.229-231]).

2 Other functions spaces

We consider the group $(T(t))_{t \in I\!\!R}$ defined as follows:

$$T(t) : L^2(I\!\!T^m, I\!\!R^N) \longrightarrow L^2(I\!\!T^m, I\!\!R^N), \ \ T(t)u := \tau_{t\omega} u.$$

For each $t \in \mathbb{R}$, $T(t)$ is linear and is an isometry. $(T(t))_{t\in\mathbb{R}}$ is a strongly continuous group in $\mathcal{L}(L^2(\mathbb{T}^m, \mathbb{R}^N), L^2(\mathbb{T}^m, \mathbb{R}^N))$ in the sense of the theory of the semi-groups. The infinitesimal generator of $(T(t))_{t\in\mathbb{R}}$, denoted by ∇_ω, is defined as follows:

$$\nabla_\omega u := \lim_{t\to 0} t^{-1}(T(t)u - u) \text{ in } L^2(\mathbb{T}^m, \mathbb{R}^N). \tag{6}$$

Definition. *The domain of definition of ∇_ω in $L^2(\mathbb{T}^m, \mathbb{R}^N)$ is denoted by $H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$.*

By using the general theory of semi-groups [21, Chapter VIII, Section 1], the following result holds.

Proposition 1. *$H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$ is a vector subspace of $L^2(\mathbb{T}^m, \mathbb{R}^N)$, $\nabla_\omega : H^1_\omega(\mathbb{T}^m, \mathbb{R}^N) \longrightarrow L^2(\mathbb{T}^m, \mathbb{R}^N)$ is a linear operator, $H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$ is dense into $L^2(\mathbb{T}^m, \mathbb{R}^N)$, and the graph of ∇_ω is closed in $L^2(\mathbb{T}^m, \mathbb{R}^N) \times L^2(\mathbb{T}^m, \mathbb{R}^N)$.*

When $u, v \in H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$, we set:

$$\langle u \mid v \rangle_\omega := \int_{\mathbb{T}^m} u(x) \cdot v(x) dx + \int_{\mathbb{T}^m} \nabla_\omega u(x) \cdot \nabla_\omega v(x) dx, \tag{7}$$

and $\|u\|_\omega$ denotes its associated norm.

Proposition 2. *$(H^1_\omega(\mathbb{T}^m, \mathbb{R}^N), \langle . \mid . \rangle_\omega)$ is a Hilbert space.*

Proof. The linearity of ∇_ω (Proposition 1) permits us to verify that $\langle . \mid . \rangle_\omega$ is an inner product, and the completedness is a consequence of the closedness of the graph of ∇_ω (Proposition 1).$\blacksquare$

When $u \in C^0(\mathbb{T}^m, \mathbb{R}^N)$, the Gâteaux variation of u at x for the increment ω (when it exists) is:

$$Du(x; \omega) := \lim_{t\to 0} t^{-1}(u(x + t\omega) - u(x)). \tag{8}$$

When $Du(x; \omega)$ exists for each $x \in \mathbb{R}^m$, we easily verify that $\text{per}(Du(\cdot; \omega)) \supset 2\pi\mathbb{Z}^m$. When it exists, we set $D^2u(x; \omega) := D(Du(\cdot; \omega))(x; \omega)$.

And so we can define the following functions spaces.

$$C^1_\omega(\mathbb{T}^m, \mathbb{R}^N) := \{u \in C^0(\mathbb{T}^m, \mathbb{R}^N) : Du(\cdot; \omega) \in C^0(\mathbb{T}^m, \mathbb{R}^N)\} \tag{9}$$

$$C^2_\omega(\mathbb{T}^m, \mathbb{R}^N) := \{u \in C^0(\mathbb{T}^m, \mathbb{R}^N) : D^2u(\cdot;\omega) \in C^0(\mathbb{T}^m, \mathbb{R}^N)\}. \qquad (10)$$

It is clear that $C^r(\mathbb{T}^m, \mathbb{R}^N) \subset C^r_\omega(\mathbb{T}^m, \mathbb{R}^N)$, for $r = 1, 2$. When $u \in C^1(\mathbb{R}^m, \mathbb{R}^N)$, we define:

$$\partial_\omega u(x) := u'(x).\omega = \sum_{j=1}^m \omega_j \frac{\partial u}{\partial x_j}(x) = Du(x;\omega). \qquad (11)$$

When $T \in \mathcal{D}(\mathbb{T}^m)^*$, we define:

$$\partial_\omega T := \sum_{j=1}^m \omega_j \partial_j T \qquad (12)$$

and when $T \in \mathcal{D}(\mathbb{R}^m)^*$, we define:

$$D_\omega T := \sum_{j=1}^m \omega_j D_j T. \qquad (13)$$

Proposition 3. *When $u \in C^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$, then we have $u \in H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$, and $\nabla_\omega u = Du(\cdot;\omega)$.*

Proof. Since $Du(\cdot;\omega)$ is continuous and periodic on $\mathbb{R}^m$, it is uniformly continuous on $\mathbb{R}^m$. By using the mean value inequality, we obtain:

$$\sup_{x\in\mathbb{R}^m} |t^{-1}(u(x+t\omega) - u(x)) - Du(x,\omega)| \leq$$

$$\sup_{|s|<t} \sup_{x\in\mathbb{R}^m} \|Du(x+s\omega;\omega) - Du(x;\omega)\|.|\omega|$$

The last term converges to zero when $t \to 0$ because of the uniform continuity, and since the uniform norm is greatest than the L^2-norm on Q^m, we can conclude.$\blacksquare$

Now, we study the regularization by convolution in $H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$.

Lemma 1. *Let $u \in H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$. Then the following assertions hold.*

(i) *Let $\varrho \in C^r_c(\mathbb{R}^m, \mathbb{R})$, with $r \in \mathbb{N} \cup \{\infty\}$. Then we have $\varrho * u \in C^r(\mathbb{T}^m, \mathbb{R}^N)$, and $\partial_\omega(\varrho * u) = (\partial_\omega \varrho) * u$, when $r \geq 1$.*

(ii) *Let $(\varrho_h)_h$ be a sequence of mollifiers (with values in $\mathcal{D}(\mathbb{R}^m)$) such that $supp(\varrho_h) \subset Q^m$ for each $h \in \mathbb{N}$. Then we have*

$$\lim_{h\to\infty} \|\varrho_h * u - u\|_{L^2(\mathbb{T}^m,\mathbb{R}^N)} = 0.$$

Proof.
(i) Since $u \in L^1(\mathbb{T}^m, \mathbb{R}^N)$ and $\varrho \in L^1_{loc}(\mathbb{R}^m, \mathbb{R})$, the convolution product of u and ϱ is well defined on $\mathbb{R}^m$ [24, p.17]. Since $\varrho \in C^r_c(\mathbb{R}^m, \mathbb{R})$, we have $\varrho * u \in C^r(\mathbb{R}^m, \mathbb{R})$ [24, p.17]. If $p \in 2\pi\mathbb{Z}^m$, then, for every $z \in \mathbb{R}^m$, we have

$$\varrho * u(z+p) = \int_{\mathbb{R}^m} \varrho(x)u(z+p-x)dx = \int_{\mathbb{R}^m} \varrho(x)u(z-x)dx = \varrho * u(z).$$

And so, $\mathrm{per}(\varrho * u) \supset 2\pi\mathbb{Z}^m$, and consequently, we have $\varrho * u \in C^r(\mathbb{T}^m, \mathbb{R}^N)$. By using a general property of the convolution products [38, p.122], we have

$$\frac{\partial}{\partial x_j}(\varrho * u) = \frac{\partial \varrho}{\partial x_j} * u,$$

and consequently, we have

$$\partial_\omega(\varrho * u) = \sum_{j=1}^m \omega_j \frac{\partial}{\partial x_j}(\varrho * u) = \sum_{j=1}^m \omega_j \frac{\partial \varrho}{\partial x_j} * u = (\partial_\omega \varrho) * u.$$

(ii) The proof is similar to this one of the usual case [24, pp.17-18] .■

Proposition 4. *$C^1(\mathbb{T}^m, \mathbb{R}^N)$ is dense in $H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$. Precisely, if $(\varrho_h)_h$ is a sequence of mollifiers such that $supp(\varrho_h) \subset Q^m$ for each $h \in \mathbb{N}$, then we have $\varrho_h * u \in C^r(\mathbb{T}^m, \mathbb{R}^N)$ for each h, and $\lim\limits_{h\to\infty} \|\varrho_h * u - u\|_\omega = 0$.*

Proof. First, for every $h \in \mathbb{N}_*$, for every $t \in \mathbb{R} \setminus \{0\}$, and for every $z \in \mathbb{R}^m$, we have:

$$\varrho_h * [t^{-1}(\tau_{t\omega} u - u)](z) = [t^{-1}(\tau_{t\omega}\varrho_h - \varrho_h)] * u(z).$$

Secondly, for every $h \in \mathbb{N}_*$, $t \in \mathbb{R} \setminus \{0\}$, $z \in \mathbb{R}^m$, we have:

$$|\varrho_h * (\tau_{t\omega} u - u)(z) - \varrho_h * \nabla_\omega u(z)| \le$$

$$\|\varrho_h\|_{L^2(Q^m)}.\|t^{-1}(\tau_{t\omega} u - u) - \nabla_\omega u\|_{L^2(Q^m)}.$$

Therefore, we have:

$$\lim_{t\to 0} \varrho_h * [t^{-1}(\tau_{t\omega} u - u)](z) = \varrho_h * \nabla_\omega u(z).$$

Thirdly, by using the same reasoning as above, we have:

$$\lim_{t\to 0}[t^{-1}(\tau_{t\omega}\varrho_h - \varrho_h)] * u(z) = (\partial_\omega \varrho_h) * u(z).$$

By using the three previous relations, we obtain:

$$(\partial_\omega \varrho_h) * u = \varrho_h * \nabla_\omega u.$$

By using Lemma 1, we have $\varrho_h * \nabla_\omega u = \partial_\omega(\varrho_h * u)$, and

$$\lim_{h\to\infty} \|\varrho_h * u - u\|_{L^2} = 0,$$

$$\lim_{h\to\infty} \|\partial_\omega(\varrho_h * u) - \nabla_\omega u\|_{L^2} = \lim_{h\to\infty} \|\varrho_h * \nabla_\omega u - \nabla_\omega u\|_{L^2} = 0. \blacksquare$$

Proposition 5. *The three following assertions hold.*

(i) $\forall f \in C^1(\mathbb{T}^m, \mathbb{R}^N), \ \int_{\mathbf{T}^m} \partial_\omega f(x) dx = 0.$

(ii) $\forall u \in H^1_\omega(\mathbb{T}^m, \mathbb{R}^N) \ , \int_{\mathbb{T}^m} \nabla_\omega u(x) dx = 0.$

(iii) $\forall \varphi \in C^1(\mathbb{T}^m, \mathbb{R}^N), \forall u \in H^1_\omega(\mathbb{T}^m, \mathbb{R}^N),$

$$\nabla_\omega(\varphi \cdot u) = (\partial_\omega \varphi) \cdot u + \varphi \cdot \nabla_\omega u.$$

Proof.
(i) We set $dx_{\hat{j}} := dx_1 ... dx_{j-1} dx_{j+1} ... dx_m$. By using the Fubini theorem, we have:

$$\int_{\mathbb{T}^m} \frac{\partial f}{\partial x_j}(x) dx = (2\pi)^{-m} \int_{Q^{m-1}} \left\{ \int_0^{2\pi} \frac{\partial f}{\partial x_j}(x) dx_j \right\} d\hat{x}_j = 0,$$

and consequently, we obtain

$$\int_{\mathbb{T}^m} \partial_\omega f(x) dx = \sum_{j=1}^m \omega_j \int_{\mathbb{T}^m} \frac{\partial f}{\partial x_j}(x) dx = 0.$$

(ii) By using Proposition 4, there exists a sequence $(u_n)_n$ with values in $C^1(\mathbb{T}^m, \mathbb{R}^N)$ such that $\lim_{n\to\infty} \|u - u_n\|_\omega = 0$. By using (i), we have:

$$\left|\int_{\mathbb{T}^m} \nabla_\omega u(x)dx\right| = \|\nabla_\omega u - \partial_\omega u_n\|_{L^1} \leq \|\nabla_\omega u - \partial_\omega u_n\|_{L^2}$$

which converges to zero when $n \to \infty$.

