Pseudo almost automorphic solutions for hyperbolic semilinear evolution equations in intermediate Banach spaces

G. M. N'Guérékata[1] and D. Pennequin[2]

[1]Department of mathematics, Morgan State University, USA. Email: Gaston.N'Guerekata@morgan.edu

[2]Université Paris 1 Panthéon-Sorbonne, Laboratoire Marin Mersenne, FRANCE. Email: pennequi@univ-paris1.fr

Abstract. We are concerned in this paper with the pseudo almost automorphy of mild solutions for the semilinear evolution equation $x'(t) = Ax(t) + f(t, x)$ where A is a sectorial operator not necessarily densely defined in X generating an hyperbolic semigroup $(T(t))_{t\geq 0}$ in a Banach space X and $f : \mathbb{R} \times X_\alpha \to X$, where X_α is an intermediate space. We prove the existence and uniqueness of a pseudo almost automorphic solution in X_α, when the function $f : \mathbb{R} \times X_\alpha \longrightarrow X$ is pseudo almost automorphic.

Keywords. Analytic semigroup, hyperbolic semigroup, pseudo almost automorphic function, semilinear evolution equation.
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1 Introduction

The concept of almost automorphic function was introduced in the literature in the mid sixties as a generalization of almost periodicity in the sense of Bohr. Since then, several extensions of the concept were introduced including asymptotic almost automorphy by N'Guérékata ([14]), *p*-almost automorphy by Diagana ([3]), and Stepanov-like almost automorphy by N'Guérékata and Pankov ([15]). Recently, J. Liang et al. have suggested the notion of pseudo almost automorphic functions, i.e. functions that can be written uniquely as a sum of an almost automorphic function and an ergodic term, i.e. a function with vanishing mean (cf [7]). This latter turns out to be more general than asymptotic almost automorphy. However it seems to be more complicated. There has been a considerable interest in the existence of (these various types of) almost automorphic solutions of evolution equations. Semigroups theory and fixed point techniques have been frequently used for semilinear evolution equations. In the present work, which is inspired by the recent papers of N'Guérékata [13], and others [7],[8], [9], we consider semilinear evolution equations of the form

$$x'(t) = Ax(t) + f(t, x(t)), \qquad t \in \mathbb{R}, \tag{1.1}$$

where A is an unbounded sectorial operator with not necessarily dense domain in a Banach space X and $f : \mathbb{R} \times X_\alpha \longrightarrow X$, where X_α, $\alpha \in (0, 1)$, is any intermediate Banach space between $D(A)$ and X. Concrete examples of X_α are the fractional power spaces $D((-A)^\alpha)$, $0 < \alpha < 1$, the reel interpolation spaces $D_A(\alpha, \infty)$, introduced by J. L. Lions and J. Peetre, and the Hölder spaces $D_A(\alpha)$ which coincide with the continuous interpolation spaces due to G. Da Prato and P. Grisvard, see Section 2. We also give the definition and some properties of pseudo almost automorphic functions in Section 2. Then we present our main result (Theorem 3.2) in Section 3.

2 preliminaries

In this section we recall some definitions and fix notations which will be used in the sequel. Throughout this paper, X is a Banach space and A is a sectorial operator with not necessarily dense domain, i.e., there are constants $\omega \in \mathbb{R}$, $\theta \in]\frac{\pi}{2}, \pi[$, $M > 0$ such that

$$(i) \qquad \rho(A) \supset S_{\theta,\omega} := \{\lambda \in \mathbb{C} : \lambda \neq \omega, |arg(\lambda - \omega)| < \theta\} \tag{2.1}$$

$$(ii) \qquad \|R(\lambda, A)\| \leq \frac{M}{|\lambda - \omega|}, \quad \lambda \in S_{\theta,\omega}. \tag{2.2}$$

Hence A generates an analytic semigroup $\mathcal{T} := (T(t))_{t\geq 0}$ on $(0, \infty)$ to $\mathcal{L}(X)$ satisfying

$$\|T(t)\| \leq M_0 e^{\omega t}, \quad t > 0, \tag{2.3}$$

$$\|t(A - \omega)T(t)\| \leq M_1 e^{\omega t}, \quad t > 0. \tag{2.4}$$

The semigroup $\mathcal{T}$ is assumed to be hyperbolic, i.e., there exist a projection P and constants $M, \delta > 0$ such that each $T(t)$ commutes with P, $\ker P$ is invariant with respect to $T(t)$, $T(t) : ImQ \longrightarrow ImQ$ is invertible and

