Singular perturbations of Partial Differential Equations on the Torus. Applications to Quasi-Periodic and Almost Periodic Solutions of Variational Problems.

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Abstract. In this paper we consider an equation of type $q''(t) = \partial_2 V(t, q(t))$, where V is quasi-periodic (q.p.) in t, uniformly w.r.t. to q, is convex w.r.t. q for each t. We look for q.p. solutions $q : \mathbb{R} \to \mathbb{H}$ (where $\mathbb{H}$ is an Hilbert space) to this equation. By using the formalism introduced by the Physician Percival, it is possible to transform this problem in a elliptic degenerate Partial Differential Equation on the torus. By using a singular perturbation method, we obtain existence results under technical assumptions on V.

1 Introduction

1.1 Some recalls and notations.

We consider a real[1] Hilbert space $\mathbb{H}$ (inner product $< .,. >$ and norm $|.|$). $AP(\mathbb{R}, \mathbb{H})$ is the space of Bohr-almost periodic (a.p.) functions from $\mathbb{R}$ to $\mathbb{H}$ [Ngue, Cord], $AP^n(\mathbb{R}, \mathbb{H})$ is the space of $f \in C^n(\mathbb{R}, \mathbb{H})$ functions s.t. each $f^{(k)} \in AP(\mathbb{R}, \mathbb{H})$, for $k \leq n$.

We recall that $f \in AP(\mathbb{R}, \mathbb{H})$ can be uniformly approximated by trigonometric polynomials, i.e. linear combinaisons of functions $t \mapsto e^{i\lambda t}$, with $\lambda \in \mathbb{R}$. We also know that for $f \in AP(\mathbb{R}, \mathbb{H})$, the mean value:

$$\mathcal{M}\{f\} = \mathcal{M}\{f(t)\}_t := \lim_{T \to +\infty} \frac{1}{T} \int_0^T f(t)dt,$$

exists in $\mathbb{H}$. These tools permits us to associate to f a Fourier-Bohr expansion:

$$f(t) \sim \sum_{\lambda \in \mathbb{R}} a_\lambda(f) e^{i\lambda t},$$

where in fact we have convergence in the quadratic mean. The Fourier-Bohr coefficients are given by:

$$a_\lambda(f) = \mathcal{M}\{f(t)e^{-i\lambda t}\}_t.$$

The set:

$$\Lambda(f) = \{\lambda \in \mathbb{R},\ a_\lambda(f) \neq 0\}$$

is countable, and we denote by $Mod(f)$ the $\mathbb{Z}-$module generated by $\Lambda(f)$. The function f is called quasi-periodic (q.p.) if there exists a finite basis $(\omega_1, \ldots, \omega_N)$ of the module i.e.:

$$\Lambda(f) = \mathbb{Z}\omega_1 + \ldots + \mathbb{Z}\omega_N,$$

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[1]For commodity, Fourier analysis will be written with functions with values in the complexification of $\mathbb{H}$, also called $\mathbb{H}$.

and $\sum_j k_j\omega_j = 0$ with $k_j \in \mathbb{Z}$ for all j implies that $k_1 = \ldots = k_N = 0$. We note by $QP^0(\mathbb{R}, \mathbb{H})$ the set of quasi-periodic functions, and by $QP^0_\omega(\mathbb{R}, \mathbb{H})$ the set of q.p. functions f s.t. $\Lambda(f) \subset \mathbb{Z}\omega_1 + \ldots + \mathbb{Z}\omega_N$, with $\omega = (\omega_1, \ldots, \omega_N)$. We note $QP^n_\omega(\mathbb{R}, \mathbb{H}) = QP^0_\omega(\mathbb{R}, \mathbb{H}) \cap AP^n(\mathbb{R}, \mathbb{H})$.

Now, for fixed ω, we consider the torus $\mathbb{T}^N = (\mathbb{R}/(2\pi\mathbb{Z}))^N$ (it is a compact abelian group). The mapping $\mathcal{Q}_\omega : C^0(\mathbb{T}^N, \mathbb{H}) \to QP^0_\omega(\mathbb{R}, \mathbb{H})$ defined by $\mathcal{Q}_\omega(u) = [t \mapsto u(t\omega)]$ is a bijection. Now we define the set $C^1_\omega(\mathbb{T}^N, \mathbb{H})$ of functions $u \in C^0(\mathbb{T}^N, \mathbb{H})$ s.t. at every $x \in \mathbb{T}^N$, the directional derivative:

$$\partial_\omega u(x) = \lim_{t \to 0} \frac{u(x + t\omega) - u(x)}{t}$$

exists. Recurcively, we set:

$$C^n_\omega(\mathbb{T}^N, \mathbb{H}) = \{u \in C^1_\omega(\mathbb{T}^N, \mathbb{H}); \partial_\omega u \in C^{n-1}_\omega(\mathbb{T}^N, \mathbb{H})\}.$$