(iii) Since $\varphi' \in C^0(\mathbb{T}^m, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^N))$, φ' is uniformly continuous on $\mathbb{R}^m$, and so, for every $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that, for every $\xi, \zeta \in \mathbb{R}^m$, we have

$$|\xi - \zeta| \leq \eta(\varepsilon) \Longrightarrow \|\varphi'(\xi) - \varphi'(\zeta)\| \leq \varepsilon.$$

When $|t| \leq |\omega|^{-1}\eta(\varepsilon)$, we have

$$|t^{-1}(\tau_{t\omega}\varphi(x) - \varphi(x)) \cdot u(x + t\omega) - \partial_\omega\varphi(x) \cdot u(x + t\omega)| \leq$$

$$\left(\sup_{\xi\in(x,x+t\omega)} \|\varphi'(\xi) - \varphi'(x)\|.|\omega|\right).|u(x + t\omega)| \leq \varepsilon|u(x + t\omega)|,$$

that implies

$$\|[t^{-1}(\tau_{t\omega}\varphi - \varphi) - \partial_\omega\varphi] \cdot \tau_{t\omega}u\|_{L^2} \leq \varepsilon\|\tau_{t\omega}u\|_{L^2} = \varepsilon\|u\|_{L^2},$$

and so, we have proven

$$\lim_{t\to 0} \|[t^{-1}(\tau_{t\omega}\varphi - \varphi) - \partial_\omega\varphi] \cdot \tau_{t\omega}u\|_{L^2} = 0. \tag{14}$$

Since $\lim_{t\to 0} \|\tau_{t\omega}u - u\|_{L^2} = 0$, and since

$$|\partial_\omega\varphi(x) \cdot [u(x + t\omega) - u(x)]| \leq |\partial_\omega\varphi(x)|.|u(x + t\omega) - u(x)|,$$

we have

$$\|\partial_\omega\varphi \cdot [\tau_{t\omega}u - u]\|_{L^2} \leq \|\partial_\omega\varphi\|_{L^\infty}.\|\tau_{t\omega}u - u\|_{L^2},$$

and consequently, we have

$$\lim_{t\to 0} \|\partial_\omega\varphi \cdot [\tau_{t\omega}u - u]\|_{L^2} = 0. \tag{15}$$

From (14) and (15), and from the following inequalities:

$$\|t^{-1}(\tau_{t\omega}\varphi - \varphi) \cdot \tau_{t\omega}u - \partial_\omega\varphi \cdot u\|_{L^2} \leq$$

$$\|t^{-1}(\tau_{t\omega}\varphi - \varphi) \cdot \tau_{t\omega}u - \partial_\omega \varphi \cdot \tau_{t\omega}u\|_{L^2} + \|\partial_\omega \varphi \cdot \tau_{t\omega}u - \partial_\omega \varphi \cdot u\|_{L^2},$$

we obtain:

$$\lim_{t \to 0} \|t^{-1}(\tau_{t\omega}\varphi - \varphi) \cdot \tau_{t\omega}u - \partial_\omega \varphi \cdot u\|_{L^2} = 0. \tag{16}$$

Since

$$\|[t^{-1}(\tau_{t\omega}u - u) - \nabla_\omega u] \cdot \varphi\|_{L^2} \le \|t^{-1}(\tau_{t\omega}u - u) - \nabla_\omega u\|_{L^2} \cdot \|\varphi\|_{L^\infty},$$

we have

$$\lim_{t \to 0} \|[t^{-1}(\tau_{t\omega}u - u) - \nabla_\omega u] \cdot \varphi\|_{L^2} = 0. \tag{17}$$

We note that

$$\|t^{-1}(\tau_{t\omega}[\varphi \cdot u] - \varphi \cdot u) - (\partial_\omega \varphi) \cdot u - \varphi \cdot \nabla_\omega u\|_{L^2} \le$$

$$\|t^{-1}(\tau_{t\omega}\varphi - \varphi) \cdot \tau_{t\omega}u - (\partial_\omega \varphi) \cdot u\|_{L^2} + \|t^{-1}(\tau_{t\omega}u - u) \cdot \varphi - \nabla_\omega u \cdot \varphi\|_{L^2},$$

and then, by using (16) and (17), we obtain the announced result.■

Theorem 1. *Let $u \in L^2(\mathbb{T}^m, \mathbb{R}^N)$. Then the four following assertions are equivalent.*

- **(i)** $u \in H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$.
- **(ii)** $\sum_{k \in \mathbb{Z}^m} (k \cdot \omega)^2 |\hat{u}(k)|^2 < \infty$.
- **(iii)** *There exists $v \in L^2(\mathbb{T}^m, \mathbb{R}^N)$ such that, for every $\varphi \in C^1(\mathbb{T}^m, \mathbb{R}^N)$, we have*
$$\int_{\mathbb{T}^m} v(x) \cdot \varphi(x) dx = - \int_{\mathbb{T}^m} u(x) \cdot \partial_\omega \varphi(x) dx.$$
- **(iv)** *The distribution $\partial_\omega u$ belongs to $L^2(\mathbb{T}^m, \mathbb{R}^N)$.*

Moreover, when these assertions hold, we have:

$$\nabla_\omega u(x) = v(x) = \partial_\omega u(x) = \sum_{k \in \mathbb{Z}^m} i(k \cdot \omega)\hat{u}(k) e^{ik \cdot x}.$$

Proof.
(i $\Longrightarrow$ iii) By using Proposition 5, ii, iii, for every $\varphi \in C^1(\mathbb{T}^m, \mathbb{R}^N)$, we have

$$\int_{\mathbb{T}^m} \nabla_\omega u \cdot \varphi + \int_{\mathbb{T}^m} u \cdot \partial_\omega \varphi = \int_{\mathbb{T}^m} \nabla_\omega (u \cdot \varphi) = 0.$$

And so we can take $v = \nabla_\omega u$.

(iii $\Longrightarrow$ ii) For each $k \in \mathbb{Z}^m$, we set $\chi_k(x) := e^{ik\cdot x}$. And so, we have $\chi_k \in C^\infty(\mathbb{T}^m, \mathbb{C})$, and we verify that $\partial_\omega \chi_k(x) = i(k \cdot \omega)\chi_k(x)$. From (iii), for every $k \in \mathbb{Z}^m$, we have:

$$\hat{v}(k) = \int_{\mathbb{T}^m} \overline{\chi_k}.v = -\int_{\mathbb{T}^m} \overline{\partial_\omega \chi_k}.u = i(k\cdot\omega)\int_{\mathbb{T}^m} \overline{\chi_k}.u = i(k\cdot\omega)\hat{u}(k).$$

Since $v \in L^2(\mathbb{T}^m, \mathbb{R}^N)$, we have [37, p.248]:

$$(\hat{v}(k))_{k\in\mathbb{Z}^m} = (i(k\cdot\omega)\hat{u}(k))_{k\in\mathbb{Z}^m} \in \ell^2(\mathbb{Z}^m, \mathbb{C}^N),$$

and so $\sum\limits_{k\in\mathbb{Z}^m} (k\cdot\omega)^2 |\hat{u}(k)|^2 < \infty$.

(ii $\Longrightarrow$ i) For each $\nu \in \mathbb{N}$, we set $P_\nu(x) := \sum\limits_{|k|\leq\nu} e^{ik\cdot x}\hat{u}(k)$. And so, $P_\nu \in C^\infty(\mathbb{T}^m, \mathbb{C}^N)$, and we verify that

$$\nabla_\omega P_\nu(x) = \partial_\omega P_\nu(x) = \sum_{|k|\leq\nu} i(k\cdot\omega)e^{ik\cdot x}\hat{u}(k).$$

From (ii), we have $(i(k\cdot\omega)\hat{u}(k))_{k\in\mathbb{Z}^m} \in \ell^2(\mathbb{Z}^m, \mathbb{C}^N)$, and by using the harmonic synthesis [37, p.248], there exists $v \in L^2(\mathbb{T}^m, \mathbb{C})$ such that $\hat{v}(k) = i(k\cdot\omega)\hat{u}(k)$, for every $k \in \mathbb{Z}^m$.
Since $u(\mathbb{R}^m) \subset \mathbb{R}^N$, we have $\hat{v}(-k) = \overline{\hat{v}(k)}$, and consequently, we have $v \in L^2(\mathbb{T}^m, \mathbb{R}^N)$.
By using [37, p.248], we have

$$\lim_{\nu\to\infty} \|u - P_\nu\|_{L^2} = 0, \quad \lim_{\nu\to\infty} \|v - \nabla_\omega P_\nu\|_{L^2} = 0,$$

and since the graph of ∇_ω is closed (Proposition 1), we necessarily have $v = \nabla_\omega u$, and so $u \in H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$.

(iii $\Longrightarrow$ iv) We fix $j \in \{1, 2, ..., m\}$. For each $\sigma \in C^\infty(\mathbb{T}^m, \mathbb{R})$, we take $\varphi \in C^\infty(\mathbb{T}^m, \mathbb{R}^N)$ defined by $\varphi_j = \sigma$ and $\varphi_\ell = 0$ when $j \neq \ell$. Then, from

(iii), we obtain $\int_{\mathbb{T}^m} v_j.\sigma = -\int_{\mathbb{T}^m} u_j.\partial_\omega \sigma$.
Since the distributional derivative $\partial_\omega u_j \in \mathcal{D}(\mathbb{T}^m)^*$ satisfies $\int_{\mathbb{T}^m} \partial_\omega u_j.\sigma = -\int_{\mathbb{T}^m} u_j.\partial_\omega \sigma$ [16, Chapter I, Section 6], we have $v_j = \partial_\omega u_j$, and consequently $\partial_\omega u \in L^2(\mathbb{T}^m, \mathbb{R}^N)$.