$$\|T(t)Px\| \leq Me^{-\delta t}\|x\| \qquad \text{for } t \geq 0, \tag{2.5}$$

$$\|T(t)Qx\| \leq Me^{\delta t}\|x\| \qquad \text{for } t \leq 0, \tag{2.6}$$

where $Q := I - P$ and, for $t \leq 0$, $T(t) := (T(-t))^{-1}$.
We recall that if $\mathcal{T}$ is analytic, then $\mathcal{T}$ is hyperbolic if and only if

$$\sigma(A) \cap i\mathbb{R} = \emptyset,$$

see for instance [5, Prop 1.15, p.305].

For $\alpha \in (0, 1)$, a Banach space X_α with norm $\|\cdot\|_\alpha$, is said to be an intermediate space between $D(A)$ and X, or a space of class $\mathcal{J}_\alpha$, if $D(A) \subset X_\alpha \subset X$ and there is a constant $c > 0$ such that

$$\|x\|_\alpha \leq c\|x\|^{1-\alpha}\|x\|_A^\alpha, \qquad x \in D(A), \tag{2.7}$$

where $\|\cdot\|_A$ is the graph norm associated to A. Concrete examples of X_α are $D((-A)^\alpha)$, $\alpha \in (0, 1)$, the domains of the the fractional powers of $-A$, the real interpolation spaces $D_A(\alpha, \infty)$, $\alpha \in (0, 1)$, defined as follows

$$\begin{cases} D_A(\alpha, \infty) := \{x \in X : \sup_{0<t\leq 1} \|t^{1-\alpha}(A - \omega)e^{-\omega t}T(t)x\| < +\infty\} \\ \|x\|_\alpha = \|x\| + [x]_\alpha, [x]_\alpha = \sup_{0<t\leq 1} \|t^{1-\alpha}(A - \omega)e^{-\omega t}T(t)x\|, \end{cases}$$

and the abstract Hölder spaces $D_A(\alpha) := \overline{D(A)}^{\|\cdot\|_\alpha}$. A very important property of these last two spaces is given by the fact that they depend only on $D(A)$ and X (in contrast with the fractional power spaces of $-A$). That is for another sectorial operator B with $D(B) = D(A)$, their interpolation and Hölder spaces coincide. For more details about intermediate spaces, see for instance [5, Chap.II, Sec. 5.b] and [10].

For the hyperbolic analytic semigroup $\mathcal{T}$, we can easily check that estimations similar to (2.5) and (2.6) hold also with norms $\|\cdot\|_\alpha$. In fact, as the part of A in ImQ is bounded, it follows from the inequality (2.6) that

$$\|AT(t)Qx\| \leq c'e^{\delta t}\|x\| \qquad \text{for } t \leq 0.$$

Hence, from (2.7) there exists a constant $c(\alpha) > 0$ such that

$$\|T(t)Qx\|_\alpha \leq c(\alpha)e^{\delta t}\|x\| \qquad \text{for } t \leq 0. \tag{2.8}$$

We have also

$$\|T(t)Px\|_\alpha \leq \|T(1)\|_{\mathcal{L}(X,X_\alpha)}\|T(t-1)Px\| \qquad \text{for } t \geq 1,$$

and then from (2.5), we obtain

$$\|T(t)Px\|_\alpha \leq M'e^{-\delta t}\|x\|, \qquad t \geq 1,$$

where M' depends on α. For $t \in (0, 1]$, by (2.4) and (2.7)

$$\|T(t)Px\|_\alpha \leq M''t^{-\alpha}\|x\|.$$

Hence, there exist constants $M(\alpha) > 0$ and $\gamma > 0$ such that

$$\|T(t)Px\|_\alpha \leq M(\alpha)t^{-\alpha}e^{-\gamma t}\|x\| \qquad \text{for } t > 0. \tag{2.9}$$

Definition 2.1 *(S. Bochner) A continuous function $f : \mathbb{R} \to X$ is called almost automorphic if for every sequence $(\sigma_n)_{n\in N}$ there exists a subsequence $(s_n)_{n\in N} \subset (\sigma_n)_{n\in N}$ such that*

$$\lim_{n,m\to+\infty} f(t+s_n-s_m) = f(t) \quad \text{for each } t \in \mathbb{R}.$$

This is equivalent to

$$g(t) := \lim_{n\to+\infty} f(t+s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$f(t) = \lim_{n\to+\infty} g(t-s_n)$$

for each $t \in \mathbb{R}$.