Then $\mathcal{Q}_\omega$ is a bijection between $C^n_\omega(\mathbb{T}^N, \mathbb{H})$ and $QP^n_\omega(\mathbb{R}, \mathbb{H})$, and morevover for each $k \leq n$, we have:

$$\mathcal{Q}_\omega(\partial^k_\omega u) = (\mathcal{Q}_\omega u)^{(k)}.$$

If $u \in C^0(\mathbb{T}^N, \mathbb{H})$, it admits a Fourier expansion:

$$u \sim \sum_{\nu \in \mathbb{Z}^N} \hat{u}(\nu) e_\nu,$$

where[2] $e_\nu : x \mapsto e^{i\nu.x}$ and:

$$\hat{u}(\nu) = \int_{\mathbb{T}^N} f(x) e_{-\nu}(x) \frac{dx}{(2\pi)^N}.$$

After for commodity, we will introduce the standart normalized Haar measure μ on $\mathbb{T}^N$, which satisfies:

$$d\mu(x) = \frac{dx}{(2\pi)^N}.$$

Now, we introduce the spaces for weak solutions, with which we will be concerned after. We note $L^2 = L^2(\mathbb{T}^N; \mathbb{H})$, with standart scalar product:

$$(f, g)_{L^2} = \int_{\mathbb{T}^N} < f(x), g(x) > d\mu(x).$$

Its norm is written $\|.\|_{L^2}$.

We also consider the standart Sobolev space:

$$H^1 = H^1(\mathbb{T}^N; \mathbb{H}) = \left\{ u \in L^2, \quad \forall i = 1, \ldots, N, \quad \frac{\partial u}{\partial x_i} \in L^2 \right\},$$

endowed with the scalar product:

$$(u, v)_{H^1} = (u, v)_{L^2} + \sum_{i=1}^N \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)_{L^2}.$$

If we note $Du := \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N} \right) : \mathbb{T}^N \to \mathbb{H}^N$, we have:

$$(u, v)_{H^1} = (u, v)_{L^2} + (Du, Dv)_{L^2}.$$

For such q.p. problems, we also have to introduce (see [10]) $H^1_\omega = H^1_\omega(\mathbb{T}^N, \mathbb{H})$ defined by:

$$H^1_\omega = \left\{ u \in L^2, \partial_\omega u \in L^2 \right\},$$

[2] For $\nu \in \mathbb{Z}^N$ and $x \in \mathbb{R}^N$, we note $\nu.x = \sum_{j=1}^N \nu_j x_j$ and $|\nu| = \sqrt{\sum_{j=1}^N \nu_j^2}$.

with the inner product:

$$(f,g)_{H^1_\omega} = \int_{\mathbb{T}^N} (< u(x), v(x) > + < \partial_\omega u(x), \partial_\omega v(x) >)\, d\mu(x).$$

In fact, the operator $\mathcal{Q}_\omega$ is a bijection between $L^2(\mathbb{T}^N, \mathbb{H})$ (resp. $H^1_\omega(\mathbb{T}^N, \mathbb{H})$) and the Besicovitch space $B^2_\omega(\mathbb{R}, \mathbb{H})$ (resp. $B^{1,2}(\mathbb{R}, \mathbb{H})$) (see [10] for details). It has been done with $\mathbb{H} = \mathbb{R}^p$ but is can be adpated to an abstract Hilbert space).

The link described before, between q.p. functions and multiperiodic functions was first introduced by Percival [Per] and more precisely used in [10]. It permits to associate to a q.p. problem, for instance the search of q.p. solutions of:

$$q''(t) = F(q(t)) + e(t)$$

with $e \in QP^0(\mathbb{R}, \mathbb{H})$ a Partial Differential Equation, for which we look multiperiodic solutions:

$$\partial^2_\omega u(x) = F(u(x)) + E(x),$$

where $E = \mathcal{Q}_\omega^{-1}(e)$. The link, established in [10] is that formally u is a solution (strong of weak) of the second iff. $\mathcal{Q}_\omega(u)$ is a solution of the first equation. But one problem with the second equation is the lack of compacity in unidirectional derivative Sobolev space H^1_ω which corresponds to weak solutions in the space $B^{1,2}_\omega(\mathbb{R}, \mathbb{H})$ (see for instance lemma 1.2 for more explanations). To treat this difficulty, we shall introduce a singular pertubation in the problem to make it strongly elliptic. But also we will deal with a little bit more general problem.

Each $u \in L^2$ can be developped in a Fourier expansion:

$$u \sim \sum_{\nu \in \mathbb{Z}^N} \hat{u}(\nu) e_\nu,$$

We have $(\hat{u}(\nu)) \in \ell^2(\mathbb{Z}^N; \mathbb{H})$ and moreover:

- $\|u\|^2_{L^2} = \sum_{\nu \in \mathbb{Z}^N} |\hat{u}(\nu)|^2$.
- $u \in H^1$ iff. $\sum_{\nu \in \mathbb{Z}^N} (1 + |\nu|^2)|\hat{u}(\nu)|^2 < +\infty$, and we have $\|u\|^2_{H^1} = \sum_{\nu \in \mathbb{Z}^N} (1 + |\nu|^2)|\hat{u}(\nu)|^2$.
- $u \in H^1_\omega$ iff. $\sum_{\nu \in \mathbb{Z}^N} (1 + (\nu.\omega)^2)|\hat{u}(\nu)|^2 < +\infty$, and we have $\|u\|^2_{H^1_\omega} = \sum_{\nu \in \mathbb{Z}^N} (1 + (\nu.\omega)^2)|\hat{u}(\nu)|^2$.

We recall that $\hat{u}(\nu) = \int_{\mathbb{T}^N} u(x) e_{-\nu}(x) d\mu(x)$. In particular, $\hat{u}(0) = \int_{\mathbb{T}^N} u(x) d\mu(x)$ is the mean of function u.