(iv $\Longrightarrow$ ii) For each $k \in \mathbb{Z}^m$, by using the characterization of the distributional derivatives and the Fourier transform, we have

$$(\partial_\omega u)^\wedge(k) = \int_{\mathbb{T}^m} e^{-ik\cdot x} \partial_\omega u(x) dx = -\int_{\mathbb{T}^m} \partial_\omega (e^{-ik\cdot x}) u(x) dx$$

$$= i(k\cdot\omega) \int_{\mathbb{T}^m} e^{-ik\cdot x} u(x) dx = i(k\cdot\omega)\hat{u}(k).$$

Since $\partial_\omega u \in L^2(\mathbb{T}^m, \mathbb{R}^N)$, we have $((\partial_\omega u)^\wedge(k))_{k\in\mathbb{Z}^m} \in \ell^2(\mathbb{Z}^m, \mathbb{C}^N)$, therefore $(i(k\cdot\omega)\hat{u}(k))_{k\in\mathbb{Z}^m} \in \ell^2(\mathbb{Z}^m, \mathbb{C}^N)$, and consequently, we have (ii).$\blacksquare$

Comments. In the previous result, we have proven the equivalence between the definition of $\nabla_\omega = \partial_\omega$ as the infinitesimal generator of a semi-group of transformations, the distributional derivative and the generalized Sobolev derivative. In [7], the space $H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$ is denoted by $\hat{P}^{1,2}$.$\blacksquare$

We can iterate the previous construction and define, for $r \in \mathbb{N}$, $r \geq 2$, the following Hilbert spaces:

$$H^r_\omega(\mathbb{T}^m, \mathbb{R}^N) := \{u \in H^{r-1}_\omega(\mathbb{T}^m, \mathbb{R}^N) : \partial_\omega(\partial^{r-1}_\omega u) \in H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)\},$$

and $\partial_\omega(\partial^{r-1}_\omega) =: \partial^r_\omega$.
The inner product of $H^r_\omega(\mathbb{T}^m, \mathbb{R}^N)$ is defined as follows:

$$(u, v) \longmapsto (u \mid v)_{L^2} + \sum_{j=1}^{r} (\partial^j_\omega u \mid \partial^j_\omega u)_{L^2}.$$

3 Relations between q.p. functions and periodic functions.

In this section, we study the relations between the q.p. functions defined on $\mathbb{R}$ and the periodic functions defined on $\mathbb{R}^m$.

Lemma 2. *For $x \in I\!\!R^m$, we denote by $cl(x)$ the class of equivalence of x in the quotient space $I\!\!R^m/2\pi\mathbb{Z}^m$. Then the following assertions hold.*

(i) $\{\mathrm{cl}(x) : x \in I\!\!R^m\}$ *is dense in* $I\!\!R^m/2\pi\mathbb{Z}^m$.

(ii) *For every $x \in I\!\!R^m$, there exists a sequence $(t_n)_n$ with values in $I\!\!R$, such that, for every $z \in C^0(\mathbb{T}^m, I\!\!R^N)$, we have $z(x) = \lim_{n\to\infty} z(t_n\omega)$.*

(iii) *For every $z \in C^0(\mathbb{T}^m, I\!\!R^N)$, we have*

$$\sup_{x\in I\!\!R^m} |z(x)| = \sup_{t\in I\!\!R} |z(t\omega)|.$$

Proof.
(i) We can assimilate $I\!\!R^m/2\pi\mathbb{Z}^m$ and $\mathbf{U}^m$ [5, pp.60, 82]. $\mathbf{U}^m$ is endowed with the induced topology of $\mathbb{C}^m$. And so, the topology of the quotient space $I\!\!R^m/2\pi\mathbb{Z}^m$ coincides with the topology defined by the following metric:

$$d(\mathrm{cl}(x_1, ..., x_m), \mathrm{cl}(y_1, ..., y_m)) := \left(\sum_{j=1}^{m} |e^{ix_j} - e^{iy_j}|^2\right)^{1/2}.$$

We fix $x = (x_1, ..., x_m) \in I\!\!R^m$. Since $\omega_1, ..., \omega_m$ are $\mathbb{Z}$-linearly independent, by using a classical theorem of Kronecker [18, p.163], for every $\delta > 0$, there exists $t \in I\!\!R$ such that, for each $j = 1, ..., m$, we have $|t\omega_j - x_j| < \delta$ (modulo 2π). Since $s \mapsto e^{is}$ is uniformly continuous from $I\!\!R$ in $\mathbb{C}$, for every $\varepsilon > 0$, there exists $t \in I\!\!R$ such that $|e^{ix_j} - e^{i\omega_j}| < \varepsilon m^{-1}$, for each $j = 1, ..., m$, and therefore we have $d(\mathrm{cl}(x), \mathrm{cl}(t\omega)) < \varepsilon$.

(ii) Since we have $[\mathrm{cl}(x) = \mathrm{cl}(y) \Longrightarrow z(x) = z(y)]$, the function $\mathrm{fact}(z) : I\!\!R^m/2\pi\mathbb{Z}^m \longrightarrow I\!\!R^N$ defined by $\mathrm{fact}(z)(\mathrm{cl}(x)) := z(x)$, is continuous [34, pp.37-38]. We fix $x \in I\!\!R^m$, and by using (i), there exists a real sequence $(t_n)_n$ such that $\mathrm{cl}(t_n\omega) \to \mathrm{cl}(x)$ when $n \to \infty$. Therefore we have

$$z(t_n\omega) = \mathrm{fact}(z)(\mathrm{cl}(t_n\omega)) \longrightarrow \mathrm{fact}(z)(\mathrm{cl}(x)) = z(x) \ \ (n \to \infty).$$

(iii) By using (ii), $z(I\!\!R^m)$ is included into the closure of $z(I\!\!R\omega)$, and since we have $z(I\!\!R\omega) \subset z(I\!\!R^m)$, we obtain the announced equality.$\blacksquare$

We consider the linear operator

$$\mathcal{Q}_\omega : (\mathbb{R}^N)^{\mathbb{T}^m} \longrightarrow (\mathbb{R}^N)^{\mathbb{R}}, \;\; \mathcal{Q}_\omega(u)(t) := u(t\omega). \tag{18}$$

Theorem 2. *The following assertions hold.*

(i) $\mathcal{Q}_\omega(C^0(\mathbb{T}^m, \mathbb{R}^N)) = QP^0_\omega(\mathbb{R}^N)$*, and for every* $u \in C^0(\mathbb{T}^m, \mathbb{R}^N)$*, we have* $\|\mathcal{Q}_\omega(u)\|_\infty = \|u\|_\infty$*.*

(ii) *Let* $r \in \mathbb{N}_* \cup \{\infty\}$*. Then we have* $\mathcal{Q}_\omega(C^r_\omega(\mathbb{T}^m, \mathbb{R}^N)) = QP^r_\omega(\mathbb{R}^N)$*, and for every* $u \in C^r_\omega(\mathbb{T}^m, \mathbb{R}^N)$*, we have* $\mathcal{Q}_\omega(Du(\cdot;\omega)) = \frac{d}{dt}\mathcal{Q}_\omega(u)$*.*

(iii) $\mathcal{Q}_\omega(L^2(\mathbb{T}^m, \mathbb{R}^N)) = B^2_\omega(\mathbb{R}^N)$*, and for every* $u \in L^2(\mathbb{T}^m, \mathbb{R}^N)$*, we have* $\|\mathcal{Q}_\omega(u)\|_{B^2} = \|u\|_{L^2}$*.*

(iv) $\mathcal{Q}_\omega(H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)) = B^{1,2}_\omega(\mathbb{R}^N)$*, and for every* $u \in H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$*, we have* $\|\mathcal{Q}_\omega(u)\|_{B^{1,2}} = \|u\|_\omega$ *and* $\mathcal{Q}_\omega(\partial_\omega u) = \nabla(\mathcal{Q}_\omega(u))$*.*

(v) $\mathcal{Q}_\omega(H^2_\omega(\mathbb{T}^m, \mathbb{R}^N)) = B^{2,2}_\omega(\mathbb{R}^N)$*, and for every* $u \in H^2_\omega(\mathbb{T}^m, \mathbb{R}^N)$*, we have* $\mathcal{Q}_\omega(\partial^2_\omega u) = \nabla^2(\mathcal{Q}_\omega(u))$*.*

Proof.
(i) By using the formula (18) we can define $\mathcal{Q}_\omega : \mathbb{C}^{\mathbf{T}^m} \longrightarrow \mathbb{C}^{\mathbb{R}}$. The inclusion $\mathcal{Q}_\omega(P(\mathbb{T}^m, \mathbb{R}^N)) \subset TP(\mathbb{R}^N, \omega)$ is evident, and by using the density of $P(\mathbb{T}^m, \mathbb{R}^N)$ in $C^0(\mathbb{T}^m, \mathbb{R}^N)$ [35, p.2], we obtain the inclusion

$$\mathcal{Q}_\omega(C^0(\mathbb{T}^m, \mathbb{R}^N)) \subset QP^0_\omega(\mathbb{R}^N).$$

Moreover by using Lemma 2, iii, we have, for every $u \in C^0(\mathbb{T}^m, \mathbb{R}^N)$, the equality $\|\mathcal{Q}_\omega(u)\|_\infty = \|u\|_\infty$.
If $g \in TP(\mathbb{C}, \omega)$, $g(t) = \sum\limits_{\ell=1}^{L} c_\ell e^{ik_\ell \cdot \omega t}$, where $c_\ell \in \mathbb{C}$, and $k_\ell \in \mathbb{Z}^m$, and if we set $h(x) := \sum\limits_{\ell=1}^{L} c_\ell e^{ik_\ell \cdot x}$, then we have $h \in P(\mathbb{T}^m, \mathbb{C})$ and $\mathcal{Q}_\omega(h) = g$.
Now we fix $f \in QP^0_\omega(\mathbb{C})$. By using the Bohr-Weierstrass theorem [8, p.50], there exists a sequence $(f_n)_n$ with values in $TP(\mathbb{C}, \omega)$ such that $\lim\limits_{n\to\infty} \|f - f_n\|_\infty = 0$. Because of the previous remark, for each $n \in \mathbb{N}$, there exists

$u_n \in P(\mathbb{T}^m,\mathbb{C})$ such that $\mathcal{Q}_\omega(u_n) = f_n$.
We note that, for every $p, q \in \mathbb{N}$, we have

$$\|f_p - f_q\|_\infty = \|\mathcal{Q}_\omega(u_p) - \mathcal{Q}_\omega(u_q)\|_\infty = \|\mathcal{Q}_\omega(u_p - u_q)\|_\infty = \|u_p - u_q\|_\infty.$$

Since $(f_n)_n$ is convergent, it is a Cauchy sequence, and consequently, $(u_n)_n$ is also a Cauchy sequence with values in the complete space $C^0(\mathbb{T}^m,\mathbb{C})$. Therefore there exists $u \in C^0(\mathbb{T}^m,\mathbb{C})$ such that $\lim_{n\to\infty} \|u_n - u\|_\infty = 0$.
Furthermore, we have

$$\|\mathcal{Q}_\omega(u) - f\|_\infty \leq \|\mathcal{Q}_\omega(u) - f_n\|_\infty + \|f - f_n\|_\infty$$

$$= \|\mathcal{Q}_\omega(u) - \mathcal{Q}_\omega(u_n)\|_\infty + \|f - f_n\|_\infty = \|u_n - u\|_\infty + \|f - f_n\|_\infty$$

that converges to zero when $n \to \infty$, and therefore we have $\mathcal{Q}_\omega(u) = f$.
That proves the inclusion $QP^0_\omega(\mathbb{C}) \subset \mathcal{Q}_\omega(C^0(\mathbb{T}^m,\mathbb{C}))$. We have proven that $\mathcal{Q}_\omega$ is a linear isomorphism and an isometry between $C^0(\mathbb{T}^m,\mathbb{C})$ and $QP^0_\omega(\mathbb{C})$. From this result, we deduce that $\mathcal{Q}_\omega$ is a linear isomorphism and an isometry between $C^0(\mathbb{T}^m, \mathbb{R})$ and $QP^0_\omega(\mathbb{R})$. After that, by taking the cartesian products, we obtain the announced result.

