On Stepanov almost-periodic oscillations and their discretizations

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Abstract. The relationship between Carathéodory almost-periodic solutions and their discretizations is clarified for differential equations and inclusions in Banach spaces. Our investigation was stimulated by an old result of G. H. Meisters [30] about Bohr almost-periodic solutions which we generalize in several directions. Unlike for functions, Stepanov and Bohr almost-periodic sequences are shown to coincide. A particular attention is paid to purely (i.e. non-uniformly continuous) Stepanov almost-periodic solutions. Many ideas are explained in detail by means of illustrating examples.

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1. Introduction

Already a half century ago, G. H. Meisters [30] formulated the following interesting theorem.

THEOREM 1. *Let $x(\cdot)$ be a vector solution of the differential equation*

$$x' = F(t, x), \tag{1}$$

where

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(i) $F(\cdot, x)\colon \mathbb{R} \to \mathbb{C}^n$ is an almost-periodic function, for each x in some connected open subset D of $\mathbb{C}^n$, where $\mathbb{C}^n$ is an n-dimensional complex vector space, endowed with the standard Euclidean metric,

(ii) $F(t, \cdot)\colon \mathbb{C}^n \to \mathbb{C}^n$ satisfies, for all $x, y \in D$, the Lipschitz condition

$$|F(t,x) - F(t,y)| \leq L|x-y|, \quad t \in \mathbb{R},$$

and let D contain the closure of the range of $x(\cdot)$. Then a necessary and sufficient condition that $x(\cdot)$ be an almost-periodic solution in $\mathbb{C}^n$ is that $\{x(k)\}_{k\in\mathbb{Z}}$ be an almost-periodic sequence in $\mathbb{C}^n$.

The main aim of the present paper is to extend and improve Theorem 1 in several directions.

The Stepanov almost-periodicity [37] is the most natural generalization of the notion of a uniform (Bohr) almost-periodicity, mainly because of two reasons: (i) uniformly continuous Stepanov almost-periodic functions are exactly uniformly almost-periodic ones (cf. [2, 5, 10, 12, 29]), (ii) the Bochner transform of Stepanov almost-periodic functions becomes also uniformly almost-periodic (cf. [1, 3, 32]).

On the other hand, unlike for functions, the notions of Stepanov and uniformly almost-periodic sequences rather surprisingly coincide (see Section 3). Moreover, discretizations of even smooth Stepanov almost-periodic functions need not be almost-periodic (see Example 4 below) and, reversely, not necessarily Stepanov almost-periodic functions can obviously admit almost-periodic discretizations.

Although differential equations involving Stepanov almost-periodic forcing terms are usually examined in order to possess uniformly almost-periodic Carathéodory (i.e. locally absolutely continuous) solutions (see e.g. [1, 3, 11, 32, 35]), it is a question whether or not it has a meaning to consider purely (i.e. not uniformly continuous) Stepanov almost-periodic solutions (see examples in Section 4). The problem of purely Stepanov almost-periodic solutions of difference equations has however, in view of the above arguments, no meaning.

Hence, the first candidate for a generalization of Theorem 1 might seem to be, in equation (1), an essentially bounded Stepanov almost-periodic mapping $F(\cdot, x)\colon \mathbb{R} \to \mathbb{C}^n$, for each x in a suitable connected open subset D of $\mathbb{C}^n$. The related result is formulated in Theorem 3.

If equation (1) takes the particular additive form (4), then Theorem 3 can be significantly improved into Theorem 2, where $\mathbb{C}^n$ can be even replaced by a suitable (e.g. reflexive) possibly infinite-dimensional Banach space and p need not be essentially bounded. For finite-dimensional Banach spaces, Theorem 2 can be still formally extended to inclusions (9) in Corollary 1.

The delicate problem of purely Stepanov almost-periodic solutions was already addressed in the series of papers [17, 18, 19, 20, 21] of

Z. Hu and A. B. Mingarelli. However, under their conditions, Stepanov almost-periodic solutions become uniformly almost-periodic (cf. also the arguments in [38] concerning the Favard type results in [18, 19, 20]). Here, purely Stepanov almost-periodic solutions with Stepanov derivatives will be, rather curiously, shown to exist only provided none of their discretizations is (Stepanov) almost-periodic. In other words, if at least one discretized solution is (Stepanov) almost-periodic, then there is no chance to obtain purely Stepanov almost-periodic solutions. In [4], we proved accordingly that, under natural assumptions, purely Stepanov almost-periodic solutions do not occur.

2. Some facts about Stepanov almost-periodic (shortly, S_{ap}-) functions

For locally integrable functions $f, g \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$, let us recall the *Stepanov norms* and *distances*:

$$\|f\|_{\mathrm{S}^p_{\mathrm{L}}} := \sup_{x\in\mathbb{R}} \left[\frac{1}{L}\int_x^{x+L} |f(t)|^p\,dt\right]^{\frac{1}{p}},$$

$$\mathrm{D}_{\mathrm{S}^p_{\mathrm{L}}}[f,g] := \|f-g\|_{\mathrm{S}^p_{\mathrm{L}}} = \sup_{x\in\mathbb{R}} \left[\frac{1}{L}\int_x^{x+L} |f(t)-g(t)|^p\,dt\right]^{\frac{1}{p}}.$$

Without any loss of generality, one can take $L = 1$, i.e. we can work with the Stepanov norms $\|\cdot\|_{\mathrm{S}^p} = \|\cdot\|_{\mathrm{S}^p_1}$ and distances $\mathrm{D}_{\mathrm{S}^p}[\cdot,\cdot] = \mathrm{D}_{\mathrm{S}^p_1}[\cdot,\cdot]$. For $p = 1$, we shall simply write $\|\cdot\|_{\mathrm{S}}$ $(= \|\cdot\|_{\mathrm{S}^1})$ and $\mathrm{D}_{\mathrm{S}}[\cdot,\cdot]$ $(= \mathrm{D}_{\mathrm{S}^1}[\cdot,\cdot])$.

Definition 1. A function $f \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$ is said to be *almost-periodic in the sense of Stepanov* ($\mathrm{S}^p_{\mathrm{ap}}$) if, for every $\varepsilon > 0$, there corresponds a relatively dense set $\{\tau\}_\varepsilon$ of ε-Stepanov almost-periods such that

$$\mathrm{D}^p_{\mathrm{S}}[f(t+\tau), f(t)] < \varepsilon, \quad \text{for all } \tau \in \{\tau\}_\varepsilon.$$

In Banach spaces, the definition of $\mathrm{S}^p_{\mathrm{ap}}$-functions is formally the same as in Definition 1.

The Banach space of S_{ap}-functions can be equivalently defined as the one of S^p-normal functions or as the closure of all trigonometric polynomials w.r.t to the norm $\|\cdot\|_{\mathrm{S}^p}$ (for more details, see e.g. [2, 3]).

According to the Bochner theorem (see e.g. [5]), uniformly (Bohr) almost-periodic (shortly, a.p.) functions can be characterized as entirely uniformly continuous $\mathrm{S}^p_{\mathrm{ap}}$-functions, where $p \geq 1$.