Now, for each $u \in L^2$ we consider the decomposition $u = \bar{u} + \tilde{u}$ where $\bar{u} = \int_{\mathbb{T}^N} u(x) d\mu(x)$ is the mean of u. We set:

$$\tilde{L^2} = \{u \in L^2, \bar{u} = 0\}.$$

It is easy to verify that $\tilde{L^2}$ is a closed subspace of L^2, and that the sum $\mathbb{H} + \tilde{L^2}$ is direct and orthogonal. So we have:

$$\|u\|^2_{L^2} = |\bar{u}|^2 + \|\tilde{u}\|^2_{L^2}.$$

We also introduce $\tilde{H^1} = H^1 \cap \tilde{L^2}$ and $\tilde{H^1_\omega} = H^1_\omega \cap \tilde{L^2}$.

Lemma 1.1 *We have the following assertions:*

- $\forall u \in H^1, \quad \|\tilde{u}\|_{L^2} \leq \|Du\|_{L^2}$.
- $\forall u \in \tilde{H^1}, \quad \|Du\|_{L^2} \leq \|u\|_{H^1} \leq \sqrt{2}\|Du\|_{L^2}$.
- *There does not exists $C > 1$ s.t. for all $u \in \tilde{H^1_\omega}$, $\|u\|_{H^1_\omega} \leq C\|\partial_\omega u\|_{L^2}$.*

Let us prove the first assertion. In fact, we have, for all $\nu \in \mathbb{Z}^N \setminus \{0\}$, $1 \le |\nu|$. And so:

$$\|\tilde{u}\|_{L^2}^2 = \sum_{\nu \in \mathbb{Z}^N \setminus \{0\}} |\hat{u}(\nu)|^2 \le \sum_{\nu \in \mathbb{Z}^N \setminus \{0\}} |\nu|^2 |\hat{u}(\nu)|^2 = \|Du\|_{L^2}^2.$$

For the second inequality of the second assertion, the argument is the same. Now let us see why the third assertion is right (it says that the second property is not true on H^1_ω). It is related to the small divisors problem. For each $\varepsilon > 0$, we can find $\nu_\varepsilon \in \mathbb{Z}^N \setminus \{0\}$ s.t. $|\nu_\varepsilon . \omega| \le \varepsilon$. Let us choose $u_\varepsilon := e_{\nu_\varepsilon} \in \tilde{H^1_\omega}$. If $C > 1$ satisfies:

$$\|u_\varepsilon\|_{H^1_\omega} \le C \|\partial_\omega u_\varepsilon\|_{L^2},$$

we obtain:

$$C^2 - 1 \ge \frac{1}{(\nu_\varepsilon . \omega)^2} \ge \frac{1}{\varepsilon^2},$$

and so, when $\varepsilon \to +\infty$, we obtain that $C = +\infty$.

We recall some facts about the Rellich property. It is adapted from [Willem].

Lemma 1.2 *If $(u_m)_m \in (H^1)^{\mathbb{N}}$ is s.t. $u_m \rightharpoonup 0$ weakly in H^1, then $u_m \to 0$ strongly in L^2. This result is false in H^1_ω.*

Following [32], let us give the proof of the first property. Since $u_m \rightharpoonup 0$ weakly in H^1, the sequence $(u_m)_m$ is bounded in H^1 and so we can consider $C = \sup_m \|u_m\|_{H^1} \in \mathbb{R}^+$. Moreover, $\hat{u}_m(\nu) = (e_\nu, u_m) \to 0$ when $m \to +\infty$. Given $R \in \mathbb{N}$, we have:

$$\|u_m\|_{L^2}^2 = \sum_{|\nu| \le R} |u_m(\nu)|^2 + \sum_{|\nu| > R} |u_m(\nu)|^2 \le \sum_{|\nu| \le R} |u_m(\nu)|^2 + \frac{1}{1+R^2} \sum_{|\nu| > R} (1 + |\nu|^2) |u_m(\nu)|^2$$

$$\le \sum_{|\nu| \le R} |u_m(\nu)|^2 + \frac{C}{1+R^2}.$$

Given $\varepsilon > 0$, we first choose R s.t. $\frac{C}{1+R^2} \le \varepsilon$. After we take m sufficiently large to have $\sum_{|\nu| \le R} |u_m(\nu)|^2 \le \varepsilon$ (we have here a finite sum of terms going to 0). And so for sufficienlty large m, $\|u_m\|_{L^2}^2 \le 2\varepsilon$. The Rellich property is true in H^1. Now, we consider a sequence $(\nu_j)_j$ s.t. $|\nu_j . \omega| \le 1$ and $|\nu_j| > |\nu_{j-1}|$. We introduce the sequence $(u_m)_m \in H^1_\omega$ as follows:

$$\forall m \ge 1, \quad u_m = \frac{1}{\sqrt{m}} \sum_{j=1}^{m} e_{\nu_j}.$$

We note that $\|u_m\|_{L^2} = 1$ for each m, and so (u_m) is bounded in L^2. Moreover $\|u_m\|_{H^1_\omega} \le \sqrt{2}$ and for each ν, $\hat{u}_m(\nu) \to 0$ when $m \to +\infty$. With the same calculations as preeceding, if $v \in H^1_\omega$, we arrive at:

$$|(v, u_m)_{H^1_\omega}| \le \sum_{|\nu| \le R} (1 + |\nu|^2) | < \hat{v}(\nu), u_m(\nu) > | + \sqrt{2} \sum_{|\nu| > R} (1 + |\nu|^2) |\hat{v}(\nu)|^2,$$

from which we obtain $u_m \rightharpoonup 0$ weakly in H^1_ω. But since $\|u_m\|_{L^2} = 1$ for all m, we don't have $u_m \to 0$ strongly in L^2.