(ii) Let $u \in C^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$. Then for every $t \in \mathbb{R}$, we have $\partial_\omega u(t\omega) = \frac{d}{dt}(\mathcal{Q}_\omega(u))(t)$, i.e. $\mathcal{Q}_\omega(\partial_\omega u) = \frac{d}{dt}\mathcal{Q}_\omega(u)$. And so, we have $\mathcal{Q}_\omega(u) \in QP^1_\omega(\mathbb{R}^N)$. Now we reasone with $N = 1$. The general case is a simple consequence of the case $N = 1$.
Conversely, if $\rho \in QP^1_\omega(\mathbb{R})$, then we have $\rho, \dot\rho \in QP^0_\omega(\mathbb{R})$, and by using (i), there exists $u, v \in C^0(\mathbb{T}^m, \mathbb{R})$ such that $\mathcal{Q}_\omega(u) = \rho$ and $\mathcal{Q}_\omega(v) = \dot\rho$, and for every $t \in \mathbb{R}$, we have $\frac{d}{dt}u(t\omega) = v(t\omega)$.
We fix $x \in \mathbb{R}^m$. By using Lemma 2, ii, there exists a real sequence $(t_n)_n$ such that, for every $t \in \mathbb{R}$, $u(t_n\omega + t\omega) \to u(x + t\omega)$ and $u(t_n\omega) \to u(x)$ when $n \to \infty$.
For each $n \in \mathbb{N}$, we have

$$u(t_n\omega + t\omega) - u(t_n\omega) = \int_{t_n}^{t_n+t} \partial_\omega u(s\omega)ds = \int_{t_n}^{t_n+t} v(s\omega)ds,$$

therefore, for each $t \in \mathbb{R}$, there exists $\gamma(t, n) \in [0, 1]$ such that

$$u(t_n\omega + t\omega) - u(t_n\omega) = v([(1 - \gamma(t, n))t_n + \gamma(t, n)(t_n + t)]\omega).t$$

$$= v(t_n\omega + \gamma(t,n)t\omega).t.$$

We fix $t \in I\!R$. Since $(\gamma(t,n))_n$ takes its values in the compact set $[0,1]$, there exists $n \mapsto j_n$, a monotonically increasing function from $I\!N$ in $I\!N$, and there exists $\Gamma(t) \in [0,1]$ such that $\gamma(t,j_n) \to \Gamma(t)$ when $n \to \infty$. Consequently, we obtain

$$u(t_{j_n}\omega + t\omega) - u(t_{j_n}\omega) = v(t_{j_n}\omega + \gamma(t,j_n)t\omega).t,$$

that implies, when $n \to \infty$:

$$u(x + t\omega) - u(x) = v(x + \Gamma(t)t\omega).t.$$

Consequently, we obtain

$$\lim_{t\to 0} t^{-1}(u(x+t\omega) - u(x)) = v(x).$$

And so, for every $x \in I\!R^m$, $Du(x;\omega)$ exists and $Du(x;\omega) = v(x)$, that implies: $u \in C^1_\omega(\mathbb{T}^m, I\!R)$.

(iii) Since $\omega_1, ..., \omega_m$ are $\mathbb{Z}$-linearly independent, the function $k \mapsto k\cdot\omega$ is an isomorphism of $\mathbb{Z}$-moduli between $\mathbb{Z}^m$ and $\mathbb{Z}\langle\omega\rangle$. Consequently $\ell^2(\mathbb{Z}^m,\mathbb{C})$ and $\ell^2(\mathbb{Z}\langle\omega\rangle,\mathbb{C})$ are isomorphic and isometric.
By using the harmonic synthesis [37, p.248], the Fourier transform

$$\mathcal{F} : L^2(\mathbb{T}^m,\mathbb{C}^N) \longrightarrow \ell^2(\mathbb{Z}^m,\mathbb{C}^N),\ \mathcal{F}(u) := (\hat{u}(k))_{k\in\mathbb{Z}^m}$$

is a linear isomorphism and an isometry.
By using the Riesz-Fisher-Besicovitch theorem [8, p.110], the Fourier-Bohr transform

$$\mathcal{A} : B^2_\omega(\mathbb{C}^N) \longrightarrow \ell^2(\mathbb{Z}\langle\omega\rangle,\mathbb{C}^N),\ \mathcal{A}(f) := (\mathsf{a}(f;\lambda))_{\lambda\in\mathbb{Z}\langle\omega\rangle}$$

is a linear isomorphism and an isometry.
And so $\mathcal{A}^{-1}\circ\mathcal{F} : L^2(\mathbb{T}^m,\mathbb{C}^N) \longrightarrow B^2_\omega(\mathbb{C}^N)$ is a linear isomorphism and an isometry. To conclude it is sufficient to remark that we have $\mathcal{A}^{-1}\circ\mathcal{F} = \mathcal{Q}_\omega$, since

$$\mathcal{A}^{-1}\circ\mathcal{F}(u) = \mathcal{A}^{-1}((\hat{u}(k))_{k\in\mathbb{Z}^m}) = [t \longmapsto \sum_{k\in\mathbb{Z}^m} \hat{u}(k)e^{ik\cdot\omega t}] = \mathcal{Q}_\omega(u).$$

(iv) Let $u \in H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$. Then we have $u, \partial_\omega u \in L^2(\mathbb{T}^m, \mathbb{R}^N)$, and by using (iii), we know that $\mathcal{Q}_\omega(u), \mathcal{Q}_\omega(\partial_\omega u) \in B^2_\omega(\mathbb{R}^N)$.
We note that, for every $t \in \mathbb{R}$, we have

$$\mathcal{Q}_\omega(\tau_{t\omega} u) = \tau_t \mathcal{Q}_\omega(u). \tag{19}$$

By using (ii), for every $t \in \mathbb{R} \setminus \{0\}$, we have

$$\begin{aligned}\|t^{-1}(\tau_{t\omega} u - u) - \partial_\omega u\|_{L^2} &= \|\mathcal{Q}_\omega(t^{-1}(\tau_{t\omega} u - u) - \partial_\omega u)\|_{B^2} \\ &= \|t^{-1}(\tau_t \mathcal{Q}_\omega(u) - \mathcal{Q}_\omega(u)) - \mathcal{Q}_\omega(\partial_\omega u)\|_{B^2}.\end{aligned}$$

And since $u \in H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$, we have

$$\lim_{t \to 0} \|t^{-1}(\tau_t \mathcal{Q}_\omega(u) - \mathcal{Q}_\omega(u)) - \mathcal{Q}_\omega(\partial_\omega u)\|_{B^2} = 0,$$

i.e. $\mathcal{Q}_\omega(u) \in B^{1,2}(\mathbb{R}^N)$ and $\nabla \mathcal{Q}_\omega(u) = \mathcal{Q}_\omega(\partial_\omega u)$.
Since $B^{1,2}_\omega(\mathbb{R}^N) = B^{1,2}(\mathbb{R}^N) \cap B^2_\omega(\mathbb{R}^N)$, we have $\mathcal{Q}_\omega(u) \in B^{1,2}_\omega(\mathbb{R}^N)$.

Conversely, let $f \in B^{1,2}_\omega(\mathbb{R}^N)$. Then we have $f, \nabla f \in B^2_\omega(\mathbb{R}^N)$, and by using (ii), we can assert that there exists $u, v \in L^2(\mathbb{T}^m, \mathbb{R}^N)$ such that $\mathcal{Q}_\omega(u) = f$ and $\mathcal{Q}_\omega(v) = \nabla f$.
By using a previous calculation, for every $t \in \mathbb{R} \setminus \{0\}$, we have

$$\begin{aligned}\|t^{-1}(\tau_{t\omega} u - u) - v\|_{L^2} &= \|\mathcal{Q}_\omega(t^{-1}(\tau_{t\omega} u - u) - v)\|_{B^2} \\ &= \|t^{-1}(\mathcal{Q}_\omega(\tau_{t\omega} u) - \mathcal{Q}_\omega(u)) - \mathcal{Q}_\omega(v)\|_{B^2} \\ &= \|t^{-1}(\tau_t \mathcal{Q}_\omega(u) - \mathcal{Q}_\omega(u)) - \mathcal{Q}_\omega(v)\|_{B^2} \\ &= \|t^{-1}(\tau_t t - f) - \nabla f\|_{B^2} \longrightarrow 0 \quad (t \to 0),\end{aligned}$$

therefore, we have $v = \partial_\omega u$, and so $u \in H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$.
For each $u \in H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$, we have

$$\|u\|^2_\omega = \|u\|^2_{L^2} + \|\partial_\omega u\|^2_{L^2} = \|\mathcal{Q}_\omega(u)\|^2_{B^2} + \|\nabla \mathcal{Q}_\omega(u)\|^2_{B^2} = \|\mathcal{Q}_\omega(u)\|^2_{B^{1,2}}.$$

(v) Let $u \in H^2_\omega(\mathbb{T}^m, \mathbb{R}^N)$, then we have $u, \partial_\omega u \in H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$, and by using (iv), we have $\mathcal{Q}_\omega(u), \mathcal{Q}_\omega(\partial_\omega u) \in B^{1,2}_\omega(\mathbb{R}^N)$. Since $\nabla \mathcal{Q}_\omega(u) = \mathcal{Q}_\omega(\partial_\omega u)$, we have $\mathcal{Q}_\omega(u) \in B^{2,2}_\omega(\mathbb{R}^N)$.

Conversely, let $f \in B^{2,2}_\omega(\mathbb{R}^N)$. Then we have $f, \nabla f \in B^{1,2}_\omega(\mathbb{R}^N)$, and by using (iv), we can assert that there exists $u, v \in H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$ such that

$\mathcal{Q}_\omega(u) = f$, $\mathcal{Q}_\omega(v) = \nabla f$, $\mathcal{Q}_\omega(\partial_\omega u) = \nabla f$, and $\mathcal{Q}_\omega(\partial_\omega v) = \nabla^2 f$. Therefore, by using (iii), we have $v = \partial_\omega u$ and consequently $u \in H^2_\omega(\mathbb{T}^m, \mathbb{R}^N)$.■

Comments. The proof of (i) was given to us by [17]. Another way to prove (i) is provided by adapting the method of [19, Section 7]. Briefly we give some indications about this way. We consider the spectra of the commutative algebra $QP^0_\omega(\mathbb{C})$, denoted by $X(QP^0_\omega(\mathbb{C}))$. This spectra $X(QP^0_\omega(\mathbb{C}))$ is isomorphic to $\mathbf{U}^m$, and the Gelfand transformation [20, p.302]:

$$\mathcal{G} : QP^0_\omega(\mathbb{C}) \longrightarrow C^0(X(QP^0_\omega(\mathbb{C})),\mathbb{C}),\ \mathcal{G}(f)(\varphi) = \varphi(f),$$

is an isomorphism and an isometry (in this special case). We can assimilate $\mathcal{G}$ and $\mathcal{Q}_\omega^{-1}$. This method is closed to this one of [32, Chapter I, Section 3].■

4 Extension to quasi-periodic functions depending uniformly on parameters

In the following, $\mathbb{E}$ is a Banach space, $b\mathbb{R}$ is the Bohr compactification of $\mathbb{R}$ and P is a compact subset or an open subset of $\mathbb{R}^k$, for $k \geq 1$.
If P is open, let us consider the family:

$$K_n := \{x \in P \quad : \quad \|x\| \leq n \text{ and } d(x; P^c) \geq 1/n\}.$$