It is well-known (see e.g. [7, 16, 27, 28]) that the following *Bohl–Bohr theorem* holds for the indefinite integral $F(x) = \int_0^x f(t)\,dt$ of an S_{ap}-function f. F is namely a.p., provided it is S^p-bounded (i.e. bounded w.r.t. the norm $\|\cdot\|_{\mathrm{S}^p}$). Thus, in particular, such an F is a.p. iff it is bounded. In a Banach space, according to the Kadets theorem [26], it should not contain an isomorphic copy of c_0, the scalar null sequences, for the same goal. For further generalizations, see e.g. [6, 16].

On the other hand, unlike for periodic functions, the indefinite integrals of a.p. functions with a zero mean value can be unbounded (see e.g. [23, 24]).

Furthemore, the sum of two $\mathrm{S}^p_{\mathrm{ap}}$-functions is an $\mathrm{S}^p_{\mathrm{ap}}$-function and the product of an $\mathrm{S}^p_{\mathrm{ap}}$-function and an $\mathrm{S}^q_{\mathrm{ap}}$-function, where $\frac{1}{p}+\frac{1}{q}=1$, is an S_{ap}-function. For $p=1$, in particular, the product of an S_{ap}-function and an a.p. function is an S_{ap}-function. If an a.p. function g satisfies $\inf_{x\in\mathbb{R}}|g(x)|>0$, then $\frac{1}{g}$ is a.p., and subsequently the quotient f/g of two a.p. functions f and g, where g has the same property, is a.p. For more details, see e.g. [5, 10, 12, 29].

The composition $F(f(\cdot))$ of a continuous and linearly bounded function F and an S_{ap}-function f is again S_{ap}. More generally, if $f\colon\mathbb{R}\to X$ is an S_{ap}-function, where X is a complete metric space, and $F\colon X\to Y$ is a continuous linearly bounded mapping, where Y is a metric space, then $F(f(\cdot))\colon\mathbb{R}\to Y$ is an S_{ap}-function (see e.g. [13]). On the other hand, in order $F(f(\cdot))$ to be an a.p. function, where f is a.p., F should be uniformly continuous (see e.g. [10]).

It immediately follows from the Bohl–Bohr theorem mentioned above that *there is no bounded* S_{ap}*-function, which is at the same time not a.p., such that its derivative is an (unbounded)* S_{ap}*-function.* This implication has two important consequences.

CONSEQUENCE 1. *Derivative of any bounded* S_{ap} *(but not a.p.)-function cannot be* S_{ap}.

CONSEQUENCE 2. *In order to have an* S_{ap}*-derivative of an* S_{ap}*-function, such function must be either unbounded or a.p.*

EXAMPLE 1 (*of a smooth unbounded* S_{ap}*-function* f *with an unbounded derivative* f'*; cf.* [21]). Modifying an example in [39], the authors of [21] constructed a C^1-Stepanov a.p. function f which is obviously not a.p., as a series of $2n$-periodic functions f_n, as follows[2]:

$$f(x)=\sum_{n=1}^{\infty}f_n(x),$$

where

$$f_n(x)=\begin{cases}\frac{2(x-y_{n,k}+\varepsilon_n)^2}{\varepsilon_n^2}, & \text{for } x\in(y_{n,k}-\varepsilon_n,y_{n,k}-\frac{\varepsilon_n}{2}],\\ 1-\frac{2(x-y_{n,k})^2}{\varepsilon_n^2}, & \text{for } x\in(y_{n,k}-\frac{\varepsilon_n}{2},y_{n,k}+\frac{\varepsilon_n}{2}],\\ \frac{2(x-y_{n,k}-\varepsilon_n)^2}{\varepsilon_n^2}, & \text{for } x\in(y_{n,k}+\frac{\varepsilon_n}{2},y_{n,k}+\varepsilon_n],\\ 0, & \text{otherwise,}\end{cases}$$

$$y_{n,k}=(2k+1)n,\quad k=0,\pm1,\pm2,\dots,$$

[2]In [21], it was assumed the less restrictive condition: $0<\varepsilon_n<1,\quad n=1,2,\dots,\quad\sum_{n=1}^{\infty}\varepsilon_n<\infty$.

$$0 < \varepsilon_n < \frac{1}{2}, \quad n = 1, 2, \dots, \quad \sum_{n=1}^{\infty} \varepsilon_n < \infty.$$

One can readily check that

$$f'(x) = \sum_{n=1}^{\infty} f'_n(x),$$

where

$$f'_n(x) = \begin{cases} \frac{4(x-y_{n,k}+\varepsilon_n)}{\varepsilon_n^2}, & \text{for } x \in (y_{n,k} - \varepsilon_n, y_{n,k} - \frac{\varepsilon_n}{2}], \\ -\frac{4(x-y_{n,k})}{\varepsilon_n^2}, & \text{for } x \in (y_{n,k} - \frac{\varepsilon_n}{2}, y_{n,k} + \frac{\varepsilon_n}{2}], \\ \frac{4(x-y_{n,k}-\varepsilon_n)}{\varepsilon_n^2}, & \text{for } x \in (y_{n,k} + \frac{\varepsilon_n}{2}, y_{n,k} + \varepsilon_n], \\ 0, & \text{otherwise.} \end{cases}$$

EXAMPLE 2 (*of a smooth bounded non-uniformly continuous* $\mathrm{S_{ap}}$*-function whose derivative* f' *is not* $\mathrm{S_{ap}}$*; cf.* [29, pp. 212–213]). Taking

$$f(x) = \sin\left(\frac{1}{g(x)}\right),$$

where

$$g(x) = 2 + \cos x + \cos\sqrt{2}x,$$

Levitan [29] had shown that f is $\mathrm{S_{ap}}$, but not a.p.

Since the derivative f' of f takes the form

$$f'(x) = \cos\left(\frac{1}{g(x)}\right)\left(\frac{\sin x + \sqrt{2}\sin\sqrt{2}x}{g^2(x)}\right),$$

it follows from the Bohl–Bohr theorem that f' cannot be $\mathrm{S_{ap}}$, because f as its primitive is bounded, and so contradictionally a.p. (cf. Consequence 1).

Function $\cos\left(\frac{1}{g(x)}\right)$ can be proved quite analogously as $\sin\left(\frac{1}{g(x)}\right)$ to be an $\mathrm{S_{ap}}$-function (cf. [29]).

EXAMPLE 3 (*of a smooth a.p. function* h *whose non-uniformly continuous derivative* h' *is* $\mathrm{S_{ap}}$). Let g be the same function as in Example 2. The product

$$h(x) = g^2(x)\sin\left(\frac{1}{g(x)}\right)$$

of g^2 and f is obviously a continuous bounded $\mathrm{S_{ap}}$-function and, unlike f' in Example 2, so is its derivative

$$h'(x) = g'(x)\left[2g(x)\sin\left(\frac{1}{g(x)}\right) - \cos\left(\frac{1}{g(x)}\right)\right],$$

as a product of

$$g'(x) = -\sin x - \sqrt{2}\sin\sqrt{2}x$$

and a continuous bounded $\mathrm{S_{ap}}$-function

$$2g(x)\sin\left(\frac{1}{g(x)}\right)-\cos\left(\frac{1}{g(x)}\right).$$

This, however, means that h itself is in fact a.p.

3. Some facts about (Stepanov) almost-periodic sequences

In this section, we will firstly show that the notions of uniformly almost-periodic (a.p.) and Stepanov almost-periodic ($\mathrm{S_{ap}}$) sequences rather surprisingly coincide. We also collect some properties of (Stepanov) a.p. sequences.