1.2 The problem

Let us now consider an operator $V : \mathbb{R} \times \mathbb{H} \to \mathbb{H}$ such that $V(t,.)$ is C^1 and convex for all t and $V(.,y)$ is q.p. uniformly w.r.t. the parameter y. Let us consider $\omega = (\omega_1, \ldots, \omega_N)$ a common $\mathbb{Z}-$basis of the $\mathbb{Z}-$module. We know that there exists a unique $A \in C^0(\mathbb{T}^N \times \mathbb{H}; \mathbb{H})$

s.t. $A(t\omega, y) = V(t, y)$ for all $(t, y) \in \mathbb{R} \times \mathbb{H}$. The aim is, under conditions, to give existence theorems of q.p. solutions q of the equation:

$$q''(t) = \frac{\partial V}{\partial y}(t, q(t)). \tag{1}$$

Following what said before, we can formally associate to this equation a P.D.E. on the torus $\mathbb{T}^N$:

$$\partial_\omega^2 u(x) = \frac{\partial A}{\partial y}(x, u(x)), \tag{2}$$

We note that equation (2) can be written:

$$\sum_{1 \le j,k \le N} \omega_j \omega_k \frac{\partial^2 u}{\partial x_j \partial x_k}(x) = \frac{\partial A}{\partial y}(x, u(x)). \tag{3}$$

This equation, with a one-direction derivative, is a degenerate elliptic P.D.E. In order to solve it, we introduce a perturbative term to make it strongly elliptic. We will solve in fact the familly of equations depending on $m \in \mathbb{N}^*$[3]:

$$\sum_{1 \le j,k \le N} \omega_j \omega_k \frac{\partial^2 u}{\partial x_j \partial x_k}(x) + \frac{1}{m}\left(\sum_{j=1}^N \frac{\partial^2 u(x)}{\partial x_j^2} - u(x)\right) = \frac{\partial A}{\partial y}(x, u(x)).$$

So we introduce for all $m \in \mathbb{N}^*$:

$$a_{jk}^m = \begin{cases} \omega_j \omega_k & \text{if} \quad j \neq k, \\ \omega_j^2 + \frac{1}{m} & \text{if} \quad j = k. \end{cases}$$

and we know consider:

$$\sum_{1 \le j,k \le N} a_{jk}^m \frac{\partial^2 u}{\partial x_j \partial x_k}(x) - \frac{1}{m} u(x) = \frac{\partial A}{\partial y}(x, u(x)). \tag{4}$$

The assumptions. We know introduce more precisely the assumptions:

(A1) A is measurable, and for all $x \in \mathbb{T}^N$, the function $A(x, .) : \mathbb{T}^N \to \mathbb{R}$ is of class C^1 and convex,

(A2) There exists $\varphi_0 \in L^2$ s.t. $A(., \varphi_0(.)) \in L^1(\mathbb{T}^N; \mathbb{H})$ and $\frac{\partial A}{\partial y}(., \varphi_0(.)) \in L^2$,

(A3) $\exists a \in L^2, \quad \exists b \in L^1(\mathbb{T}^N; \mathbb{R}), \quad A(x, y) \geq < a(x), y > + b(x)$,

(A4) $\exists (\alpha_1, \beta_1, \gamma_1) \in \mathbb{R}_*^+ \times L^2(\mathbb{T}^N; \mathbb{R}) \times L^1(\mathbb{T}^N; \mathbb{R}), \forall (x, y) \in \mathbb{T}^N \times \mathbb{H}$:

$$\left\langle \frac{\partial A(x, y)}{\partial y}, y \right\rangle \geq \alpha_1 |y|^2 - \beta_1(x)|y| - \gamma_1(x),$$

and[4] $\|\beta_1\|_{L^2}^2 + 4 \min\{1, \alpha_1\} \left(\int_{\mathbb{T}^N} \gamma_1 d\mu\right) \geq 0$,

(A5) $\exists (c, d) \in L^2(\mathbb{T}^N; \mathbb{R}) \times L^1(\mathbb{T}^N, \mathbb{R})$:

$$\left| \frac{\partial A}{\partial y}(x, y) \right| \leq c(x)|y| + d(x) \qquad a.e.$$

[3] When $\mathbb{H} = \mathbb{R}$, the perturbative term is : $\frac{1}{m}(\Delta u - u)$.
[4] Note that the following inequality is true when $\int_{\mathbb{T}^N} \gamma_1 d\mu \geq 0$.

We give a particular example of V s.t. the associate A satisfies these assumptions. We assume that V is of the form:

$$V(t,y) = \varphi(t)f(y) + \psi(t)y.$$

Here we are concerned with an equation:

$$q''(t) = \varphi(t)f'(q(t)) + \psi(t).$$

We assume:

(i) $\varphi, \psi \in AP^0(\mathbb{R}, \mathbb{R})$ and $\inf \varphi > 0$.