All K_n are compact in $\mathbb{R}^k$, $P = \cup_n K_n$ and $K_n \subset \text{Int}(K_{n+1})$ for all n. It is known that in general, if the function $f : \mathbb{R} \times P \to \mathbb{R}^\ell$ is such that $f(\cdot, \alpha)$ is almost periodic for all $\alpha \in P$, and $\phi : \mathbb{R} \to P$ is almost periodic, the function $[t \mapsto f(t, \phi(t))]$ may not be almost periodic. For this reason, we usually set the definition of almost periodic functions depending uniformly on parameters (see [39, p.5-6]). We denote by $APU(\mathbb{R}; P; \mathbb{E})$ the subset of all these functions. If P is compact, $APU(\mathbb{R}; P; \mathbb{E})$ is a Banach space with the norm:

$$\|f\|_{APU} := \sup_{(t,\alpha)\in\mathbb{R}\times P} \|f(t, \alpha)\|$$

and if P is open, $APU(\mathbb{R}; \mathbb{E})$ is a Fréchet space with the family of seminorms $(p_n)_n$, where: $p_n(f) := \sup_{(t,\alpha)\in\mathbb{R}\times K_n} \|f(t, \alpha)\|$.
The space $C^0(b\mathbb{R} \times P; \mathbb{E})$ is endowed with the norm

$$\|f\|_{C^0(\mathbb{R}\times P;\mathbb{E})} := \sup_{(t,\alpha)\in\mathbb{R}\times P} \|f(t, \alpha)\|$$

if P is compact, and with the family $(\pi_n)_n$, where
$\pi_n(f) := \sup_{(t,\alpha)\in \mathbb{R}\times K_n} \|f(t,\alpha)\|$ if P is open.

Proposition 6 *There exists an isometrical isomorphism of Fréchet spaces beetween $APU(\mathbb{R};P;\mathbb{E})$ and $C^0(b\mathbb{R};\mathbb{E})$.*

Proof. It is sufficient to prove it when P is compact. We will set the proof into three steps.
1. The mapping $f \mapsto (f(t,\cdot))_t$ is an isometrical isomorphism of Fréchet spaces beetween $APU(\mathbb{R};P;\mathbb{E})$ and $AP(\mathbb{R};C^0(P;E))$.
Indeed, it is clearly an homomorphism which is isometric, so one-to-one. It is also onto: if we consider $\phi \in AP(\mathbb{R};C^0(P;E))$, and if we set f such that $f(t,\alpha) := \phi(t)(\alpha)$. One has $f \in APU(\mathbb{R};P;\mathbb{E})$ and $\underline{x}$ is solution of the problem.
2. There exists an isometrical isomorphism of Fréchet spaces beetween $AP(\mathbb{R};C^0(P;\mathbb{E}))$ and $C^0(b\mathbb{R};C^0(P;\mathbb{E}))$.
3. There exists an isometrical isomorphism of Fréchet spaces beetween $C^0(b\mathbb{R}\times P;\mathbb{E}))$ and $C^0(b\mathbb{R};C^0(P;\mathbb{E}))$. Let us consider $f \longmapsto [t \mapsto f(t,\cdot)]$. It is clearly well defined and an isometrical homomorphism. We next prove that this homomorphism is onto. Consider $\lambda \in C^0(b\mathbb{R};C^0(P;\mathbb{E}))$ and $(t_0;\alpha_0) \in b\mathbb{R}\times P$. If we put $f(t,\alpha) := (\lambda(t))(\alpha)$, one has:

$$\|f(t,\alpha) - f(t_0,\alpha_0)\| \leq \|f(t,\alpha) - f(t_0,\alpha)\| + \|f(t_0,\alpha) - f(t_0,\alpha_0)\| \leq$$

$$\leq \|\lambda(t) - \lambda(t_0)\|_{C^0(P;\mathbb{E})} + \|\lambda(t_0)(\alpha) - \lambda(t_0)(\alpha_0)\|.$$

Consider $\varepsilon > 0$. Since $\lambda(t_0) \in C^0(P;\mathbb{E})$, there exists a neighbourhood V_2 of α_0 in P such that if $\alpha \in V_2$, one has: $\|\lambda(t_0)(\alpha) - \lambda(t_0)(\alpha_0)\| \leq \varepsilon/2$. Since $\lambda \in C^0(b\mathbb{R};C^0(P;\mathbb{E}))$, there exists a neighbourhood V_1 of t_0 in P such that if $t \in V_1$, one has $\|\lambda(t) - \lambda(t_0)\|_{C^0(P;\mathbb{E})} \leq \varepsilon/2$. If $(t,\alpha) \in V_1 \times V_2$, one has: $\|f(t,\alpha) - f(t_0,\alpha_0)\| \leq \varepsilon$, so $f \in C^0(\mathbb{R}\times P;\mathbb{E})$.$\blacksquare$

Remark. *The previous result shows that the condition $f \in APU(\mathbb{R};P;\mathbb{E})$ is also necessary to have $(\phi \in AP^0(P)) \Rightarrow ([t \mapsto f(t,\phi(t))] \in AP^0(\mathbb{E}))$.*

Now we set:

$$QPU_\omega(\mathbb{R},P,\mathbb{E}) := \{f \in APU(\mathbb{R},P,\mathbb{E}) \quad : \quad (\forall \alpha \in P)\ f(\cdot,\alpha) \in QP^0_\omega(\mathbb{E})\}.$$

The fact that Theorem 2, i, remains valid with a Banach space in place of $\mathbb{R}^N$ and the same proof as above show that $C^0(\mathbb{T}^m \times P; \mathbb{E})$ is canonically isomorphic to $QPU_\omega(\mathbb{R}, P, \mathbb{E})$. Moreover, the isomorphism, also denoted by $\mathcal{Q}_\omega$, is defined by: $u \mapsto [(t, \alpha) \mapsto u(t\omega, \alpha)]$.

5 About the absolute continuity

$\mathbb{R}^m$ is endowed with its standard inner product. We set $b_1 := |\omega|^{-1}\omega$, we consider $b_2, ..., b_m \in \mathbb{R}^m$ such that $(b_1, b_2, ..., b_m)$ is an orthonormal basis of $\mathbb{R}^m$. We denote by $(b_1^*, b_2^*, ..., b_m^*)$ the dual basis, and we consider the automorphism β of $\mathbb{R}^m$ defined by $\beta(x) := (b_1^*(x), b_2^*(x), ..., b_m^*(x))$. β is a linear isometry since we can consider β as the change of orthonormal bases, from the canonical basis to the basis $(b_j)_j$. We note that

$$\omega^\perp := \{y \in \mathbb{R}^m : y \cdot \omega = 0\} = \mathrm{span}\{b_2, ..., b_m\}.$$

Let $u \in C^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$. We associate to u the new function $v := u \circ \beta^{-1} \in C^0(\mathbb{R}^m, \mathbb{R}^N)$. For every $(y_1, ..., y_m) \in \mathbb{R}^m$, we have:

$$v(y_1, ..., y_m) = u\left(y_1|\omega|^{-1}\omega + \sum_{j=2}^m y_j b_j\right). \tag{20}$$

The function v is periodic and $\mathrm{per}(v) \supset 2\pi\beta(\mathbb{Z}^m)$. The function $\frac{\partial v}{\partial y_1}$ is defined and continuous on $\mathbb{R}^m$, and for every $(y_1, ..., y_m) \in \mathbb{R}^m$, we have:

$$\frac{\partial v}{\partial y_1}(y_1, ..., y_m) = |\omega|^{-1}\partial_\omega u\left(\sum_{j=1}^m y_j b_j\right). \tag{21}$$

To generalize the correspondence described by (18), we need some tools. We recall that, when $\psi \in C^r_c(\mathbb{R}^m, \mathbb{R}^N)$, $r \in \mathbb{N} \cup \{\infty\}$, the periodic transform [38, pp.61-62] of ψ is:

$$\varpi(\psi) := \sum_{k \in \mathbb{Z}^m} \tau_{2\pi k}\psi \in C^r(\mathbb{T}^m, \mathbb{R}^N). \tag{22}$$

We introduce another operator of periodic transformation.
When $\varphi \in C^r_c(\mathbb{R}^m, \mathbb{R}^N)$, we set:

$$\varpi_1(\varphi) := \sum_{k \in \mathbb{Z}^m} \tau_{2\pi\beta(k)}\varphi \in C^r(\mathbb{T}^m, \mathbb{R}^N). \tag{23}$$

We denote by $\langle .,. \rangle_{(m)}$ the duality bracket between $\mathcal{D}(I\!R^m)^*$ and $\mathcal{D}(I\!R^m)$. When $z \in L^1_{loc}(I\!R^m, I\!R)$, $T_z \in \mathcal{D}(I\!R^m)^*$ denotes the regular distribution built on z.

Lemma 3. *The following assertions hold.*

(i) *If $u \in L^2(T\!\!T^m, I\!R)$ and $\psi \in \mathcal{D}(I\!R^m)$, then we have:*

$$\langle D_\omega T_u, \psi \rangle_{(m)} = -\int_{Q^m} u(x).\varpi(\psi)(x)dx.$$

(ii) *If $v \in L^2_{loc}(I\!R^m, I\!R)$ is such that $per(v) \supset 2\pi\beta(Z\!\!Z^m)$ and if $\varphi \in \mathcal{D}(I\!R^m)$, then we have:*

$$\langle D_\omega T_v, \varphi \rangle_{(m)} = -\int_{\beta(Q^m)} v(y).\varpi_1(\varphi)(y)dy.$$

(iii) *When $\varphi \in C^1_c(I\!R^m, I\!R)$, we have:*

$$\frac{\partial \varpi_1(\varphi)}{\partial y_1} = \varpi_1\left(\frac{\partial \varphi}{\partial y_1}\right), \qquad \varpi_1(\varphi) \circ \beta = \varpi(\varphi \circ \beta),$$

$$\frac{\partial \varpi_1(\varphi)}{\partial y_1} \circ \beta = |\omega|^{-1}\partial_\omega(\varpi(\varphi \circ \beta)).$$

Proof.