Let $\mathbb{Z}$ denote, as usual, the set of integers and let E be a Banach space endowed with the norm $|\cdot|_E$. Taking $\underline{x} := \{x_k\}_{k\in\mathbb{Z}} \in E^{\mathbb{Z}}$, we start with the definition of an a.p. sequence (cf. [8, 10, 12, 33]).

Definition 2. A sequence $\underline{x}$ is called *almost-periodic* (a.p.) if, for any $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon) \in \mathbb{N}$ such that, in any set of N consecutive integers $\{m,\dots,m+N\}$, there exists $p \in \{m,\dots,m+N\}$ such that $|x_{k+p}-x_k|_E < \varepsilon$, for all $k \in \mathbb{Z}$. The number p is called an ε-*almost-period* of $\underline{x}$.

Defining the continuous piece-wise linear function $f_{\underline{x}}\colon \mathbb{R}\to E$ by $f_{\underline{x}}(k+\theta) := x_k + \theta(x_{k+1}-x_k)$, for all $k\in\mathbb{Z}$, $\theta\in[0,1]$, we know that the following properties are equivalent:

- $\underline{x}$ is an a.p. sequence;
- the *canonical extension* $f_{\underline{x}}$ of $\underline{x}$ is an a.p. function (cf. [8, 33]);
- there exists an a.p. function $f\colon\mathbb{R}\to E$ such that $f(k) = x_k$, for all $k\in\mathbb{Z}$ (cf. [8, 12, 33]);
- for $E=\mathbb{R}$, any sequence of translates $\{x_k+m_\ell\}_{\ell\in\mathbb{N}}$ contains a subsequence $\{x_k+m_{\ell_l}\}$ which is uniformly (w.r.t. $k\in\mathbb{Z}$) convergent (cf. [10, 12]).

By a *discretization* of a function f, we mean in the entire text a sequence $\{f(ak+b)\}_{k\in\mathbb{Z}}$, $a>0$. By *standard discretizations*, we mean those with $a=1$, $b=0$. One can always find a set of integer ε-translation numbers for an a.p. function (see [5]). Every set of integer ε-translation numbers of a fucntion is a set of integer ε-translation numbers of its standard discretization. Moreover, every set of (integer) ε-translation numbers for $\underline{x}$ is a set of ε-translation numbers for $f_{\underline{x}}$. So, the set of (integer) ε-translation numbers for $\underline{x}$ and $f_{\underline{x}}$ are the same.

Setting

$$\|\underline{x}\|_{\mathcal{S}^1_{\mathrm{T}}} := \sup_{n\in\mathbb{Z}}\left(\frac{1}{T+1}\sum_{k=n}^{n+T}|x_k|_E\right)\in[0,\infty],\quad \text{for } T\in\mathbb{N},$$

one can readily check that $\|\underline{x}\|_{\mathcal{S}^1_0} = \|\underline{x}\|_\infty$.

We also set
$$\mathcal{S}^1_{\mathrm{T}} := \{\underline{x} \mid \|\underline{x}\|_{\mathcal{S}^1_{\mathrm{T}}} < \infty\}.$$
For functions $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}, E)$, we still define:
$$\|f\|_{\mathrm{S}^1_1} := \sup_{a\in\mathbb{R}} \left(\int_a^{a+1} |f(t)|_E \, dt\right) \in [0, \infty],$$
$$\|f\|_{\mathrm{S}^1_{1,\mathbb{Z}}} := \sup_{n\in\mathbb{Z}} \left(\int_n^{n+1} |f(t)|_E \, dt\right) \in [0, \infty].$$
The function spaces S^1_1 and $\mathrm{S}^1_{1,\mathbb{Z}}$ can be defined respectively as
$$\mathrm{S}^1_1 := \{f \mid \|f\|_{\mathrm{S}^1_1} < \infty\} \quad \text{and} \quad \mathrm{S}^1_{1,\mathbb{Z}} := \{f \mid \|f\|_{\mathrm{S}^1_{1,\mathbb{Z}}} < \infty\}.$$
If S^1_1 (resp. $\mathrm{S}^1_{1,\mathbb{Z}}$) is endowed with $\|.\|_{\mathrm{S}^1_1}$ (resp. $\|.\|_{\mathrm{S}^1_{1,\mathbb{Z}}}$), then S^1_1 (resp. $\mathrm{S}^1_{1,\mathbb{Z}}$) becomes a vector space.

The following lemma is obvious.

Lemma 1. *The following relations hold, for all $\underline{x} \in E^{\mathbb{Z}}$, $T \in \mathbb{N}$ and $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}, E)$:*

(i) $\|\underline{x}\|_{\mathcal{S}^1_0} = \|\underline{x}\|_\infty$;

(ii) $\frac{1}{T+1}\|\underline{x}\|_\infty \leq \|\underline{x}\|_{\mathcal{S}^1_{\mathrm{T}}} \leq \|\underline{x}\|_\infty$;

(iii) $\|f\|_{\mathrm{S}^1_{1,\mathbb{Z}}} \leq \|f\|_{\mathrm{S}^1_1} \leq 2\|f\|_{\mathrm{S}^1_{1,\mathbb{Z}}}$;

(iv) $\|f\|_{\mathrm{S}^1_1} \leq \|f\|_\infty$;

(v) $\|f_{\underline{x}}\|_\infty = \|\underline{x}\|_\infty$.

As a consequence of the properties (i), (ii), we have $\mathrm{S}^1_T = \ell^\infty$, for any $T \in \mathbb{N}$, and all the above considered norms on this space $\mathrm{S}^1_T = \ell^\infty$ are equivalent. Furthermore, the property (iv) implies that $\mathrm{S}^1_1 = \mathrm{S}^1_{1,\mathbb{Z}}$.
We can state the following proposition.

Proposition 1. *The following properties are equivalent:*

- $\underline{x} \in \ell^\infty$;
- $\underline{x} \in \mathcal{S}^1_{\mathrm{T}}$, *for any* $T \in \mathbb{N}$;
- $\underline{x} \in \mathcal{S}^1_{\mathrm{T}}$, *for some* $T \in \mathbb{N}$;
- $f_{\underline{x}} \in \mathrm{S}^1_1$;
- $f_{\underline{x}} \in \mathrm{S}^1_{1,\mathbb{Z}}$;
- $f_{\underline{x}} \in L^\infty$.

Moreover, all the norms $\|\cdot\|_\infty$, $\|\cdot\|_{\mathcal{S}^1_{\mathrm{T}}}$, $\underline{x} \to \|f_{\underline{x}}\|_\infty$, $\underline{x} \to \|f_{\underline{x}}\|_{\mathrm{S}^1_1}$ are equivalent.

Proof. Because of the basic properties (i)–(v) in Lemma 1, we should only show the existence of a constant $C > 0$ such that, for any $\underline{x} \in E^{\mathbb{Z}}$, there is
$$\|f_{\underline{x}}\|_{\mathrm{S}^1_{1,\mathbb{Z}}} \geq C\|f_{\underline{x}}\|_\infty.$$

If this is true, we are able to write:

$$C\|f_{\underline{x}}\|_\infty \le \|f_{\underline{x}}\|_{\mathrm{S}^1_{1,\mathbb{Z}}} \le \|f_{\underline{x}}\|_{\mathrm{S}^1_1} \le 2\|f_{\underline{x}}\|_{\mathrm{S}^1_{1,\mathbb{Z}}} \le 2\|f_{\underline{x}}\|_\infty = 2\|\underline{x}\|_\infty,$$

which gives the claim.