(ii) $f \in C^1(\mathbb{R}, \mathbb{H})$ is convex.

(iii) $\exists(\alpha, \beta) \in \mathbb{R}_*^+ \times \mathbb{R}$, $y \mapsto < f'(y), y > -\alpha|y|^2 + \beta|y|$ is bounded from below.

(iv) $\exists(\mu, \nu) \in (\mathbb{R}_*^+)^2$, $\forall y \in \mathbb{R}$, $|f'(y)| \leq \mu|y| + \nu$.

Then assumptions **(A1)-(A5)** hold for the associated A. Let us see it. A is defined by:

$$A(x,y) = \Phi(x)f(y) + \Psi(x)y,$$

where $\Phi, \Psi \in C^0(\mathbb{T}^N, \mathbb{H})$ are uniquely defined by $\Phi(t\omega) = \varphi(t)$ and $\Psi(t\omega) = \psi(t)$ for all $t \in \mathbb{R}$. These functions are bounded, and $\inf \Phi = \inf \varphi > 0$ because of the density of $\{t\omega, t \in \mathbb{R}\}$ on $\mathbb{T}^N$. Assumption **(A1)** is clear. For **(A2)**, we can choose for instance $\varphi_0 = 0$. For **(A3)**, we see that the convexity of f gives:

$$\forall y, \quad f(y) \geq < f'(0), y > + f(0),$$

from what we deduce:

$$\forall y, \quad A(x,y) \geq < \Phi(x)f'(0), y > + \Psi(x)f(0).$$

For **(A4)**, we know that there exists $\gamma \in \mathbb{R}^+$ s.t.:

$$\forall y \in \mathbb{H}, \quad < f'(y), y > -\alpha|y|^2 + \beta|y| \geq -\gamma.$$

And so:

$$\forall y \in \mathbb{H}, \quad < f'(y), y > \geq \alpha|y|^2 - \beta|y| - \gamma.$$

But $\frac{\partial A(x,y)}{\partial y} = \Phi(x)f'(y) + \Psi(x)$. So, by setting $m_1 := \inf \Phi > 0$ and $m_2 := \inf \Psi \in \mathbb{R}$, we obtain:

$$\left\langle \frac{\partial A(x,y)}{\partial y}, y \right\rangle = \Phi(x) < f'(y), y > + \Psi(x) \geq m_1 < f'(y), y > + m_2 \geq m_1\alpha|y|^2 - m_1\beta|y| - (m_1\gamma - m_2).$$

Now by increasing γ if necessary, we can suppose that $m_1\gamma - m_2 \geq 0$ (see that the more we increase γ, the more the property defining γ is true). And so, by taking constant functions:

$$\alpha_1(x) := m_1\alpha > 0, \qquad \beta_1(x) := -m_1\beta, \qquad \gamma_1(x) := m_1\gamma - m_2,$$

we see that **(A4)** is true. And **(A5)** follows immediatly from **(iv)**, since Φ and Ψ are bounded.

2 The Perturbative Equation.

Now we define $\phi_m : H^1 \to \overline{\mathbb{R}}$ as follows:

$$\phi_m(u) := \begin{cases} \frac{1}{2}\sum a^m_{jk}\left(\frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_k}\right)_{L^2} + \frac{1}{2m}\|u\|^2_{L^2} + \int_{\mathbb{T}^N} A(x,u(x))d\mu(x) & \text{if} \quad A(.,u(.)) \in L^1(\mathbb{T}^N,\mathbb{R}), \\ +\infty \quad \text{if} \quad A(.,u(.)) \notin L^1(\mathbb{T}^N,\mathbb{R}) \end{cases},$$

where $\sum$ stands for $\sum_{1\le j,k\le N}$.

Lemma 2.1 *Under* **(A1), (A2), (A3)**, *ϕ_m is a l.s.c. differential, and its subdifferential $\partial\phi_m(u)$ is the set given by:*

$$\left\{ v \mapsto \sum a^m_{jk}\left(\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_k}\right)_{L^2} + \frac{1}{m}(u,v)_{L^2} + \int_{\mathbb{T}^N} < \frac{\partial A}{\partial y}(x,u(x)), v(x) > d\mu(x) \right\},$$

if $\frac{\partial A}{\partial y}(.,u(.)) \in L^2(\mathbb{T}^N,\mathbb{H})$ and is $\emptyset$ if $\frac{\partial A}{\partial y}(.,u(.)) \notin L^2(\mathbb{T}^N,\mathbb{H})$.

The functional ϕ_m can be written as: $\phi_m = Q_m + I$, where:

$$Q_m(u) := \frac{1}{2}\sum_{1\le j,k\le N} a^m_{jk}\left(\frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_k}\right)_{L^2} + \frac{1}{2m}\|u\|^2_{L^2}$$

and:

$$I(u) := \begin{cases} \int_{\mathbb{T}^N} A(x,u(x))d\mu(x) & \text{if} \quad A(.,u(.)) \in L^1(\mathbb{T}^N;\mathbb{R}) \\ +\infty & \text{elsewhere.} \end{cases}$$

Q_m is a convex quadratic and continuous functional (and so C^1), and we have:

$$Q'_m(u).v = \sum_{1\le j,k\le N} a^m_{jk}\left(\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_k}\right)_{L^2} + \frac{1}{m}(u,v)_{L^2}.$$

Moreover, Propositions 2.7 and 2.8 (chapter 2) in [2] , can be quickly adapted to our problem to say that I is a convex l.s.c. functional with subdifferential:

$$\partial I(u) = \left\{\frac{\partial A}{\partial y}(.,u(.))\right\} \cap L^2(\mathbb{T}^N,\mathbb{H}).$$

So the subdifferential is reduced to the function $\frac{\partial A}{\partial y}(.,u(.))$ if this one is $L^2(\mathbb{T}^N;\mathbb{H})$, and emptyset if not.