The space $AA(\mathbb{R}, X)$ of all almost automorphic functions $f : \mathbb{R} \to X$ is a Banach space under the supremum norm (see [1], [11], or [13], for more information on such functions and their applications in abstract differential equations).

Definition 2.2 *Let X, Y be Banach sapces.*

1. *A bounded continuous function with vanishing mean value can be defined as*

$$AA_0(\mathbb{R}, X) = \left\{ \phi \in BC(\mathbb{R}, X) : \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \|\phi(\sigma)\| d\sigma = 0 \right\}.$$

2. *Similarly we define $AA_0(\mathbb{R}\times Y, X)$ to be the collection of all functions $f : t \mapsto f(t,x) \in BC(\mathbb{R}\times Y, X)$ satisfying*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \|f(\sigma, x)\| d\sigma = 0$$

uniformly for x in any bounded subset of X.

We are now ready to introduce the sets $PAA(\mathbb{R}, X)$ and $PAA(\mathbb{R}\times Y, X)$ of pseudo almost automorphic functions:

$$PAA(\mathbb{R}, X) = \left\{ \begin{array}{l} f = g + \phi \in BC(\mathbb{R}, X), \\ g \in AA(\mathbb{R}, X) \ \text{ and } \ \phi \in AA_0(\mathbb{R}, X) \end{array} \right\};$$

$$PAA(\mathbb{R}\times Y, X) = \left\{ \begin{array}{l} f = g + \phi \in BC(\mathbb{R}\times Y, X), \\ g \in AA(\mathbb{R}\times Y, X) \ \text{ and } \ \phi \in AA_0(\mathbb{R}\times Y, X) \end{array} \right\}.$$

In both cases above, g and ϕ are called respectively the principal and the ergodic terms of f. Note that in [7], $Y = X$. Thus our definition here is more general.

Proposition 2.3 *Let $f, f_1, f_2 \in PAA(\mathbb{R}, X)$. Then we have*

- *(i) $\lambda f \in PAA(\mathbb{R}, X)$, for any scalar λ.*
- *(ii) $f_1 + f_2 \in PAA(\mathbb{R}, X)$*
- *(iii) $g(t) := f(-t) \in PAA(\mathbb{R}, X)$*
- *(iv) $f_a(t) := f(t+a) \in PAA(\mathbb{R}, X)$, for any $a \in \mathbb{R}$.*
- *(v) $PAA(\mathbb{R}, X)$ is a Banach space under the supremum norm.*

Proof. (i)-(iv) are easy to check. For (v) see [9] Theorem 2.2.

3 Main results

In this Section, we assume $f \in PAA(\mathbb{R}\times X_\alpha, X)$ $(0 < \alpha < 1)$, with principal term g and ergodic term ψ.

Consider the semilinear evolution equation

$$x'(t) = Ax(t) + f(t, x(t)), \qquad t \in \mathbb{R}, \tag{3.1}$$

We also assume the following assumptions.

- H1. A is the generator of an hyperbolic analytic semigroup $(T(t))_{t\geq 0}$
- H2. $f(t,x)$ is uniformly continuous on every bounded subset $K \subset X_\alpha$ uniformly in $t \in \mathbb{R}$
- H3. $g(t,x)$ is is uniformly continuous on every bounded subset $K \subset X_\alpha$ uniformly in $t \in \mathbb{R}$
- H4. f satisfies the condition

$$\|f(t,x) - f(t,y)\| \leq k(t)\|x-y\|_\alpha$$

for every $t \in \mathbb{R}$ and $x, y \in X_\alpha$ and some function $k \in L^p(\mathbb{R}, \mathbb{R}^+)$ with $p \in (\frac{1}{1-\alpha}; \infty]$, such that

$$\left[M(\alpha)(\gamma q)^\alpha (\Gamma(1-\alpha q))^{1/q} + \frac{c(\alpha)}{(\gamma q)^{1/q}} \right] \|k\|_p < 1 \tag{3.2}$$

where q is the conjugate of p (note that $1 - q\alpha > 0$ since $p > \frac{1}{1-\alpha}$).

A mild solution of (3.1) is a continuous function $x : \mathbb{R} \to X_\alpha$ satisfying

$$x(t) = T(t-s)x(s) + \int_s^t T(t-\sigma)f(\sigma, x(\sigma))\, d\sigma \tag{3.3}$$

for all $t \geq s$ and all $s \in \mathbb{R}$.