Fixing $n \in \mathbb{Z}$, we have

$$\begin{aligned}\int_n^{n+1} |f_{\underline{x}}(t)|_E \, dt &= \int_0^1 |(1-\theta)x_n + \theta x_{n+1}|_E \, d\theta \\ &\ge \int_0^1 \big|(1-\theta)\,|x_n|_E - \theta\,|x_{n+1}|_E\big| d\theta. \end{aligned} \tag{2}$$

Setting $u := |x_n|_E$, $v := |x_{n+1}|_E$, we have $(u,v) \in (\mathbb{R}^+)^2$, and since

$$\int_0^1 |(1-\theta)u - \theta v| \, d\theta = \int_0^1 |u - \theta(u+v)| \, d\theta,$$

the last integral can be easily computed to be equal to $\frac{u^2+v^2}{2(u+v)}$.

Assuming, for instance, that $u \ge v$, we have $u^2 + v^2 \ge u^2$ and $2(u+v) \le 4u$, and we can deduce that

$$\int_0^1 |(1-\theta)u - \theta v| \, d\theta \ge \frac{u}{4}.$$

This yields

$$\int_n^{n+1} |f_{\underline{x}}(t)|_E \, dt \ge \frac{\max\{|x_n|_E, |x_{n+1}|_E\}}{4},$$

and so, when taking the sup-norm,

$$\|f_{\underline{x}}\|_{\mathrm{S}^1_{1,\mathbb{Z}}} \ge \frac{1}{4}\|\underline{x}\|_\infty = \frac{1}{4}\|f_{\underline{x}}\|_\infty.$$

Thus, for $C = \frac{1}{4}$, the claim is true. □

Remark 1. For $E = \mathbb{R}$, the proof of Proposition 1 is more straightforward, because the integral in (2) can be computed.

Proposition 1 has the following important consequence.

Consequence 3. *Sequence $\underline{x}$ is Stepanov a.p. iff it is a.p. iff $f_{\underline{x}}$ is a.p. iff $f_{\underline{x}}$ is* S_{ap}.

In fact, we can say that $\underline{x}$ is $\mathcal{S}^1_{\mathrm{T}}$ almost-periodic (shortly, $\mathcal{S}^1_{\mathrm{T}}$-a.p.) if it has the following property: for any $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that, in any set of N consecutive integers $\{m, \dots, m+N\}$, there exists $p \in \{m, \dots, m+N\}$ such that

$$\|x_{.+p} - x_.\|_{\mathcal{S}^1_{\mathrm{T}}} < \varepsilon.$$

On the other hand, in view of conditions (i) and (ii) in Lemma 1, the norms $\|.\|_\infty$, $\|.\|_{\mathcal{S}^1_1}$, $\|.\|_{\mathcal{S}^1_{\mathrm{T}}}$ are equivalent. Thus, the definition does not change provided we employ one of these norms. Moreover, since

$$\forall (k,p) \in \mathbb{Z}^2: \quad f_{\underline{x}(.+p)-\underline{x}}(k) = f_{\underline{x}}(k+p) - f_{\underline{x}}(k) = x_{k+p} - x_k,$$

we obtain the same definition with the norms $\underline{x} \to \|f_{\underline{x}}\|_\infty$ or $\underline{x} \to \|f_{\underline{x}}\|_{\mathrm{S}_1^1}$.

REMARK 2. In view of the Hölder inequality

$$\int_a^{a+1} |f(t)|\, dt \le \left(\int_a^{a+1} |f(t)|^p\, dt \right)^{1/p}, \tag{3}$$

Proposition 1 and Consequence 3 can be extended to $\mathrm{S}_{\mathrm{ap}}^p$-sequences.

In view of Proposition 1 resp. Consequence 3, examples from the foregoing section can be now continued in the following way.

EXAMPLE 4 (*of an* S_{ap}*-function whose one discretization is* S_{ap} *and another one is not* S_{ap}). Consider firstly the sequence $\{f(k)\}_{k\in\mathbb{Z}}$, where f is the S_{ap}-function defined in Example 1. We will show that $\{f(k)\}_{k\in\mathbb{Z}}$ is not S_{ap}.

Since

$$f_m(m(2k+1)) = 1, \quad \text{for } m = 1, 2, \ldots \text{ and } k = 0, \pm 1, \pm 2, \ldots,$$

we obtain

$$\begin{aligned} &f_m(1) = 1,\ f_m(6) + f_{2m}(6) = 2, \\ &f_m(20) + f_{2m}(20) + f_{2^2 m}(20) = 3,\ \ldots, \end{aligned}$$

i.e. in general:

$$\sum_{j=0}^{k} f_{2^j m}\left(2^k(2k+1)\right) = k+1, \quad \text{for all } k, m \in \mathbb{N}.$$

Thus, in view of nonnegativity of f, we have (for any $m \in \mathbb{N}$)

$$\begin{aligned} \limsup_{k\to\infty} f(k) &= \limsup_{k\to\infty} \sum_{n=1}^{\infty} f_n(k) \ge \limsup_{k\to\infty} \sum_{j=0}^{k} f_{2^j m}(k) \\ &\ge \lim_{k\to\infty} \sum_{j=0}^{k} f_{2^j m}\left(2^k(2k+1)\right) = \lim_{k\to\infty} (k+1) = \infty, \end{aligned}$$

by which the discretization $\{f(k)\}_{k\in\mathbb{Z}}$, of the S_{ap}-function f is not S_{ap}, as claimed.

Now, consider the sequence $\{f(k+\frac{1}{2})\}_{k\in\mathbb{Z}}$, where f is the same function as above. We will show that $\{f(k+\frac{1}{2})\}_{k\in\mathbb{Z}}$ is this time constant, and so S_{ap}. We can see that:

$$f(x) = \sum_{n\ge 1} \sum_{k\in\mathbb{Z}} \varphi_n(x - y_{n,k}),$$

where φ_n is defined as follows:

$$\varphi_n(x) = \begin{cases} \frac{2(x+\varepsilon_n)^2}{\varepsilon_n^2}, & \text{for } x \in (-\varepsilon_n, -\frac{\varepsilon_n}{2}], \\ 1 - \frac{2x^2}{\varepsilon_n^2}, & \text{for } x \in (-\frac{\varepsilon_n}{2}, +\frac{\varepsilon_n}{2}], \\ \frac{2(x-\varepsilon_n)^2}{\varepsilon_n^2}, & \text{for } x \in (\frac{\varepsilon_n}{2}, \varepsilon_n], \\ 0, & \text{otherwise.} \end{cases}$$

Take $\bar{x} = \frac{1}{2} + p$, $p \in \mathbb{Z}$. If $\varphi_n(\bar{x} - y_{n,k}) \neq 0$, it means that $|\bar{x} - y_{n,k}| < \varepsilon_n$, by which $d(\bar{x}, \mathbb{Z}) < \varepsilon_n$. But $d(\bar{x}, \mathbb{Z}) = \frac{1}{2} > \varepsilon_n$, and we have that $\varphi_n(\bar{x} - y_{n,k}) = 0$, for every $k \in \mathbb{Z}$ and every positive integer n. Thus, $f(\bar{x}) = 0$. This is true, for any $p \in \mathbb{Z}$, and we obtain that $f(\frac{1}{2} + p) = 0$, for each $p \in \mathbb{Z}$, as claimed.