By sum, ϕ_m is a l.s.c. convex functional. But since $I(\varphi_0)$ is finite and Q_m is continuous at φ_0, we can say that:

$$\partial(Q_m + I)(u) = \partial Q_m(u) + \partial I(u).$$

Remark 2.2 *In fact, if we also assume the condition* **(A5)**, *the Nemytskii operator on $\frac{\partial A}{\partial y}$ maps continuously L^2 in L^2, and so in this case, ϕ_m is finite, continuous and Gâteaux-differentiable everywhere, with Gâteaux-differential:*

$$D_G\phi_m(u).v = \sum_{1\le j,k\le N} a^m_{jk}\left(\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_k}\right)_{L^2} + \frac{1}{m}(u,v)_{L^2} + \int_{\mathbb{T}^N} < \frac{\partial A}{\partial y}(x,u(x)), v(x) > d\mu(x)$$

Lemma 2.3 *Under* **(A1), (A2), (A4)**, *ϕ_m is coercive on $H^1(\mathbb{T}^N;\mathbb{H})$.*

Since $A(x,.)$ is convex for any x, have the following:

$$\forall (y_1,y_2) \in \mathbb{H}^2, \quad A(x,y_2) - A(x,y_1) \ge \left\langle \frac{\partial A}{\partial y}(x,y_1), (y_2 - y_1)\right\rangle.$$

Let us take $y_1 = \varphi_0(x)$ and $y_2 = u(x)$ in this relation, and after we integrate to obtain:

$$\int_{\mathbb{T}^N} (A(x, u(x)) - A(x, \varphi_0(x)))d\mu(x) \geq \int_{\mathbb{T}^N} \left\langle \frac{\partial A}{\partial y}(x, \varphi_0(x)), u(x) - \varphi_0(x) \right\rangle d\mu(x).$$

Note that $K := \| \frac{\partial A}{\partial y}(., \varphi_0(.))\| \in \mathbb{R}^+$ by assumption **(A2)**. The preeceeding inequality gives:

$$\int_{\mathbb{T}^N} (A(x, u(x)) - A(x, \varphi_0(x)))d\mu(x) \geq -K\|u - \varphi_0\|_{L^2}.$$

But $\|u - \varphi_0\|_{L^2} \leq \|u\|_{L^2} + \|\varphi_0\|_{L^2}$ and so:

$$\int_{\mathbb{T}^N} (A(x, u(x)) - A(x, \varphi_0(x)))d\mu(x) \geq -K(\|u\|_{L^2} + \|\varphi_0\|_{L^2}).$$

Introducing the real $K' := \int_{\mathbb{T}^N} A(x, \varphi_0(x))d\mu(x) - K\|\varphi_0\|_{L^2}$, we obtain:

$$I(u) \geq -K\|u\|_{L^2} + K'.$$

But:

$$Q_m(u) = \frac{1}{2}\|\partial_\omega u\|^2_{L^2} + \frac{1}{2m}\|Du\|^2_{L^2} + \frac{1}{2m}\|u\|^2_{L^2} \geq \frac{1}{2m}\|u\|^2_{H^1}.$$

We deduce from this that:

$$\phi_m(u) \geq \frac{1}{2m}\|u\|^2_{H^1} - K\|u\|_{L^2} + K' \geq \|u\|^2_{H^1} - K\|u\|_{H^1} + K'.$$

It follows from this inequality that $\lim_{\|u\|_{H^1} \to +\infty} \phi_m(u) = +\infty$, the coercivity of ϕ_m.

Proposition 2.4 *Under* **(A1), (A2), (A3)***, the equation (4) admits a solution* $u_m \in H^1(\mathbb{T}^N; \mathbb{H})$*.*

Since $H^1(\mathbb{T}^N; \mathbb{H})$ is a Hilbert space, it is reflexive. On this space, ϕ_m is convex and coercive and l.s.c, and so it admits a minimum $u_m \in H^1$:

$$\exists u_m \in H^1(\mathbb{T}^N; \mathbb{H}), \quad \forall u \in H^1(\mathbb{T}^N; \mathbb{H}), \quad \phi_m(u) \geq \phi_m(u_m).$$

Since u_m is a minimum of ϕ_m, we have: $0 \in \partial\phi_m(u_m)$. So, first we obtain that $\partial\phi_m(u_m)$ is not empty from which we deduce that $\frac{\partial A}{\partial y}(., u_m(.)) \in L^2$, and equation $0 \in \partial\phi_m(u_m)$ says also that:

$$\forall v \in H^1, \quad \sum_{1 \leq j,k \leq N} a^m_{jk} \left(\frac{\partial u_m}{\partial x_j}, \frac{\partial v}{\partial x_k} \right)_{L^2} + \frac{1}{m}(u_m, v)_{L^2} + \int_{\mathbb{T}^N} \frac{\partial A}{\partial y}(x, u_m(x)).v(x)d\mu(x) = 0. \quad (5)$$