(i) We fix $j \in \{1, ..., m\}$. Since $\text{supp}(\psi)$ is compact, there exists a finite list of distinct numbers, namely $\lambda_1, ..., \lambda_\nu \in 2\pi Z\!\!Z^m$ such that

$$\text{supp}\left(\frac{\partial \psi}{\partial x_j}\right) \subset \text{supp}(\psi) \subset \bigcup_{1 \le \ell \le \nu} (Q^m + \lambda_\ell).$$

When $\lambda \in 2\pi Z\!\!Z^m \setminus \{\lambda_\ell : \ell = 1, ..., \nu\}$, we have $\tau_\lambda \psi = 0$ on $\text{Int}Q^m$, and so, on $\text{Int}Q^m$, we have

$$\varpi(\psi) = \sum_{\ell=1}^{\nu} \tau_{\lambda_\ell}\psi, \text{ and } \varpi\left(\frac{\partial \psi}{\partial x_j}\right) = \sum_{\ell=1}^{\nu} \tau_{\lambda_\ell}\left(\frac{\partial \psi}{\partial x_j}\right).$$

And so, we have

$$\langle D_j T_u, \psi \rangle_{(m)} = -\left\langle T_u, \frac{\partial \psi}{\partial x_j} \right\rangle_{(m)} = -\int_{\mathbb{R}^m} u(x) \frac{\partial \psi}{\partial x_j}(x) dx$$

$$= -\sum_{\ell=1}^{\nu} \int_{Q^m + \lambda_\ell} u(x) \frac{\partial \psi}{\partial x_j}(x) dx$$

$$= -\sum_{\ell=1}^{\nu} \int_{Q^m} u(x).\tau_{\lambda_\ell} \left(\frac{\partial \psi}{\partial x_j} \right)(x) dx$$

$$= -\int_{Q^m} u(x).\varpi \left(\frac{\partial \psi}{\partial x_j} \right)(x) dx.$$

Consequently, we obtain:

$$\langle D_\omega T_u, \psi \rangle_{(m)} = -\int_{Q^m} u(x) \varpi \left(\frac{\partial \psi}{\partial x_j} \right)(x) dx.$$

(ii) Since $\text{supp}(\varphi)$ is compact, there exists a finite list of distinct numbers, namely $\alpha_1, \alpha_2, ..., \alpha_\nu \in 2\pi\beta(\mathbb{Z}^m)$ such that

$$\text{supp}\left(\frac{\partial \varphi}{\partial y_1} \right) \subset \text{supp}(\varphi) \subset \bigcup_{1 \le \ell \le \nu} (\beta(Q^m) + \alpha_\ell).$$

When $\alpha \in 2\pi\beta(\mathbb{Z}^m) \setminus \{\alpha_\ell : \ell = 1, ..., \nu\}$, we have $\tau_\alpha \varphi = 0$ on $\text{Int}\beta(Q^m) = \beta(\text{Int}Q^m)$ and therefore we have

$$\varpi_1(\varphi) = \sum_{\ell=1}^{\nu} \tau_{\alpha_\ell} \varphi, \quad \text{and} \quad \varpi_1 \left(\frac{\partial \varphi}{\partial y_1} \right) = \sum_{\ell=1}^{\nu} \tau_{\alpha_\ell} \frac{\partial \varphi}{\partial y_1}.$$

And so, we have:

$$\langle D_1 T_v, \varphi \rangle_{(m)} = -\left\langle T_v, \frac{\partial \varphi}{\partial y_1} \right\rangle_{(m)} = -\int_{\mathbb{R}^m} v(y) \frac{\partial \varphi}{\partial y_1}(y) dy$$

$$= -\sum_{\ell=1}^{\nu} \int_{\beta(Q^m)+\alpha_\ell} v(y) \frac{\partial \varphi}{\partial y_1}(y) dy = -\sum_{\ell=1}^{\nu} \int_{\beta(Q^m)} v(y + \alpha_\ell) \frac{\partial \varphi}{\partial y_1}(y + \alpha_\ell) dy$$

$$= -\sum_{\ell=1}^{\nu} \int_{\beta(Q^m)} v(y).\tau_{\alpha_\ell}\left(\frac{\partial\varphi}{\partial y_1}\right)(y)dy = -\int_{\beta(Q^m)} v(y).\varpi_1\left(\frac{\partial\varphi}{\partial y_1}\right)(y)dy.$$

(iii) By using arguments described in the proof of (ii), we have, on $\beta(\mathrm{Int}Q^m)$, the following equalities:

$$\varpi_1\left(\frac{\partial\varphi}{\partial y_1}\right) = \sum_{\ell=1}^{\nu}\tau_{\mu_\ell}\left(\frac{\partial\varphi}{\partial y_1}\right) = \sum_{\ell=1}^{\nu}\frac{\partial(\tau_{\mu_\ell}\varphi)}{\partial y_1} = \frac{\partial}{\partial y_1}\left(\sum_{\ell=1}^{\nu}\tau_{\mu_\ell}\varphi\right) = \frac{\partial\varpi_1(\varphi)}{\partial y_1}.$$

By using the continuity, the previous equality (between the extreme terms) holds on $\beta(Q^m)$, and by using the periodicity, it is true on $I\!R^m$. And so the first equality is proven.

For every $x \in I\!R^m$, we have:

$$\varpi_1(\varphi)\circ\beta(x) = \sum_{\alpha\in 2\pi\beta(\mathbb{Z}^m)}\tau_\alpha\varphi\circ\beta(x) = \sum_{\alpha\in 2\pi\beta(\mathbb{Z}^m)}\varphi(\beta(x)+\alpha)$$

$$= \sum_{\lambda\in 2\pi\mathbb{Z}^m}\varphi(\beta(x)+\beta(\lambda))$$

$$= \sum_{\lambda\in 2\pi\mathbb{Z}^m}\varphi\circ\beta(x+\lambda) = \sum_{\lambda\in 2\pi\mathbb{Z}^m}\tau_\lambda\varphi\circ\beta(x) = \varpi(\varphi\circ\beta)(x).$$

And so the second equality is proven.

By using successively (21) and the second equality, we have:

$$\frac{\partial\varpi_1(\varphi)}{\partial y_1}\circ\beta = |\omega|^{-1}\partial_\omega(\varpi_1(\varphi)\circ\beta) = |\omega|^{-1}\partial_\omega(\varpi(\varphi\circ\beta)).\blacksquare$$

Lemma 4. *Let $u \in H^1_\omega(\mathbb{T}^m, I\!R^N)$ and $v := u\circ\beta^{-1}$. Then we have $\frac{\partial v}{\partial y_1} \in L^2_{loc}(I\!R^m, I\!R^N)$ ($\frac{\partial v}{\partial y_1}$ is taken in the distributional sense), v and $\frac{\partial v}{\partial y_1}$ are periodic with $per(v) \supset 2\pi\beta(\mathbb{Z}^m)$, $per(\frac{\partial v}{\partial y_1}) \supset 2\pi\beta(\mathbb{Z}^m)$, and $\frac{\partial v}{\partial y_1}\circ\beta = |\omega|^{-1}\partial_\omega u$.*

Proof. We reason in the case $N = 1$; the general case is a simple consequence of this special case by working on the components.
By using Lemma 3, ii, when $\varphi \in \mathcal{D}(I\!R^m)$, we have:

$$\langle D_1 T_v, \varphi\rangle_{(m)} = -\int_{\beta(Q^m)} v(y)\frac{\partial\varpi_1(\varphi)}{\partial y_1}(y)dy.$$

Since β is orthogonal, by using the formula of the change of variable, we obtain:

$$\langle D_1 T_v, \varphi\rangle_{(m)} = -\int_{Q^m} v \circ \beta(x). \frac{\partial \varpi_1(\varphi)}{\partial y_1} \circ \beta(x) dx,$$

and by using Lemma 3, i, iii, we obtain:

$$\langle T_v, \varphi\rangle_{(m)} = -\int_{Q^m} u(x)|\omega|^{-1}\partial_\omega(\varpi(\varphi \circ \beta))(x)dx = |\omega|^{-1}\langle D_\omega T_u, \varphi \circ \beta\rangle_{(m)}$$

$$= |\omega|^{-1}\langle T_{\partial_\omega u}, \varphi \circ \beta\rangle_{(m)} = |\omega|^{-1}\int_{I\!R^m} \partial_\omega u(x).\varphi \circ \beta(x)dx$$

$$= |\omega|^{-1}\int_{I\!R^m} \partial_\omega u \circ \beta^{-1}(y).\varphi(y)dy = |\omega|^{-1}\langle T_{\partial_\omega u \circ \beta^{-1}}, \varphi\rangle_{(m)}.$$

Consequently, we obtain the following equality in $\mathcal{D}(I\!R^m)^*$:

$$D_1 T_v = |\omega|^{-1} T_{\partial_\omega u \circ \beta^{-1}}.$$

Since $\partial_\omega u \circ \beta^{-1} \in L^2_{loc}(I\!R^m, I\!R)$, we have:

$$\frac{\partial v}{\partial y_1} = D_1 T_v \in L^2_{loc}(I\!R^m, I\!R), \text{ and } \frac{\partial v}{\partial y_1} = |\omega|^{-1}\partial_\omega u \circ \beta^{-1}.$$

Since $\text{per}(u)$ and $\text{per}(\partial_\omega u)$ contain $2\pi Z\!\!\!Z^m$, the moduli $\text{per}(v)$ and $\text{per}(\frac{\partial v}{\partial y_1})$ contain $2\pi\beta(Z\!\!\!Z^m)$.$\blacksquare$

Theorem 3. *Let $u \in H^p_\omega(T\!\!\!T^m, I\!R^N)$. We note that, when $\xi \in I\!R^m$, we have $\mathcal{Q}_\omega(\tau_\xi u) = [t \mapsto u(t\omega + \xi)]$.*
Then the set $\omega^\perp \setminus \{\xi \in \omega^\perp : \mathcal{Q}_\omega(\tau_\xi u) \in H^p_{loc}(I\!R, I\!R^N)\}$ is Lebesgue negligible in $\omega^\perp$.

Proof.
Following the notations of Lemma 4, we set $v := u \circ \beta^{-1}$. The Lebesgue measure on $I\!R^\ell$ is denoted by μ_ℓ.

(i) the case $p = 1$**.** For each $n \in I\!N_*$, we set $I_n := (-n, n) \subset I\!R$, and for each $p \in I\!N_*$, we set $C_p := (-p, p)^{m-1} \subset I\!R^{m-1}$. For each $(n, p) \in I\!N^2_*$, we set

$\Omega_{p,n} := I_n \times C_p$. $\Omega_{p,n}$ is an open bounded subset of $I\!R^m$. The restrictions of v and $\frac{\partial v}{\partial y_1}$ belong to $L^2(\Omega_{p,n}, I\!R^N)$.
By using [31, pp.61-62] we can assert that there exists $A_{p,n} \subset C_p$ such that $\mu_{m-1}(A_{p,n}) = \mu_{m-1}(C_p)$, and such that for every $(y_2, ..., y_m) \in A_{p,n}$, we have

$$v(\cdot, y_2, ..., y_m)_{|I_n} \in H^1(I_n, I\!R^N).$$

We set:

$$A_p := \{(y_2, ..., y_m) \in C_p : \forall n \in I\!N_*, v(\cdot, y_2, ..., y_m)_{|I_n} \in H^1(I_n, I\!R^N)\}$$
$$= \{(y_2, ..., y_m) \in C_p : v(\cdot, y_2, ..., y_m) \in H^1_{loc}(I\!R, I\!R^N)\}.$$

Then we have $A_p = \bigcap_{n \geq 1} A_{p,n}$, and:

$$\mu_{m-1}(C_p \setminus A_p) = \mu_{m-1}\left(\bigcup_{n\geq 1}(C_p \setminus A_{p,n})\right) \leq \sum_{n=1}^{\infty} \mu_{m-1}(C_p \setminus A_{p,n}) = 0.$$

After that, we set:

$$A := \{(y_2, ..., y_m) \in I\!R^{m-1} : v(\cdot, y_2, ..., y_m) \in H^1_{loc}(I\!R, I\!R^N)\}.$$

We verify that $A = \bigcup_{p\geq 1} A_p$, and since $I\!R^{m-1} = \bigcup_{p \geq 1} C_p$, we have

$$I\!R^{m-1} \setminus (\bigcup_{p\geq 1} A_p) \subset \bigcup_{q\geq 1}(C_q \setminus A_q)$$

and therefore

$$\mu_{m-1}(I\!R^{m-1} \setminus A) \leq \mu_{m-1}\left(\bigcup_{q\geq 1} C_q \setminus A_q)\right) \leq \sum_{q=1}^{\infty} \mu_{m-1}(C_q \setminus A_q) = 0,$$

and consequently $I\!R^{m-1} \setminus A$ is Lebesgue negligible in $I\!R^{m-1}$.
We note that

$$\beta^{-1}(0 \times A) = \{\xi \in \omega^{\perp} : \mathcal{Q}_{\omega}(\tau_{\xi} u) \in H^1_{loc}(I\!R, I\!R^N)\}.$$

Since β is orthogonal, β preserves the Lebesgue measure, and so, the complement of $\beta^{-1}(0 \times A)$ in $\omega^{\perp} = \beta^{-1}(0 \times I\!R^m)$ is Lebesgue negligible in $\omega^{\perp}$.