Example 5 (*of an* $\mathrm{S_{ap}}$*-function whose discretization is a.p.*). Consider the sequence $\{f\big(\pi\big(k + \frac{1}{2}\big)\big)\}_{k\in\mathbb{Z}}$, where f is the $\mathrm{S_{ap}}$-function defined in Example 2. We will show that

$$f\left(\pi\big(k + \tfrac{1}{2}\big)\right) = \sin\left(\frac{1}{2 + \cos\left(\sqrt{2}\pi\big(k + \frac{1}{2}\big)\right)}\right)$$

is a.p.

Since $g_0(x) = 2 + \cos\sqrt{2}x \geq 1$ is a $\sqrt{2}\pi$-periodic function, so is $\sin\left(\frac{1}{g_0(x)}\right)$; in particular, it is a.p. Its discretization is, by the definition, a.p., too.

Since the values of $f(x) = \sin\left(\frac{1}{g(x)}\right)$ and $\sin\left(\frac{1}{g_0(x)}\right)$ coincide, for $x = \pi\big(k + \frac{1}{2}\big)$, $k = 0, \pm 1, \pm 2, \ldots$, $\{f\left(\pi\big(k + \frac{1}{2}\big)\right)\}_{k\in\mathbb{Z}}$ must be a.p., as claimed.

Remark 3. It is not immediately visible whether or not there exists another discretization of f in Example 2 which is not a.p. More generally, it is a question what are the sufficient conditions in order every discretization of a non-uniformly continuous $\mathrm{S_{ap}}$-function to be or not to be a.p.

Remark 4. Although the discretizations $\{f(k)\}_{k\in\mathbb{Z}}$ of $\mathrm{S_{ap}}$-functions f can be not necessarily a.p. (see Example 4), those $\{f^b(k)\}_{k\in\mathbb{Z}} = \{f(k+\eta)\}_{k\in\mathbb{Z}}$, $\eta \in [0,1]$, of their (a.p.) *Bochner transforms* (for more details, see [1, 3, 31, 32]) $f^b(t) = f(t+\eta)$, $\eta \in [0,1]$, are a.p. in the Banach space $\mathrm{L}_\infty(\mathbb{R}, L([0,1], E))$, by the definition.

4. Some remarks on $\mathrm{S_{ap}}$-solutions of differential equations

A natural question arises for $\mathrm{S_{ap}}$-solutions of differential equations, namely under which assumptions they reduce to uniformly a.p. solutions or reversely when they become purely $\mathrm{S_{ap}}$-solutions (?).

In order to answer these two questions, let us consider (for the sake of transparency) the particular form of equation (1)

$$x' = \varphi(x) + p(t), \tag{4}$$

where $\varphi \in \mathrm{C}(\mathbb{R}^n, \mathbb{R}^n)$ and $p \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$. By a *solution* of (4), we mean a Carathéodory one, i.e. $x(\cdot) \in AC_{loc}(\mathbb{R}, \mathbb{R}^n)$ which satisfies (4) almost everywhere. If $p \in C(\mathbb{R}, \mathbb{R}^n)$, then $x(\cdot) \in C^1(\mathbb{R}, \mathbb{R}^n)$ becomes obviously a classical solution.

PROPOSITION 2. *If $x(\cdot)$ is an S_{ap}-solution of (4), where φ is bounded and p is essentially bounded, then $x(\cdot)$ is uniformly a.p. If an S_{ap}-solution $x(\cdot)$ of (4) is additionally bounded and p is either essentially bounded or S_{ap}, then $x(\cdot)$ is also uniformly a.p.*

PROOF. The boundedness of functions φ, p implies a.e. the same for $x'(\cdot)$ which means that $x(\cdot)$ is uniformly continuous, and so a.p. The boundedness of $x(\cdot)$ implies the same for the composition $\varphi(x(\cdot))$. Thus, if p is not essentially bounded (otherwise, the same conclusion holds as above), but S_{ap}, then so is $x'(\cdot)$. According to Consequence 2, a bounded S_{ap}-solution $x(\cdot)$ must be a.p. $\square$

REMARK 5. Equation (4) with $\varphi(x) \equiv 0$ and $p(t) = h'(t)$, where h was defined in Example 3, has the a.p. solution $x(t) = h(t)$. This case concerns all conditions in Proposition 2. Equation (4) with $\varphi(x) = -x$ and $p(t) = h(t) + h'(t)$, having the same a.p. solution $x(t) = h(t)$, will be still treated below. It is related to the last two possibilities in Proposition 2. More sophisticated examples of this type can be deduced from the results in [1] (cf. [3, Theorem III.10.12]). The last possibility in Proposition 2 can be also nontrivially guaranteed by [11, Theorem 2].

Necessary conditions for purely S_{ap}-solutions of (4) can be formulated as follows.

PROPOSITION 3. *The necessary condition in order an S_{ap}-solution $x(\cdot)$ of (4) not to be uniformly a.p. consists in satisfying one of the following three possibilities: (i) $x(\cdot)$ and φ are unbounded or (ii) $x(\cdot)$ is unbounded and p is not essentially bounded or (iii) p is neither essentially bounded nor S_{ap}.*

PROOF. It follows immediately from Proposition 2 that the following possibilities may occur in order an S_{ap}-solution $x(\cdot)$ not to be uniformly a.p.: (i) or (ii) above or, for a bounded $x(\cdot)$, p to be either not essentially bounded or not S_{ap}. However, if p is not essentially bounded but S_{ap}, then a (not essentially bounded) S_{ap}-derivative $x'(\cdot)$ implies that, according to Consequence 2, a bounded $x(\cdot)$ must be a.p. Thus, condition (iii) represents the only remaining necessity. $\square$

REMARK 6. Equation (4) with $\varphi(x) \equiv 0$ and $p(t) = \sum_{n=1}^{\infty} f_n'(t)$, where f_n were defined in Example 1, has the purely S_{ap}-solution $x(t) = \sum_{n=1}^{\infty} f_n(t)$. This case is related to condition (ii) in Proposition 3. Equation (4) with $\varphi(x) = -x$ and $p(t) = \sum_{n=1}^{\infty} f_n(t) + f_n'(t)$, having the same S_{ap}-solution $x(t) = \sum_{n=1}^{\infty} f_n(t)$, will be still treated below. It is related to condition (i) in Proposition 3. Equation (4) with $\varphi(x) = -x$ and

$$p(t) = \sin\left(\frac{1}{g(t)}\right) + \cos\left(\frac{1}{g(t)}\right)\left(\frac{\sin t + \sqrt{2}\sin\sqrt{2}t}{g^2(t)}\right),$$

where $g(t) = 2 + \cos t + \cos\sqrt{2}t$, has the purely $\mathrm{S_{ap}}$-solution $x(t) = \sin\left(\frac{1}{g(t)}\right)$ and will be also still treated below. It is related to condition (iii) in Proposition 3.