The preeceeding equation implies:

$$\forall v \in H^1(\mathbb{T}^N; \mathbb{H}), \quad \int_{\mathbb{T}^N} \left[\sum_{1 \leq j,k \leq N} a^m_{jk} \left\langle \frac{\partial u_m(x)}{\partial x_j}, \frac{\partial v(x)}{\partial x_k} \right\rangle + \frac{1}{m} < u_m(x), v(x) > + < \frac{\partial A}{\partial y}(x, u_m(x)), v(x) > \right] d\mu(x) =$$

So, $\sum_{1 \leq j,k \leq N} a^m_{jk} \frac{\partial^2 u_m}{\partial x_j \partial x_k}$ is L^2 and we can write:

$$\forall v \in H^1(\mathbb{T}^N; \mathbb{H}), \quad \int_{\mathbb{T}^N} \left\langle -\sum_{1 \leq j,k \leq N} a^m_{jk} \frac{\partial^2 u_m(x)}{\partial x_j \partial x_k} + \frac{1}{m}u_m(x) + \frac{\partial A}{\partial y}(x, u_m(x)), v(x) \right\rangle d\mu(x) = 0,$$

i.e.:

$$\sum_{1 \leq j,k \leq N} a^m_{jk} \frac{\partial^2 u_m(x)}{\partial x_j \partial x_k} - \frac{1}{m}u_m(x) = \frac{\partial A}{\partial y}(x, u_m(x)),$$

and so u_m is solution of (4). This ends the proof of the proposition.

Proposition 2.5 *Under* **(A1), (A2), (A3), (A4)**, $(u_m)_m$ *is bounded on* $H^1_\omega(\mathbb{T}^N;\mathbb{H})$.

To prove this, we first take $v = u_m$ in (5). This shows that:

$$\sum_{1\le j,k\le N} a^m_{jk} \left\|\frac{\partial u_m}{\partial x_j}\right\|^2_{L^2} + \frac{1}{m}\|u_m\|^2_{L^2} + \int_{\mathbb{T}^N} \left\langle \frac{\partial A}{\partial y}(x,u_m(x)), u_m(x)\right\rangle d\mu(x) = 0.$$

But:

$$\sum_{1\le j,k\le N} a^m_{jk} \left\|\frac{\partial u_m}{\partial x_j}\right\|^2_{L^2} = \|\partial_\omega u_m\|^2_{L^2} + \frac{1}{m}\|Du_m\|^2_{L^2},$$

and so we obtain:

$$\|\partial_\omega u_m\|^2_{L^2} + \frac{1}{m}\|Du_m\|^2_{L^2} + \frac{1}{m}\|u_m\|^2_{L^2} = -\int_{\mathbb{T}^N} \left\langle \frac{\partial A}{\partial y}(x,u_m(x)), u_m(x)\right\rangle d\mu(x). \tag{6}$$

So:

$$\|\partial_\omega u_m\|^2_{L^2} + \frac{1}{m}\|u_m\|^2_{H^1} + \frac{1}{m}\|u_m\|^2_{L^2} = -\int_{\mathbb{T}^N} \left\langle \frac{\partial A}{\partial y}(x,u_m(x)), u_m(x)\right\rangle d\mu(x) \le$$

$$\int_{\mathbb{T}^N} \left(-\alpha_1|u_m(x)|^2 + \beta_1(x)u_m(x) + \gamma_1(x)\right) d\mu(x) \le -\alpha_1\|u_m\|^2_{L^2} + \|\beta_1\|_{L^2}\|u_m\|_{L^2} + \int_{\mathbb{T}^N} \gamma_1 d\mu.$$

By setting $\delta_1 = \min\{1,\alpha_1\} > 0$ and $\gamma_2 = \int_{\mathbb{T}^N} \gamma_1 d\mu$, we deduce:

$$\delta_1\|u_m\|^2_{H^1_\omega} \le \|\partial_\omega u_m\|^2_{L^2} + \alpha_1\|u_m\|^2_{L^2} \le \|\beta_1\|_{L^2}\|u_m\|_{L^2} + \gamma_2 \le \|\beta_1\|_{L^2}\|u_m\|_{H^1_\omega} + \gamma_2.$$

We introduce the polynom $P(T) = \delta_1 T^2 - \|\beta_1\|_{L^2}T - \gamma_2$. Since by assumption $\|\beta_1\|^2_{L^2} + 4\gamma_2\delta_1 \ge 0$, this polynom has its zeros real. The greatest one is:

$$R_1 = \frac{\|\beta_1\|_{L^2} + \sqrt{\|\beta_1\|^2_{L^2} + 4\gamma_2\delta_1}}{2\delta_1}.$$

But we have $P(\|u_m\|_{H^1_\omega}) \le 0$. From this, we obtain that $\|u_m\|_{H^1_\omega} \le R_1$. This proves that $(u_m)_m$ is bounded on H^1_ω.

Remark 2.6 *We also obtain that:*

$$\sup\left[\frac{\|u_m\|_{H^1}}{\sqrt{m}}\right] \le \sqrt{\|\beta_1\|_{L^2}R_1 + \gamma_2} < +\infty.$$

But we don't know if $(u_m)_m$ *is bounded on* H^1.