Let's first consider the following inhomogeneous problem

$$\frac{d}{dt}x(t) = Ax(t) + h(t), \qquad t \in \mathbb{R}. \tag{3.4}$$

Proposition 3.1 *If $h \in PAA(\mathbb{R}, X)$, then there is a unique mild solution $x(\cdot)$ of (3.4) in $PAA(\mathbb{R}, X_\alpha)$ given by*

$$x(t) = \int_{-\infty}^{t} T(t-s)Ph(s)ds - \int_t^{+\infty} T(t-s)Qh(s)ds, \quad t \in \mathbb{R}. \tag{3.5}$$

Proof. First note that the function given by (3.5) is a mild solution of Equation (3.4) (cf [2]). By assumption, there exists $\beta \in AA(\mathbb{R}, X)$ and $\phi \in AA_0(\mathbb{R}, X)$ such that $h = \beta + \phi$. In view of [2] Proposition 3.1., the function

$$\xi(t) := \int_{-\infty}^{t} T(t-s)P\beta(s)ds - \int_t^{+\infty} T(t-s)Q\beta(s)ds, \quad t \in \mathbb{R}$$

is in $AA(\mathbb{R}, X_\alpha)$

It remains to prove that

$$\theta(t) := \int_{-\infty}^{t} T(t-s)P\phi(s)ds - \int_t^{+\infty} T(t-s)Q\phi(s)ds, \quad t \in \mathbb{R}$$

is in $AA_0(\mathbb{R}, X_\alpha)$.

Clearly $\theta(t) \in BC(\mathbb{R}, X_\alpha)$. Now let's prove that $\lim_{T\to\infty} \frac{1}{2T}\int_T^T \|\theta(s)\|_\alpha ds = 0$. Note that

$$\frac{1}{2T}\int_T^T \|\theta(s)\|_\alpha ds \leq \frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{t} \|T(t-s)P\phi(s)\|_\alpha dsdt +$$
$$\frac{1}{2T}\int_{-T}^{T}\int_t^{\infty} \|T(t-s)Q\phi(s)\|_\alpha dsdt = I_1 + J_1 + I_2 + J_2$$

where $I_i, J_i, i = 1.2$ are as follows.

$$\begin{aligned} I_1 &:= \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{t} \|T(t-s)P\phi(s)\|_\alpha dsdt \\ &\leq \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{t} M(\alpha)(t-s)^{-\alpha}e^{-\gamma(t-s)}\|\phi(s)\|dsdt \\ &= \frac{M(\alpha)}{2T}\int_{-T}^{T}\|\phi(s)\|ds\int_s^T (t-s)^{-\alpha}e^{-\gamma(t-s)}dt \\ &= \frac{M(\alpha)}{2T}\int_{-T}^{T}\|\phi(s)\|ds\int_0^{T-s} r^{-\alpha}e^{-\gamma r}dr \\ &\leq \frac{M(\alpha)}{2T}\int_{-T}^{T}\|\phi(s)\|ds\int_0^{\infty} r^{-\alpha}e^{-\gamma r}dr \\ &= \frac{M(\alpha)\Gamma(1-\alpha)}{\gamma^{\alpha-1}}\frac{1}{2T}\int_{-T}^{T}\|\phi(s)\|ds, \end{aligned}$$

which shows that $\lim_{T\to\infty} I_1 = 0$.

Now let's show that $\lim_{T\to\infty} J_1 = 0$.

Set $K = \sup_{t\in\mathbb{R}} \|\phi(t)\|$. Let us introduce $X > 0$. We have:

$$J_1 := \frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{-T} \|T(t-s)P\phi(s)\|_\alpha dsdt$$

$$\leq \frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{-T} M(\alpha)(t-s)^{-\alpha}e^{-\gamma(t-s)}\|\phi(s)\|dsdt$$
$$\leq \frac{M(\alpha)K}{2T}\int_{-T}^{T}\int_{-\infty}^{-T}(t-s)^{-\alpha}e^{-\gamma(t-s)}dsdt.$$

Let $X>0$. Let us introduce:

$$D=\{(s,t)\in\mathbb{R}^2,\quad s\leq -T,\quad \text{and}\quad |t|\leq T\},$$
$$D_1=\{(s,t)\in D, t-s\geq X\},\qquad D_2=D\setminus D_1.$$