(ii) The induction argument. We assume the property for 1 and $p-1$ ($p \geq 2$), and show it for p. We define the following sets:

$$N_1 := \{\xi \in \omega^\perp : \mathcal{Q}_\omega(\tau_\xi u) \in H^1_{loc}(I\!R, I\!R^N)\}$$

$$N_2 := \{\xi \in \omega^\perp : \mathcal{Q}_\omega(\tau_\xi(\partial^{p-1}_\omega u)) \in H^1_{loc}(I\!R, I\!R^N)\}.$$

Since $u \in H^p_\omega(T\!\!T^m, I\!R^N)$, we have $u, \partial^{p-1}_\omega u \in H^1_\omega(T\!\!T^m, I\!R^N)$, and by induction, we can say that $\omega^\perp \setminus N_1$ and $\omega^\perp \setminus N_2$ are Lebesgue negligible in $\omega^\perp$. Consequently, $\omega^\perp \setminus N_1 \cap N_2$ is Lebesgue negligible in $\omega^\perp$. But we have

$$N_1 \cap N_2 = \{\xi \in \omega^\perp : \mathcal{Q}_\omega(\tau_\xi u) \in H^p_{loc}(I\!R, I\!R^N)\}$$

since, from Theorem 1, ii, we have

$$\frac{d}{dt}\mathcal{Q}_\omega(\tau_\xi u) = \mathcal{Q}_\omega(\partial_\omega(\tau_\xi u)) = \mathcal{Q}_\omega(\tau_\xi(\partial_\omega u)).$$

Theorem 4. *If we assume that $u \in H^p_\omega(T\!\!T^m, I\!R^N) \cap L^\infty(T\!\!T^m, I\!R^N)$, then the set: $\omega^\perp \setminus \{\xi \in \omega^\perp : \mathcal{Q}_\omega(\tau_\xi u) \in H^p_{loc}(I\!R, I\!R^N), \sup_{t \in I\!R} |u(t\omega + \xi)| \leq \|u\|_{L^\infty}\}$ is Lebesgue negligible in $\omega^\perp$.*

Proof. We use the notations of the previous proof.
(i) The case $p = 1$. We set

$$M := \{(y_1, y_2, ..., y_m) \in I\!R \times I\!R^{m-1} : |v(y_1, y_2, ..., y_m)| \leq \|u\|_{L^\infty}\}.$$

We have $\mu_m(M^c) = 0$. Since $\mu_m = \mu_1 \otimes \mu_{m-1}$, by using the Fubini theorem [22, pp.147-148], there exists a subset $B_1 \subset I\!R^{m-1}$ such that $I\!R^{m-1} \setminus B_1$ is Lebesgue negligible in $I\!R^{m-1}$, and such that for every $(y_2, ..., y_m) \in B_1$, the complement of $\{y_1 \in I\!R : (y_1, y_2, ..., y_m) \in M\}$ is Lebesgue negligible in $I\!R$. And so for every $(y_2, ..., y_m) \in B_1$, we have $|v(\cdot, y_2, ..., y_m)| \leq \|u\|_{L^\infty}$ L.a.e. on $I\!R$. Therefore the complement of $A \cap B_1$ is Lebesgue negligible in $I\!R^{m-1}$. Furthermore, when $(y_2, ..., y_m) \in A \cap B_1$, we have $v(\cdot, y_2, ..., y_m) \in C^0(I\!R, I\!R^N)$, and therefore for every $y_1 \in I\!R$, we have $|v(y_1, y_2, ..., y_m)| \leq \|u\|_{L^\infty}$. Since

$$\beta^{-1}(0 \times (A \cap B_1)) =$$

$$\{\xi \in \omega^\perp : \mathcal{Q}_\omega(\tau_\xi u) \in H^1_{loc}(I\!R, I\!R^N), \sup_{t \in I\!R} |u(t\omega + \xi)| \leq \|u\|_{L^\infty}\}$$

we obtain the announced result.

(ii) By using (i) and Theorem 3, the complement of $A \cap B_1 \cap (N_1 \cap N_2)$ in $I\!R^{m-1}$ is Lebesgue negligible in $I\!R^{m-1}$. To conclude, it is sufficient to note that:

$$\beta^{-1}(0 \times [A \cap B_1 \cap (N_1 \cap N_2)]) =$$
$$\{\xi \in \omega^{\perp} : \mathcal{Q}_\omega(\tau_\xi u) \in H^p_{loc}(I\!R, I\!R^N),\ \sup_{t \in I\!R} |u(t\omega + \xi)| \leq \|u\|_{L^\infty}\}. \blacksquare$$

6 Relations between (2) and (3).

We assume that $F \in QPU_\omega(I\!R, I\!R^{pN}, I\!R^N)$. Thus $\Phi := \mathcal{Q}_\omega^{-1}(F) \in C^0(T\!\!T^m \times I\!R^{(p+1)N}, I\!R^N)$ and that F satisfies (1). Φ satisfies the same Lipschitz condition. From this we build two Nemytskii operators:

$$\mathcal{N}_{F,1} : (B^2_\omega(I\!R^N))^p \longrightarrow (I\!R^N)^{I\!R},\ \ \mathcal{N}_{F,1}(q) := [t \mapsto F(t, q(t))].$$

$$\mathcal{N}_{\Phi,2} : (L^2(T\!\!T^m, I\!R^N))^p \longrightarrow (I\!R^N)^{T\!\!T^m},\ \ \mathcal{N}_{\Phi,2}(u) := [x \mapsto \Phi(x, u(x))].$$

Lemma 5. *Under (1), the following assertions hold.*

(i) $\mathcal{N}_{F,1}((B^2_\omega(I\!R^N))^p) \subset B^2_\omega(I\!R^N)$, *and* $\mathcal{N}_{F,1} \in C^0((B^2_\omega(I\!R^N))^p, B^2_\omega(I\!R^N))$.

(ii) $\mathcal{N}_{\Phi,2}((L^2(T\!\!T^m, I\!R^N))^p) \subset L^2(T\!\!T^m, I\!R^N)$, *and* $\mathcal{N}_{\Phi,2} \in C^0((L^2(T\!\!T^m, I\!R^N))^p), L^2(T\!\!T^m, I\!R^N))$.

(iii) *We consider* $\mathcal{Q}_\omega : (L^2(T\!\!T^m, I\!R^N))^j \longrightarrow B^2_\omega(I\!R^N)$ *for* $j = 1, p$. *Then we have:*

$$\mathcal{Q}_\omega \circ \mathcal{N}_{\Phi,2} = \mathcal{N}_{F,1} \circ \mathcal{Q}_\omega.$$

Proof.
(i) By using [13,Theorem 2], we have $\mathcal{N}_{F,1}((B^2(I\!R^N))^p) \subset B^2(I\!R^N)$, and $\mathcal{N}_{F,1} \in C^0((B^2(I\!R^N))^p, B^2(I\!R^N))$. Since $\mathcal{N}_{F,1}((QP^0_\omega(I\!R^N))^p) \subset QP^0_\omega(I\!R^N)$ [9, Proposition 3], we obtain the announced results.

(ii) cf. [27, Chapter I, Section 2].

(iii) If $u \in C^0(\mathbb{T}^m; \mathbb{R}^N)^p$, we have:

$$\mathcal{Q}_\omega \circ \mathcal{N}_{\Phi,2}(u) = \mathcal{Q}_\omega[x \mapsto \Phi(x, u(x))] = [t \mapsto \Phi(t\omega, u(t\omega))] = [t \mapsto F(t, u(t\omega))]$$

and:

$$\mathcal{N}_{F,1} \circ \mathcal{Q}_\omega(u) = \mathcal{N}_{F,1}[t \mapsto u(t\omega)] = [t \mapsto F(t, u(t\omega))]$$

and so $\mathcal{Q}_\omega \circ \mathcal{N}_{\Phi,2} = \mathcal{N}_{F,1} \circ \mathcal{Q}_\omega$ on $C^0(\mathbb{T}^m; \mathbb{R}^N)^p$. Since $C^0(\mathbb{T}^m; \mathbb{R}^N)^p$ is dense on $L^2(\mathbb{T}^m; \mathbb{R}^N)^p$ and since $\mathcal{Q}_\omega$, $\mathcal{N}_{\Phi,2}$ and $\mathcal{N}_{F,1}$ are continuous respectively on $L^2(\mathbb{T}^m; \mathbb{R}^N)^j$ $(j = 1, p)$, $L^2(\mathbb{T}^m; \mathbb{R}^N)^p$ and $B^2_\omega(\mathbb{R}^N)$, we can conclude.$\blacksquare$

Proposition 7. *Let $F \in QPU_\omega(\mathbb{R}, \mathbb{R}^{pN}, \mathbb{R}^N)$ and $q \in QP^p_\omega(\mathbb{R}^N)$. If q is a q.p. solution of (2), then $u := \mathcal{Q}_\omega^{-1}(q) \in C^p_\omega(\mathbb{T}^m, \mathbb{R}^N)$ is a periodic solution of (3).*

Proof. Since for every $t \in \mathbb{R}$, we have $q^{(p)}(t) = F(t, q(t), ..., q^{(p-1)}(t))$, we also have

$$\frac{d^p}{dt^p} u(t\omega) = \Phi(t\omega, u(t\omega), ..., \frac{d^{p-1}}{dt^{p-1}} u(t\omega))$$

i.e.

$$\partial^p_\omega u(t\omega) = \Phi(t\omega, u(t\omega), ..., \partial^{p-1}_\omega u(t\omega)).$$

Since $u, ..., \partial^p_\omega u$ and $\Phi(\cdot, \alpha_1, ..., \alpha_p)$ are continuous and periodic on $\mathbb{R}^m$, by using Lemma 2, we obtain, for every $x \in \mathbb{R}^m$, $\partial^p_\omega u(x) = \Phi(x, u(x), ..., \partial^{p-1}_\omega u(x))$, i.e. u is a periodic solution of (2).$\blacksquare$

Proposition 8. *Under (1), let $q \in B^{p,2}_\omega(\mathbb{R}^N)$. If q is a weak q.p. solution of (2) (i.e. q is a solution of (4)), then $u := \mathcal{Q}_\omega^{-1}(q) \in H^p_\omega(\mathbb{T}^m, \mathbb{R}^N)$ is a weak periodic solution of (3) (i.e. u is a solution of (5)).*

Proof. By using Theorem 2, we can assert that $\nabla^p q \sim_2 F(., q, ..., \nabla^{p-1} q)$ implies $\mathcal{Q}_\omega^{-1}(\nabla^p q) = \mathcal{Q}_\omega^{-1}(F(., q, ..., \nabla^{p-1} q))$ in $L^2(\mathbb{T}^m, \mathbb{R}^N)$. Since $\mathcal{Q}_\omega^{-1}(\nabla^p q) = \partial^p_\omega u$ and since $\mathcal{Q}_\omega^{-1}(F(., q, ..., \nabla^{p-1} q)) = \mathcal{Q}_\omega^{-1}(\mathcal{N}_{F,1}(q, ..., \nabla^{p-1} q)) = \mathcal{N}_{\Phi,2} \circ \mathcal{Q}_\omega^{-1}(q, ..., \nabla^{p-1} q)$ (Lemma 5, iii), we have $\partial^p_\omega u = \Phi(., u, ..., \partial^{p-1}_\omega u)$.$\blacksquare$