Theorem 2.7 *Under* **(A1), (A2), (A3), (A4), (A5)**, *the equation (1) admits a weak solution.*

Now we set $\phi : H^1 \to \overline{\mathbb{R}}$ as follows:

$$\phi(u) := \frac{1}{2}\sum_{1\le j,k\le N} \omega_j\omega_k \left(\frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_k}\right)_{L^2} + \int_{\mathbb{T}^N} A(x,u(x))d\mu(x).$$

We see that for each $u \in H^1$, the sequence $(\phi_m(u))_m$ is decreasing to $\phi(u)$. So we have:

$$\inf_{u\in H^1}\phi(u)=\inf_{u\in H^1}\inf_{m\in\mathbb{N}^*}\phi_m(u)=\inf_{m\in\mathbb{N}^*}\inf_{u\in H^1}\phi_m(u).$$

Now, since $(u_m)_m$ is bounded in the Hilbert space H^1_ω, it has a weakly convergent subsequence. For simplicity, we continue to write it $(u_m)_m$. Set U the weak limit. Since ϕ is l.s.c. and $\phi\le\phi_m$, we have:

$$\phi(U)\le\liminf_m\phi(u_m)\le\liminf_m\phi_m(u_m)=\liminf_m\inf_{v\in H^1}\phi_m(v)=\inf_{u\in H^1}\phi(u).$$

But now with **(A5)**, we can say that ϕ is everywhere finite on H^1_ω, and so continuous. Since H^1 is dense on H^1_ω and ϕ is continuous, we have $\inf_{u\in H^1}\phi(u)=\inf_{u\in H^1_\omega}\phi(u)$. So, U is a minimum of the convex functional ϕ on H^1_ω.

As for ϕ_m, it can be proved that ϕ is Gâteaux-differentiable, and we can calculate this differential. Writing that $D_G\phi(U)=0$, we obtain:

$$\forall v\in H^1_\omega,\quad\int_{\mathbb{T}^N}\left\langle-\sum_{1\le j,k\le N}\omega_j\omega_k\frac{\partial^2U(x)}{\partial x_j\partial x_k}+\frac{\partial A}{\partial y}(x,U(x)),v(x)\right\rangle d\mu(x)=0.$$

i.e.:

$$\partial^2_\omega U(x)=\frac{\partial A}{\partial y}(x,U(x)).$$

Remark 2.8 *Since $u_m\rightharpoonup U$ in H^1_ω, we have:*

$$\|U\|_{H^1_\omega}\le\liminf\|u_m\|_{H^1_\omega}\le R_1.$$

Now, if we set $q(t)=U(t\omega)$, we have:

$$q''(t)=\partial^2_\omega U(t\omega)=\frac{\partial A}{\partial y}(t\omega,U(t\omega))=\frac{\partial V}{\partial y}(t,U(t\omega))=\frac{\partial V}{\partial y}(t,q(t)).$$

3 A.p. solutions

We consider now the problem of finding a.p. solutions of the equation:

$$q''(t)=F'(q(t))+b(t)\tag{7}$$

where $b\in AP^0(\mathbb{R},\mathbb{H})$ is a.p. and $F\in C^1(\mathbb{H},\mathbb{R})$ is convex and satisfies:

$$\exists(\alpha,\beta,\gamma)\in\mathbb{R}^+_*\times\mathbb{R}\times\mathbb{R},\qquad <F'(y),y>\ge\beta^2+4\gamma\min\{1,\alpha\}\ge0.$$

Theorem 3.1 *Under these assumptions, equation (7) admits an a.p. solution.*

To prove this, we know that one can find a sequence of trigonometric polynomials $(b_n)_n$ s.t. $b_n\to b$ uniformly on $\mathbb{R}$. Let us consider for each n, $V_n:\mathbb{R}\times\mathbb{H}\to\mathbb{H}$ defined by:

$$V_n(t,y)=F'(y)+b_n(t).$$

Each V_n is q.p. uniformly w.r.t. y. So, let us now see if V_n satisfies assumptions **(H1)-(H4)**. **(H1)** is clear, and **(H2)** is true, for instance with $\varphi_0=0$. For **(H3)** we write for instance by convexity of $V_n(t,.)$:

$$V_n(t,y)\ge V_n(t,0)+\left\langle\frac{\partial V_n}{\partial y}(t,0),y\right\rangle=b_n(t)+<F'(0),y>.$$

And for **(H4)** there is no problem since

$$\left\langle \frac{\partial V_n(t,y)}{\partial y}, y \right\rangle =< F'(y), y > .$$

So, by using the preeceeding facts, each problem:

$$q''(t) = F(q(t)) + b_n(t)$$

admits a solution q_n. But if U_n is associated to q_n, we can say that:

$$\|q_n\|_{B^{1,2}} = \|U_n\|_{B^{1,2}} = \|U_n\|_{H^1_\omega} \leq R_1,$$

where $R_1 = \frac{\beta+\sqrt{\beta^2+4\gamma\delta}}{2\delta}$ does not depends on n.

So, $(q_n)_n$ is bounded in the Hilbert space $B^{1,2}$, and so we can find a weakly-convergent subsequence (also written $(q_n)_n$) to $q \in B^{1,2}$.

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