So we have:

$$J_1=\frac{M(\alpha)K}{2T}\int\int_{D}(t-s)^{-\alpha}e^{-\gamma(t-s)}dsdt$$
$$=\frac{M(\alpha)K}{2T}\int\int_{D_1}(t-s)^{-\alpha}e^{-\gamma(t-s)}dsdt+$$
$$\frac{M(\alpha)K}{2T}\int\int_{D_2}(t-s)^{-\alpha}e^{-\gamma(t-s)}dsdt.$$

The first integral yields:

$$\frac{M(\alpha)K}{2T}\int\int_{D_1}(t-s)^{-\alpha}e^{-\gamma(t-s)}dsdt\leq\frac{M(\alpha)K}{2TX^{\alpha}}\int\int_{D_1}e^{-\gamma(t-s)}dsdt$$
$$\leq\frac{M(\alpha)K}{2TX^{\alpha}}\int\int_{D}e^{-\gamma(t-s)}dsdt=\frac{M(\alpha)K}{2T\gamma^2X^{\alpha}}\left(1-e^{-2\gamma T}\right).$$

And the second:

$$\frac{M(\alpha)K}{2T}\int\int_{D_2}(t-s)^{-\alpha}e^{-\gamma(t-s)}dsdt=$$
$$\frac{M(\alpha)K}{2T}\int_{-T-X}^{-T}\int_{-T}^{s+X}(t-s)^{-\alpha}e^{-\gamma(t-s)}dtds$$
$$=\frac{M(\alpha)K}{2T}\int_{-T-X}^{-T}\int_{-T-s}^{X}u^{-\alpha}e^{-\gamma u}duds$$
$$\leq\frac{M(\alpha)K}{2T}\int_{-T-X}^{-T}\int_{0}^{X}u^{-\alpha}e^{-\gamma u}duds$$
$$=\frac{M(\alpha)K}{2T}X\int_{0}^{X}u^{-\alpha}e^{-\gamma u}du.$$

Now, since $\int_0^\infty u^{-\alpha}e^{-\gamma u}du$ is convergent, given $\varepsilon>0$, it is possible to choose $X>0$ s.t.:

$$\frac{M(\alpha)K}{2}X\int_0^X u^{-\alpha}e^{-\gamma u}du\leq\frac{\varepsilon}{2}.$$

Let us take such an X. For all $T\geq 1$, we obtain:

$$J_1\leq\frac{M(\alpha)K}{2T\gamma^2X^{\alpha}}\left(1-e^{-2\gamma T}\right)+\frac{\varepsilon}{2T}\leq\frac{M(\alpha)K}{2T\gamma^2X^{\alpha}}+\frac{\varepsilon}{2}.$$

Now, for sufficiently large T, we have:

$$\frac{M(\alpha)K}{2T\gamma^2X^{\alpha}}\leq\frac{\varepsilon}{2},$$

and so:

$$J_1\leq\varepsilon.$$

The proof for J_1 is complete.

Now, let us consider I_2.

$$I_2:=\frac{1}{2T}\int_{-T}^{T}\int_{t}^{T}\|T(t-s)Q\phi(s)\|_{\alpha}dsdt$$
$$\leq\frac{1}{2T}\int_{-T}^{T}\int_{t}^{T}c(\alpha)e^{\delta(t-s)}\|\phi(s)\|dsdt$$
$$\leq\frac{c(\alpha)}{2T}\int_{-T}^{T}\|\phi(s)\|ds\int_{s-T}^{0}e^{\delta r}dr$$
$$\leq\frac{c(\alpha)}{2T}\int_{-T}^{T}\|\phi(s)\|ds\int_{-\infty}^{0}e^{\delta r}dr$$
$$=\frac{c(\alpha)}{2T}\int_{-T}^{T}\|\phi(s)\|ds.\delta^{-1}.$$

Thus $\lim_{T\to\infty}I_2=0$.

Finally we have,

$$J_2:=\frac{1}{2T}\int_{-T}^{T}\int_{T}^{\infty}\|T(t-s)Q\phi(s)\|_{\alpha}dsdt$$
$$\leq\frac{c(\alpha)}{2T}\int_{-T}^{T}\int_{T}^{\infty}e^{\delta(t-s)}\|\phi(s)\|dsdt$$
$$\leq\frac{c(\alpha)K}{2T}\int_{-T}^{T}\int_{T}^{\infty}e^{\delta(t-s)}dsdt$$
$$=\frac{c(\alpha)K}{2T\delta^2}\left(1-e^{-2\delta T}\right),$$

where $K=\sup_{t\in\mathbb{R}}\|\phi(t)\|$. This shows that $\lim_{T\to\infty}J_2=0$.