Proposition 9. *Let $\Phi \in C^0(\mathbb{T}^m \times \mathbb{R}^{pN}, \mathbb{R}^N)$ and let $u \in C^p_\omega(\mathbb{T}^m, \mathbb{R}^N)$. If u is a periodic solution of (3), then $q = \mathcal{Q}_\omega(u) \in QP^p_\omega(\mathbb{R}^N)$ is a q.p. solution of (2).*

Proof. Necessarily we have, for every $t \in I\!R$, $\partial_\omega^p u(t\omega) = F(t\omega, u(t\omega), ..., \partial_\omega^{p-1} u(t\omega))$, and consequently we have $q^{(p)}(t) = F(t, q(t), ..., q^{(p-1)}(t))$.■

Theorem 5. *Under (1), let $u \in H^p_\omega(\mathbf{T}^m, I\!R^N)$. We assume that u is a weak periodic solution of (3) (i.e. u is a solution of (5)). We set $q := \mathcal{Q}_\omega(u) \in B^{p,2}_\omega(I\!R^N)$. Then the following assertions hold.*

(i) *q is a weak q.p. solution of (2), i.e. q is a solution of (4).*

(ii) *There exists a subset Ξ of $\omega^\perp$ such that the complement of Ξ in $\omega^\perp$ is Lebesgue negligible in $\omega^\perp$, and such that, for every $\xi \in \Xi$, the function*

$$\mathcal{Q}_\omega(\tau_\xi u) = [t \mapsto u(t\omega + \xi)] \in C^p(I\!R, I\!R^N) \cap B^{p,2}_\omega(I\!R^N)$$

and satisfies, for every $t \in I\!R$,

$$\frac{d^p}{dt^p} u(t\omega + \xi) = \Phi\left(t\omega + \xi, u(t\omega + \xi), ..., \frac{d^{p-1}}{dt^{p-1}} u(t\omega + \xi)\right).$$

Proof
(i) Since $\partial_\omega^p u = \Phi(\cdot, u, ..., \partial_\omega^{p-1} u)$ in $L^2(I\!\!T^m, I\!R^N)$, by using Theorem 2, iii, we have $\mathcal{Q}_\omega(\partial_\omega^p u) \sim_2 \mathcal{Q}_\omega(\Phi(\cdot, u, ..., \partial_\omega^{p-1} u))$. Since $\mathcal{Q}_\omega(\partial_\omega^p u) = \nabla^p(\mathcal{Q}_\omega u)$ and since $\mathcal{Q}_\omega(\Phi(\cdot, u, ..., \partial_\omega^{p-1} u)) = \mathcal{Q}_\omega \circ \mathcal{N}_{\Phi,2}(u) = \mathcal{N}_{F,1} \circ \mathcal{Q}_\omega(u, ..., \partial_\omega^{p-1} u) = F(q, ..., \nabla^{p-1} q)$ (Lemma 5, iii), we obtain $\nabla^p q \sim_2 F(q, ..., \nabla^{p-1} q)$.

(ii) We define Ψ by the formula:

$$\forall (y, \alpha_1, ..., \alpha_p) \in \beta(I\!R^m) \times I\!R^{pN}, \quad \Psi(y, \alpha_1, ..., \alpha_p) := \Phi(\beta^{-1}(y), \alpha_1, ..., \alpha_p)$$

and we take $v := u \circ \beta^{-1}$ like in the proof of Theorem 2. We set:

$$N := \{(y_2, ..., y_m) \in I\!R^{m-1} : v(\cdot, y_2, ..., y_m) \in H^p_{loc}(I\!R, I\!R^N)\}.$$

By using Theorem 3, $I\!R^{m-1} \setminus N$ is Lebesgue negligible in $I\!R^{m-1}$.
We know that $\frac{\partial^p v}{\partial y_1} \in L^2_{loc}(I\!R^m, I\!R^N)$ and also that $\mathcal{N}_{\Psi,2}(v, ..., \frac{\partial^p v}{\partial y_1^p}) \in L^2_{loc}(I\!R^m, I\!R^N)$.
Since u is a solution of (5), we have $\frac{\partial^p v}{\partial y_1^p} = \Psi(\cdot, v, ..., \frac{\partial^{p-1} v}{\partial y_1^{p-1}})$ L.a.e. on $I\!R^m$.

By using the Fubini theorem [22, pp.147-148] we can assert that $I\!R^{m-1} \setminus E_1$ is Lebesgue negligible in $I\!R^{m-1}$, where

$$E_1 := \left\{ (y_2, ..., y_m) \in I\!R^{m-1} : \frac{\partial^p v}{\partial y_1^p}(\cdot, y_2, ..., y_m)) = \right.$$

$$\left. \Psi \left((\cdot, y_2, ..., y_m), v(\cdot, y_2, ..., y_m), ..., \frac{\partial^{p-1} v}{\partial y_1^{p-1}}(\cdot, y_2, ..., y_m) \right) \text{ L.a.e. on } I\!R \right\}.$$

By using the transformation β, we obtain that $\omega^{\perp} \setminus E$ is Lebesgue negligible in $\omega^{\perp}$, where

$$E := \{\xi \in \omega^{\perp} : \partial_{\omega}^p u(t\omega + \xi) = \Phi(t\omega + \xi, u(t\omega + \xi), ..., \partial_{\omega}^{p-1} u(t\omega + \xi))$$
$$\text{for L.a.e. } t \in I\!R\},$$

since $\beta(\omega^{\perp} \setminus E) = I\!R^{m-1} \setminus E_1$.
Consequently, $\omega^{\perp} \setminus (E \cap N)$ is Lebesgue negligible in $\omega^{\perp}$, and we note that the assertion $\xi \in E \cap N$ means that we have simulteanously:

$$\partial_{\omega}^p u(t\omega + \xi) = \Phi(t\omega + \xi, u(t\omega + \xi), ..., \partial_{\omega}^{p-1} u(t\omega + \xi)) \text{ for L.a.e. } t \in I\!R,$$

and

$$[t \longmapsto u(t\omega + \xi)] \in H_{loc}^p(I\!R, I\!R^N),$$

therefore since Φ is continuous, by applying the usual technic of regularization of the absolutely continuous solutions of the ordinary differential equations, we can assert that $[t \longmapsto u(t\omega + \xi)] \in C^p(I\!R, I\!R^N)$ and that, for every $t \in I\!R$, we have

$$\frac{d^p}{dt^p} u(t\omega + \xi) = \Phi \left(t\omega + \xi, u(t\omega + \xi), ..., \frac{d^{p-1}}{dt^{p-1}} u(t\omega + \xi) \right).$$

And so, it is sufficient to take $\Xi := E \cap N$.$\blacksquare$

Comments. This Theorem show that there exists a particular perturbated form of equation whose solutions are regular.

Theorem 6. *We assume the hypothesis on Theorem 5 and we also assume that* $u \in L^{\infty}(T\!\!T^m, I\!R^N)$ *and that for all compact* $K \subset I\!R^N$,

$$M_K := \sup_{(t, \alpha_1, ..., \alpha_p) \in I\!R \times K \times I\!R^{(p-1)N}} |F(t, \alpha_1, ..., \alpha_p)| < \infty.$$

Then $q \in BC^p(I\!R, I\!R^N) \cap B^{p,2}_{\omega}(I\!R^N)$ *and satisfies, for every* $t \in I\!R$,

$$q^{(p)}(t) = F(t, q(t), ..., q^{(p-1)}(t)).$$

Proof. We set

$$\Xi_1 := \{\xi \in \Xi : |u(t\omega + \xi)| \leq \|u\|_{L^\infty} \text{ for L.a.e. } t \in I\!R\}.$$

By using Theorem 5 and Theorem 2 , iv, we can assert that $\omega^\perp \setminus \Xi_1$ is Lebesgue negligible in $\omega^\perp$, as the union of two Lebesgue negligible subsets of $\omega^\perp$. Consequently Ξ_1 is dense in $\omega^\perp$, and we can choose a sequence $(\xi_n)_n$ with values in Ξ_1 such that $\xi_n \to 0$ $(n \to \infty)$. We set $q_n := \mathcal{Q}_\omega(\tau_{\xi_n} u)$.
For each $n \in I\!N$, we have $\|q_n\|_\infty \leq \|u\|_{L^\infty}$, and since (cf. Theorem 5, ii):

$$q_n^{(p)} = \Phi(\cdot\omega + \xi_n, q_n, ..., q_n^{(p-1)}),$$

we have:

$$\left\|q_n^{(p)}\right\|_\infty \leq M_{\|u\|_{L^\infty}} < \infty.$$

By using [1 p.265, example 2], there exits $M > 0$ such that for every $n \in I\!N$ and for every $j \in \{0, ..., p\}$, we have:

$$\left\|q_n^{(j)}\right\|_\infty \leq M.$$

Consequently, by using the Ascoli-Arzela theorem [14, p.X.18], we can tell that there exists $\rho \in C^1(I\!R, I\!R^N)$, and $n \mapsto j_n$, a monotonically increasing function from $I\!N$ in $I\!N$, such that for every $j \in \{0, ..., p-1\}$, $(q_n^{(j)})_n$ converges to $\rho^{(j)}$ in the sense of the compact convergence.
Since the compact convergence implies the pointwise convergence, and since $u(t\omega + \xi_n) \to u(t\omega)$ when $n \to \infty$ for each $t \in I\!R$, by using the uniqueness of the limit, we have $\rho^{(j)} = q^{(j)}$ for every $j \in \{0, ..., p-1\}$.
Finally, for every $s < t$ in $I\!R$, we have:

$$q_{j_n}^{(p-1)}(t) - q_{j_n}^{(p-1)}(s) = \int_s^t \{\Phi(\sigma\omega + \xi_{j_n}, q_{j_n}(\sigma), ..., q_{j_n}^{(p-1)}(\sigma)\} d\sigma,$$

and when $n \to \infty$, we obtain:

$$q^{(p-1)}(t) - q^{(p-1)}(s) = \int_s^t \{F(\sigma, q(\sigma), ..., q^{(p-1)}(\sigma))\} d\sigma,$$

that implies: $q^{(p)}(t) = F(t, q(t), ..., q^{(p-1)}(t))$.$\blacksquare$

Conclusion

In the quasi-periodic setting, by using the $B^{p,2}$ spaces, Theorem 6 ensures that any weak q.p. solution (obtained for instance by using a variational method) which is also essentially bounded is in fact a solution in the usual sense and is also Besicovich q.p. And so, Theorem 6 provides an improvement to the approach via $B^{p,2}$ spaces.
Also, Theorem 5 provides an improvement, specific to the quasi-periodic setting, to the results, called results in density, cited in introduction: if for instance L is not depending on t, the b_ε can be choosen of the form $b_\varepsilon(\cdot) = b(\cdot\omega + \xi(\varepsilon))$, where $\xi(\varepsilon) \perp \omega$ and $\xi(\varepsilon) \to 0$ when $\varepsilon \to 0$.

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