Remark 7. It follows from Proposition 3 that the existence of purely $\mathrm{S_{ap}}$-solutions of (4) for the Bohr–Neugebauer type theorems (boundedness implies almost-periodicity) can be just related to condition (iii), i.e. p is not essentially bounded and at the same time not $\mathrm{S_{ap}}$. On the other hand, in all the Bohr–Neugebauer type results resp. Favard type results in [9, 10, 11, 12, 19, 20, 21, 17, 18, 34, 35], it is assumed that p in (4) or, more generally, $F(\cdot, x)$ in (1) is at least an $\mathrm{S_{ap}}$-function. This indicates that the question posed in [17, 18], namely "*whether boundedness of solutions can imply their (pure) Stepanov's almost-periodicity (if not a.p.) (?)*" has, under the assumptions imposed on p in (4) or on F in (1), no meaning. In particular, since $F(\cdot, x)$ is assumed in [21] to be $\mathrm{S^p_{ap}}$ and a.e. bounded, uniformly in $x \in \mathbb{R}^n$, and A, p in $x' = A(t)x + p(t)$ are assumed in [17, 18, 19, 20] to be uniformly a.p., it immediately follows that bounded solutions must be uniformly continuous, and subsequently bounded $\mathrm{S^p_{ap}}$-solutions become, in view of the Hölder inequality (3), uniformly a.p. Moreover, a Stepanov extension of Favard type results in [18, 19, 20] was shown in [38] just apparent.

Since e^y is, on any compact interval, uniformly continuous, it follows that e^f is $\mathrm{S_{ap}}$ or a.p. if so is a bounded f, respectively. In particular, the discretization $f_k = f(k)$ of a bounded f is a.p. iff so is e^{f_k}.

Therefore, if the bounded function $f \in \mathrm{AC_{loc}}$ is $\mathrm{S_{ap}}$ or a.p., then

$$x(t) = \exp\left(\int^t f'(s)\,ds\right) = \exp(f(t))$$

is an $\mathrm{S_{ap}}$-solution or an a.p. solution, respectively, of the equation

$$x' = f'(t)x. \tag{5}$$

Moreover, the discretization

$$\{x(k)\}_{k\in\mathbb{Z}} = \left\{\exp\left(\int^k f'(s)\,ds\right)\right\}_{k\in\mathbb{Z}} = \{\exp(f(k))\}_{k\in\mathbb{Z}}$$

is a.p. iff $\{f(k)\}_{k\in\mathbb{Z}}$ is a.p.

Furthermore, since $x(t) = f(t) \in \mathrm{AC_{loc}}$ is a solution of the equation

$$x' + x = f(t) + f'(t), \tag{6}$$

it is trivially $\mathrm{S_{ap}}$ or a.p. iff f is so. The same is obviously true for its discretization $\{x(k)\}_{k\in\mathbb{Z}} = \{f(k)\}_{k\in\mathbb{Z}}$.

Thus, the following possibilities can be demonstrated by means of the above examples:

• Function $x(t) = f(t)$, where f is the $\mathrm{S_{ap}}$-function defined in Example 1, is a purely $\mathrm{S_{ap}}$-solution of (6). Its discretization $\{x(k)\}_{k\in\mathbb{Z}} = \{f(k)\}_{k\in\mathbb{Z}}$ is, according to Example 4, not a.p. ($\mathrm{S_{ap}}$).

• Although the coefficient

$$f'(t) = \cos\left(\frac{1}{g(t)}\right)\left(\frac{\sin t + \sqrt{2}\sin\sqrt{2}t}{g^2(t)}\right)$$

in (5), where $g(t) = 2+\cos t+\cos\sqrt{2}t$, is (according to Example 2) not $\mathrm{S_{ap}}$, equation (5) admits a purely $\mathrm{S_{ap}}$-solution $x(t) = \exp\big(\sin\big(\frac{1}{g(t)}\big)\big)$ whose discretization

$$\left\{x\left(\pi\left(k+\tfrac{1}{2}\right)\right)\right\}_{k\in\mathbb{Z}} = \left\{\exp\left(\sin\left(\frac{1}{2+\cos\left(\sqrt{2}\pi\left(k+\frac{1}{2}\right)\right)}\right)\right)\right\}_{k\in\mathbb{Z}}$$

is, in view of Example 5, a.p.

Furthermore, equation (6), where the nonhomogenity $f(t) + f'(t) = \sin\big(\frac{1}{g(t)}\big) + f'(t)$ is not $\mathrm{S_{ap}}$, admits a purely $\mathrm{S_{ap}}$-solution $x(t) = \sin\big(\frac{1}{g(t)}\big)$ whose discretization

$$\left\{x\left(\pi\left(k+\tfrac{1}{2}\right)\right)\right\}_{k\in\mathbb{Z}} = \left\{\sin\left(\frac{1}{2+\cos\left(\sqrt{2}\pi\left(k+\frac{1}{2}\right)\right)}\right)\right\}_{k\in\mathbb{Z}}$$

is a.p.

• Although the term f' at the equation

$$x' + \frac{1}{g^2(t)}x = f'(t) + \frac{f(t)}{g^2(t)}, \tag{7}$$

where

$$f(t) = \sin\left(\frac{1}{g(t)}\right), \quad f'(t) = \cos\left(\frac{1}{g(t)}\right)\left(\frac{\sin t + \sqrt{2}\sin\sqrt{2}t}{g^2(t)}\right),$$

$g(t) = 2+\cos t+\cos\sqrt{2}t$, is (according to Example 2) not $\mathrm{S_{ap}}$, equation (7) admits a purely $\mathrm{S_{ap}}$-solution $x(t) = \sin\big(\frac{1}{g(t)}\big)$ whose discretization

$$\left\{x\left(\pi\left(k+\tfrac{1}{2}\right)\right)\right\}_{k\in\mathbb{Z}} = \left\{\sin\left(\frac{1}{2+\cos\left(\sqrt{2}\pi\left(k+\frac{1}{2}\right)\right)}\right)\right\}_{k\in\mathbb{Z}}$$

was already pointed out to be a.p. On the other hand, at the equivalent equation $g^2(t)x' + x = f'(t)g^2(t) + f(t)$, the coefficient $g^2(t) > 0$ is a.p. and the nonhomogenity $f'g^2 + f$ is $\mathrm{S_{ap}}$.

• Equation (5), where the coefficient

$$f'(t) = g'(t)\left(\cos\left(\frac{1}{g(t)}\right) - 2g(t)\right)$$

with $g(t) = 2 + \cos t + \cos\sqrt{2}t$ and $g'(t) = -\sin t - \sqrt{2}\sin\sqrt{2}t$ is (according to Example 3) an $\mathrm{S_{ap}}$-function, admits an a.p. solution

$$x(t) = \exp(f(t)) = \exp\left(g^2(t)\sin\left(\frac{1}{g(t)}\right)\right).$$

Its discretization

$$\{x(k)\}_{k\in\mathbb{Z}} = \left\{\exp\left(g^2(k)\sin\left(\frac{1}{g(k)}\right)\right)\right\}_{k\in\mathbb{Z}}$$

is, by the definition, a.p.

Furthermore, equation (6), where the nonhomogenity $f(t) + f'(t) = g^2(t)\sin\left(\frac{1}{g(t)}\right) + f'(t)$ is $\mathrm{S_{ap}}$, admits an a.p. solution $x(t) = g^2(t)\sin\left(\frac{1}{g(t)}\right)$ whose discretization

$$\{x\left(\pi\left(k+\tfrac{1}{2}\right)\right)\}_{k\in\mathbb{Z}} = \left\{g^2\left(\pi\left(k+\tfrac{1}{2}\right)\right)\sin\left(\frac{1}{g\left(\pi\left(k+\frac{1}{2}\right)\right)}\right)\right\}_{k\in\mathbb{Z}}$$

is, by the definition, a.p.

5. **Main results**

For equation (4), Theorem 1 can be significantly extended as follows.