The proof is now complete.

Theorem 3.2 *Under the assumptions H1-H4, the evolution equation (3.1) has a unique pseudo almost automorphic solution $x(\cdot)$ in X_α ($x(\cdot)\in PAA(\mathbb{R},X_\alpha)$) satisfying*

$$x(t)=\int_{-\infty}^{t}T(t-s)Pf(s,x(s))ds-\int_{t}^{+\infty}T(t-s)Qf(s,x(s))ds,\quad t\in\mathbb{R}. \tag{3.6}$$

Proof.

Consider the mapping $\mathcal{G}:PAA(\mathbb{R},X_\alpha)\to PAA(\mathbb{R},X_\alpha)$ defined for all $t\in\mathbb{R}$ by

$$(\mathcal{G}x)(t):=\int_{-\infty}^{t}T(t-s)Pf(s,x(s))ds-\int_{t}^{+\infty}T(t-s)Qf(s,x(s))ds.$$

Using H1 and H2, we deduce that $f(\cdot,x(\cdot))\in PAA(\mathbb{R}\times X_\alpha,X)$ if $x\in PAA(\mathbb{R},X_\alpha)$ (cf Theorem 4.2 [7]). Thus $\mathcal{G}$ is well-defined. Now let $u,v\in PAA(\mathbb{R},X_\alpha)$. Then we have

$$\|(\mathcal{G}u)(t)-(\mathcal{G}v)(t)\|_{\alpha}\leq\int_{-\infty}^{t}\|T(t-s)P[f(s,u(s))-f(s,v(s))]\|_{\alpha}ds$$
$$+\int_{t}^{+\infty}\|T(t-s)Q[f(s,u(s))-f(s,v(s))]\|_{\alpha}ds$$
$$\leq M(\alpha)\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\gamma(t-s)}\|[f(s,u(s))-f(s,v(s))]\|_{\alpha}ds$$
$$+c(\alpha)\int_{t}^{+\infty}e^{\delta(t-s)}\|[f(s,u(s))-f(s,v(s))]\|_{\alpha}ds$$
$$\leq M(\alpha)\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\gamma(t-s)}k(s)\ \|u(s)-v(s)]\|_{\alpha}ds$$
$$+c(\alpha)\int_{t}^{+\infty}e^{\delta(t-s)}k(s)\|u(s)-v(s)]\|_{\alpha}ds$$
$$\leq\Bigg[M(\alpha)\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\gamma(t-s)}k(s)ds+$$
$$c(\alpha)\int_{t}^{+\infty}e^{\delta(t-s)}k(s)ds\Bigg]\sup_t\|u(t)-v(t)]\|_{\alpha}.$$

Now we use Hölder's inequality. Assume first that p is finite. We can write:

$$\|(\mathcal{G}u)(t)-(\mathcal{G}v)(t)\|_{\alpha}\leq$$
$$\left[M(\alpha)\left(\int_{-\infty}^{t}(t-s)^{-q\alpha}e^{-q\gamma(t-s)}ds\right)^{\frac{1}{q}}\left(\int_{-\infty}^{t}(k(s))^{p}ds\right)^{\frac{1}{p}}+\right.$$
$$\left.c(\alpha)\left(\int_{t}^{+\infty}e^{q\delta(t-s)}ds\right)^{\frac{1}{q}}\left(\int_{t}^{\infty}(k(s))^{p}ds\right)^{\frac{1}{p}}ds\right]\sup_t\|u(t)-v(t)]\|_{\alpha}$$
$$\leq\left[M(\alpha)(\gamma q)^{\alpha}(\Gamma(1-\alpha q))^{1/q}+\frac{c(\alpha)}{(\gamma q)^{1/q}}\right]\|k\|_{p}\sup_t\|u(t)-v(t)]\|_{\alpha}.$$

When $p=\infty$, we obtain directly the same result (with $q=1$). So, it is true for any p. And so:

$$\sup_t\|(\mathcal{G}u)(t)-(\mathcal{G}v)(t)\|_{\alpha}\leq$$
$$\left[M(\alpha)(\gamma q)^{\alpha}(\Gamma(1-\alpha q))^{1/q}+\frac{c(\alpha)}{(\gamma q)^{1/q}}\right]\|k\|_{p}\sup_t\|u(t)-v(t)]\|_{\alpha}.$$

The proof is completed, by using Banach's fixed point theorem.

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email:journal@bbcr.uwaterloo.ca