DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **4**, Number **4**, Xxxx **1998**

Website: http://AIMsciences.org

pp. **735–764**

EXISTENCE OF ALMOST PERIODIC SOLUTIONS OF DISCRETE TIME EQUATIONS

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(Communicated by Jean Mawhin)

Abstract. In this paper, we study almost periodic (a.p.) solutions of discrete dynamical systems. We first adapt some results on a.p. differential equations to a.p. difference equations, on the link between boundedness of solutions and existence of a.p. solutions. After, we obtain an existence result in the space of the Harmonic Synthesis for an equation $A_t(x_t, ..., x_{t+p}) = 0$ when the dependance of A on t is a.p. and when A_t and DA_t are uniformly Lipschitz and satisfy another condition which is exactly the extension of a simple one for the basic linear system. The main tools for that are Nonlinear Functional Analysis and the Newton method.

Introduction. Almost periodic solutions (and the special case of quasi-periodic solutions) of Dynamical Systems arise in numerous theories, from Dynamical Systems [6], Dynamical Economics [10], Chaos [1], Physics [15] and their references. In [5], J. Blot and D. Pennequin have introduced and compared some current notions of almost periodic (a.p.) sequences. They have introduced a notion of a.p. sequences depending on a parameter and have shown an isomorphism between the space of a.p. sequences depending on a parameter and a space of a.p. sequences with values in a Banach space. They have built some variational principles, and they have obtained some structure results and some existence results in special cases: concave (or convex) functional, special linear equations or quantitavely perturbated equations.

Let us recall some notations and some facts [5]. We consider a Banach space $\mathbf{E}$, which will be in general $\mathbf{R}^N$. The reason for which we consider at the beginning an abstract Banach space is that spaces of a.p. sequences with parameters can be seen as spaces of a.p. sequences in a special (infinite dimensional) Banach space [5]. The norm of $\mathbf{E}$ will be denoted by $|.|$, and when $\mathbf{E} = \mathbf{R}^N$, we denote $x \cdot y$ the standard scalar product of $x, y \in \mathbf{R}^N$. For $A : \mathbf{Z} \times P \to \mathbf{E}$, and $t \in \mathbf{Z}$, we also denote by A_t the function $A(t, .)$. When it exists, the partial derivative w.r.t. the second argument is written D_2A and we also write $DA_t(x) := D_2A(t, x)$. Now, for $f : \mathbf{R} \to \mathbf{E}$ and $\alpha \in \mathbf{R}$, we denote by $\tau_\alpha f$ the function $f(. + \alpha)$; this notation is also avaliable for sequences (and $\alpha \in \mathbf{Z}$).

2000 *Mathematics Subject Classification.* 39A10, 43A60, 93C55.

Key words and phrases. Discrete Time Equations, almost periodic solutions, bounded solutions, Newton's method.

I would like thank Joël Blot for helpful comments.

A sequence $\underline{x} := (x_t)_t \in (\mathbf{E})^{\mathbf{Z}}$ is called almost periodic (a.p.) when one of the following equivalent assertions holds:

- $\forall \varepsilon > 0,\ \exists N \in \mathbf{N}^*,\ \forall m \in \mathbf{Z},\ \exists p \in \{m, \cdots, m+N\},\ \forall t \in \mathbf{Z},$
$$|x_{t+p} - x_t| \leq \varepsilon.$$
- There exists $f \in AP^0(\mathbf{R}; \mathbf{E})$ such that for any $t \in \mathbf{Z}$, $f(t) = x_t$.
- The fonction $f_{\underline{x}} : \mathbf{R} \to \mathbf{E}$ defined by $f_{\underline{x}}(t+u) = x_t + u(x_{t+1} - x_t)$ for $t \in \mathbf{Z}$, $u \in [0; 1]$ is a.p.
- There exists a function $\varphi \in C^0(b\mathbf{Z}, \mathbf{E})$ such that $(\varphi \circ in)(t) = x_t$ for all $t \in \mathbf{Z}$, where $b\mathbf{Z}$ is the Bohr compactification of $\mathbf{Z}$ and $in : \mathbf{Z} \to b\mathbf{Z}$ is the canonical injection.