Theorem 2. *Let $x(\cdot)$ be a Carathéory (i.e. $x(\cdot) \in AC_{loc}(\mathbb{R}, E)$) solution of the differential equation (4) in a Banach space $E = (E, |\cdot|_E)$ satisfying the Radon–Nikodym property (e.g. reflexivity), where*

(i) $\varphi\colon E \to E$ is a Lipschitz-continuous mapping with a constant L, i.e.

$$|\varphi(x) - \varphi(y)|_E \le L|x-y|_E, \quad \textit{for all } x, y \in E, \tag{8}$$

(ii) $p\colon \mathbb{R} \to E$ is an $\mathrm{S_{ap}}$-mapping.

Then a necessary and sufficient condition that $x(\cdot)$ be an almost-periodic solution of (4) is that $\{x(k)\}_{k\in\mathbb{Z}}$ is a (Stepanov) a.p. sequence in the sense of Definition 2.

Proof. It is well-known (cf. e.g. [3, p. 243]) that if E satisfies the Radon–Nikodym property, then the derivative $x'(\cdot)$ of a Carathéodory solution $x(\cdot) \in \mathrm{AC_{loc}}(\mathbb{R}, E)$ is locally Lebesgue–Bochner integrable and the fundamental theorem of calculus (the Newton–Leibniz formula) holds:

$$x(t) = x(0) + \int_0^t x'(s)\,ds.$$

Hence, it is enough to show that $x(\cdot)$ possesses a relatively dense set of ε almost-periods.

For $t \in \mathbb{R}$, take $k := [t]$ as the integer part $[t]$ of t. We obtain

$$
\begin{aligned}
&|x(t+\tau) - x(t)|_E = \\
&\quad = \left| \int_0^{t+\tau} [\varphi(x(s)) + p(s)]\, ds - \int_0^t [\varphi(x(s)) + p(s)]\, ds \right|_E \\
&\quad = \left| \int_0^{k+\tau} [\varphi(x(s)) + p(s)]\, ds - \int_0^k [\varphi(x(s)) + p(s)]\, ds \right. \\
&\quad \left. + \int_{k+\tau}^{t+\tau} [\varphi(x(s)) + p(s)]\, ds - \int_k^t [\varphi(x(s)) + p(s)]\, ds \right|_E \\
&\quad \le |x(k+\tau) - x(k)|_E + \int_k^t |\varphi(x(s+\tau)) - \varphi(x(s))|_E \, ds \\
&\qquad\qquad + \int_k^t |p(s+\tau) - p(s)|_E \, ds \\
&\quad \le \|x_{.+\tau} - x_{.}\|_\infty + L \int_k^t |x(s+\tau) - x(s)|_E \, ds \\
&\qquad\qquad + \int_k^{k+1} |p(t+\tau) - p(t)|_E \, dt.
\end{aligned}
$$

Now, using the notation in the proof of Proposition 1, we obtain:

$$
\|x_{.+\tau} - x_{.}\|_\infty = \|f_{\underline{x}}(.+\tau) - f_{\underline{x}}(\cdot)\|_\infty \le
$$

$$
\frac{1}{C}\|f_{\underline{x}}(.+\tau) - f_{\underline{x}}(\cdot)\|_{\mathrm{S}_1^1} = 4\|f_{\underline{x}}(.+\tau) - f_{\underline{x}}(\cdot)\|_{\mathrm{S}_1^1}.
$$

Moreover

$$
\int_k^{k+1} |p(t+\tau) - p(t)|_E \, dt \le \|p(.+\tau) - p(\cdot)\|_S.
$$

Thus, we have:

$$
\begin{aligned}
|x(t+\tau) - x(t)|_E \le 4\|f_{\underline{x}}(.+\tau) - f_{\underline{x}}(\cdot)\|_S + L \int_k^t |x(s+\tau) - x(s)|_E \, ds \\
+ \|p(.+\tau) - p(\cdot)\|_S\,.
\end{aligned}
$$

Since $\underline{x} := \{x(k)\}_{k\in\mathbb{Z}}$ is (Stepanov) a.p., its canonical extension $f_{\underline{x}}$ is a.p. as well, and so S_{ap}. Thus, there exists a relatively dense set $\{\tau\}_\varepsilon$ of integer ε-Stepanov almost-periods which is common for $f_{\underline{x}}$ and p (see e.g. [29, pp. 203–204]).

The above inequality implies that

$$
|x(t+\tau) - x(t)|_E \le 5\varepsilon + L \int_k^t |x(s+\tau) - x(s)|_E \, ds
$$

and, by virtue of the well-known Gronwall inequality, we arrive at

$$
|x(t+\tau) - x(t)|_E \le 5\varepsilon e^L,
$$

for any $t \in \mathbb{R}$. This already means that τ is a desired $5\varepsilon e^L$ almost-period of $x(\cdot)$.

The reverse implication is trivial. $\square$

REMARK 8. The equation:

$$x' + x = h(t) + h'(t),$$

where $h(\cdot)$ is the function from Example 3, represents an example of application of Theorem 2. Indeed, the function $h + h'$ is obviously an $\mathrm{S_{ap}}$-function and $x(\cdot) = h(\cdot)$ is an a.p. solution whose discretization is a.p.

REMARK 9. The equation:

$$x' + x = \tilde{f}(t) + \tilde{f}'(t),$$

where $\tilde{f}(\cdot) := f(. + \frac{1}{2})$ and f is the function from Example 1, does not represent an example of application of Theorem 2, because the function $\tilde{f} + \tilde{f}'$ (resp. $\tilde{f}'$) is not an $\mathrm{S_{ap}}$-function. Otherwise, since $\tilde{x}(\cdot) = \tilde{f}(\cdot)$ is an $\mathrm{S_{ap}}$-solution whose discretization is a.p., because $\tilde{x}(k) = 0$, for every $k \in \mathbb{Z}$, $\tilde{x}(\cdot)$ should be (uniformly continuous) a.p.

Remark 9 suggests us to formulate the following interesting proposition which makes Consequence 2 more precise.

PROPOSITION 4. *Let E be a Banach space satisfying the Radon-Nikodym property. There is no purely (i.e. not uniformly continuous) $\mathrm{S_{ap}}$-function $g \in AC_{loc}(\mathbb{R}, E)$ which has at least one a.p. discretization and whose derivative is $\mathrm{S_{ap}}$.*

PROOF. Assume that there exists such a function g. Let $a > 0$, $b \in \mathbb{R}$ be the numbers such that $\{g(ak + b)\}_{k\in\mathbb{Z}}$ is $\mathrm{S_{ap}}$. Now, set $f : x \to g(ax + b)$. We can say that f is purely $\mathrm{S_{ap}}$ and f' is $\mathrm{S_{ap}}$, and the related standard discretization is a.p. Now, consider the equation:

$$x' + x = f(t) + f'(t).$$

At this equation, we have $p(t) := f(t) + f'(t)$ which is $\mathrm{S_{ap}}$ and $\varphi(x) := -x$ which is Lipschitzean. It admits an $\mathrm{S_{ap}}$-solution $x(\cdot) = f(\cdot)$ and it is easy to see that this is the only one to be $\mathrm{S_{ap}}$. Since $\{x(k)\}_{k\in\mathbb{Z}}$ is a.p., by the hypothesis, we would have that $x(\cdot)$ is contradictionally a.p., when applying Theorem 2. This completes the proof. $\square$

CONSEQUENCE 4. *In order to have purely $\mathrm{S_{ap}}$-solutions $x(\cdot)$ for equation (4), where φ and p satify conditions (i) and (ii) in Theorem 2, none discretization $\{x(ak+b)\}_{k\in\mathbb{Z}}$ can be $\mathrm{S_{ap}}$ (a.p.). For $E = \mathbb{R}^n$, such solutions must be also unbounded.*

In a finite-dimensional Banach space $E = (E, |\cdot|_E)$, consider now the differential inclusion

$$x' \in \Phi(x) + P(t), \tag{9}$$

where

(i) $\Phi\colon E \multimap E$ is a (multivalued) mapping with nonempty, convex, compact values such that

$$d_H(\Phi(x), \Phi(y)) \le L_0|x-y|_E, \quad \text{for all } x, y \in E,$$

holds with a Lipschitz constant L_0, where d_H stands for the Hausdorff distance (cf. e.g. [3, 22]),

(ii) $P\colon \mathbb{R} \multimap E$ is a (multivalued) measurable mapping with nonempty, compact values which is $\mathrm{S_{ap}}$, i.e. that, for every $\varepsilon > 0$, there corresponds a relatively dense set $\{\tau\}_\varepsilon$ of ε-Stepanov almost-periods such that

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} d_H(P(s+\tau), P(s))\, ds < \varepsilon, \quad \text{for all } \tau \in \{\tau\}_\varepsilon\,.$$

Since Φ possesses, under (i), a single-valued Lipschitz-continuous selection $\varphi \subset \Phi$ with a constant $L = (12\sqrt{3}/5+1)L_0 \dim E$ (see e.g. [22, pp. 101–103]) and P possesses, under (ii), a single-valued $\mathrm{S_{ap}}$-selection $p \subset P$ (see [13, 14, 15]), we can immediately give the following corollary of Theorem 2.

Corollary 1. *Let conditions (i) and (ii) be satisfied for multivalued maps Φ and P in inclusion (9). Let $x(\cdot)$ be a Carathéodory solution of the differential equation (4), where $\varphi \subset \Phi$ and $p \subset P$ are the single-valued selections with the properties indicated above. Then $x(\cdot)$ is an $\mathrm{S_{ap}}$-solution of inclusion (9) if and only if $\{x(k)\}_{k\in\mathbb{Z}}$ is an a.p. sequence in the sense of Definition 2.*

Remark 10. It would be more natural and much better to assume directly that $x(\cdot)$ is a solution of the differential inclusion (9). Unfortunately, since the examples above suggest that $x(\cdot)$ could then take the form

$$x(t) = x(0) + \int_0^t [\tilde{\varphi}(x(s)) - \tilde{p}(s)]\, ds,$$

where the single-valued selections $\tilde{\varphi} \subset \Phi$ and $\tilde{p} \subset P$ need not share the indicated properties, it seems to be difficult to avoid this obstruction.

Theorem 1 can be also generalized in the following way.

Theorem 3. *Let $x(\cdot)$ be a Carathéodory (i.e. $x(\cdot) \in AC_{loc}(\mathbb{R}, E)$) solution of the differential equation (1) in a finite-dimensional Banach space E, where*

(i) $F(\cdot, x)\colon \mathbb{R} \to E$ is an essentially bounded $\mathrm{S_{ap}}$-mapping, for each x in some connected open subset D of E,

(ii) $F(t, \cdot)\colon E \to E$ satisfies, for all $x, y \in D$, the Lipschitz condition

$$|F(t,x) - F(t,y)|_E \le L|x-y|_E, \quad t \in \mathbb{R}, \tag{10}$$

and let D contain the closure of the range of $x(\cdot)$. Then a necessary and sufficient condition that $x(\cdot)$ be an a.p. solution with values in $D \subset E$ is that $\{x(k)\}_{k\in\mathbb{Z}}$ be an a.p. sequence in $D \subset E$.

Sketch of the proof. First of all, under the above assumptions, the solution $x(\cdot)$ can be shown, quite anagously as in [30], to be bounded. Let $\overline{R} \subset D$ denote the closure of the range R of $x(\cdot)$ in E. Since E is finite-dimensional, $\overline{R}$ is obviously compact.

Using the Bochner transform (for more details, see [1, 3, 31, 32]),

$$F^b(\cdot, x) := F(\cdot + \eta, x), \quad \eta \in [0,1], \ x \in \overline{R},$$

of $F(\cdot, x)$, $x \in \overline{R}$, and applying Tornehave's arguments recalled in [30], one can prove, quite analogously as for Lemma 2 in [30], that there exists a relatively dense set $\{\tau\}_\varepsilon$ of integer ε almost-periods which are common for $f^b_{\underline{x}}$ and $F^b(\cdot, x)$, $x \in \overline{R}$, in the Banach space $\mathrm{L}_\infty(\mathbb{R}, L([0,1], E))$. Since, for every $f \in \mathrm{S}^1_1$, we have that:

$$\|f^b(.+\tau) - f^b(\cdot)\|_\infty = \|f(.+\tau) - f(\cdot)\|_{\mathrm{S}^1_1},$$

we can see that the set of ε almost-periods of f^b and Stepanov ε almost-periods of f are the same. In view of the arguments in Section 3, $\{\tau\}_\varepsilon$ is so a relatively dense set of ε-Stepanov almost-periods, for $\{x_k\}_{k\in\mathbb{Z}}$ and $F(\cdot, x)$, $x \in \overline{R}$, in E.

Now, following step by step the proof of Theorem in [30], one can check, in the same way as in the proof therein and when applying the above fact about common integer ε-Stepanov almost-periods, that $x(\cdot)$ is almost-periodic in E, as claimed. $\square$

Remark 11. The equation:

$$x' = h'(t)x,$$

where $h(\cdot)$ is the function from Example 3, represents an example of application of Theorem 3, because h' is a bounded S_{ap}-function, $h'(t)(\cdot)$ is Lipschitzean with the constant $L := \max_{t\in\mathbb{R}} |h'(t)| \leq 9(1+\sqrt{2})$ and $x(\cdot) = \exp(h(\cdot))$ is an a.p. solution whose discretization is a.p.

Remark 12. Checking critically the proof of Theorem 1 again, one can observe that Theorem 3 can be still formally extended into infinite-dimensional Banach spaces satisfying the Radon–Nikodym property (cf. Theorem 2), but provided the closure of the range of $x(\cdot)$ to be contained in a compact subset C of D. However, since such an additional requirement is very drastic, we decided to present only a finite-dimensional version for equation (5).

Remark 13. It follows from Proposition 1 that, unlike for differential equations and inclusions, the problem of S_{ap}-solutions of difference equations and inclusions, which are at the same time not uniformly a.p. solutions, has no meaning. On the other hand, the results concerning a.p. solutions of difference equations and inclusions (see e.g. [8, 12, 25, 33, 36]) can be expressed in terms of equivalent conditions in Proposition 1. It is a question whether or not one can expect purely Weyl or Besicovitch a.p. solutions of difference equations and inclusions.

Remark 14. In [4], where the nonexistence of purely S_{ap}-solutions of (4) was considered in even more general situations, we have shown that Consequence 4 can be improved in the sense that, under the assumptions (i) and (ii) in Theorem 2, at least in uniformly convex spaces, there are no purely S_{ap}-solutions of equation (4).

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