We denote by $AP(\mathbf{Z}, \mathbf{E})$ the space of a.p. sequences. The function in the third Point is called the canonical extension of $\underline{x}$ to $AP^0(\mathbf{R}, \mathbf{E})$. The function φ is unique and denoted by $\varphi^{\underline{x}}$. $AP(\mathbf{Z}, \mathbf{R}^N)$ is a Banach space when it is endowed with the norm:
$$\|\underline{x}\|_\infty := \sup_{t \in \mathbf{Z}} |x_t| = \|f_{\underline{x}}\|_\infty.$$
Recall that $b\mathbf{Z}$ is a topological compact group, we denote by $\mu_{b\mathbf{Z}}$ its Haar measure. We can define the mean of the a.p. sequence by one of the equivalent formulae:
$$\mathcal{M}\{\underline{x}\} = \mathcal{M}\{x_t\}_t := \lim_{T \to +\infty} \frac{1}{T+1} \sum_{t=0}^{T} x_t = \mathcal{M}\{f_{\underline{x}}\} = \int_{b\mathbf{Z}} \varphi^{\underline{x}}(\theta) d\mu_{b\mathbf{Z}}(\theta).$$
We recall that, by denoting $e_\alpha(t) := e^{2i\pi\alpha t}$, for $\alpha \in [0; 1)$ and $t \in \mathbf{Z}$, and $\underline{e_\alpha} := (e_\alpha(t))_t$ and finally
$$c_\alpha(\underline{x}) := \mathcal{M}\{e_{-\alpha}(t) x_t\}_t$$
we have:
$$\underline{x} \sim \sum_{\alpha \in [0;1)} c_\alpha \underline{e_\alpha}.$$
$AP(\mathbf{Z}, \mathbf{R}^N)$ can be endowed with the following scalar product:
$$< \underline{x}, \underline{y} >:= \mathcal{M}\{\overline{x_t} \cdot y_t\}_t$$
and its completion is denoted by $B^2(\mathbf{Z}, \mathbf{R}^N)$. We note $\|.\|$ the norm of $L^2(b\mathbf{Z}, \mathbf{R}^N)$ and the norm of $B^2(\mathbf{Z}, \mathbf{R}^N)$. The following conditions are equivalent:

- $\underline{x} \in B^2(\mathbf{Z}, \mathbf{R}^N)$.
- $\sum_\alpha |c_\alpha(\underline{x})|^2 < +\infty$.
- $\varphi^{\underline{x}} \in L^2(b\mathbf{Z}, \mathbf{R}^N)$.

Moreover, $\|\underline{x}\| = \|\varphi^{\underline{x}}\|$. $B^2(\mathbf{Z}, \mathbf{R}^N)$ and $L^2(b\mathbf{Z}, \mathbf{R}^N)$ are similar and be viewen as spaces where Harmonic Synthesis is realized.

Let us consider a metric space P such that P is compact or $P = \cup_p K_p$ with K_p compact for all p and $K_p \subset \mathrm{Int} K_{p+1}$, where Int stands for interior. Let us consider $A : \mathbf{Z} \times P \to \mathbf{R}^N$. We say that A is u.a.p. and we note $A \in APU(\mathbf{Z}, P, \mathbf{R}^N)$ if for any $K \subset P$ compact and any $\varepsilon > 0$ we have:
$$\exists N \in \mathbf{N}^*,\ \forall m \in \mathbf{Z},\ \exists p \in \{m, ..., m+N\},\ \forall t \in \mathbf{Z},$$
$$\sup_{(t,\alpha) \in \mathbf{Z} \times K} |A(t+p, \alpha) - A(t, \alpha)| \leq \varepsilon.$$
It can be shown that $APU(\mathbf{Z}, P, \mathbf{R}^N)$, $AP(\mathbf{Z}, C^0(P, \mathbf{R}^N))$ and $C^0(b\mathbf{Z} \times P, \mathbf{R}^N)$ are isomorphic as Fréchet spaces (when they are embedded with their natural topological vector space structure).

The aim of this paper is to explore some new equations. In Section 1, we prove the link between a.p. sequences studied in [5] and the Bochner property, and extend it to a.p. sequences depending on a parameter. In Section 2, we prove a criterion of almost periodicity of bounded solutions, closed to the Amerio one in continuous time. The idea to look for a.p. solutions (in $AP(\mathbf{Z}, \mathbf{R}^N)$) by studying first bounded solutions is classical for differential equations and justified by the fact that all bounded a.p. solutions of the basic linear system $x_{t+1} = Ax_t + b_t$ (where A is constant and $\underline{b}$ a.p.) are a.p. In Section 3, we give a condition for existence of a $B^2(\mathbf{Z}, \mathbf{R}^N)$ solution to the quite general equation:

$$A_t(x_t, ..., x_{t+p}) = 0$$

when A and D_2A satisfy some growth properties and a conditions which generalizes a simple condition for the basic linear system.

1. **The Bochner condition for sequences.** A natural extension of the classical Bochner criterion for sequences is given by the following proposition.

Proposition 1.1. *A sequence $\underline{x}$ is a.p. if and only if, for any $(h_n)_n \in \mathbf{Z}^{\mathbf{N}}$, there exists a subsequence $(h_{\phi(n)})_n$ such that $(\tau_{h_{\phi(n)}}\underline{x})_n$ converges uniformly on $\mathbf{Z}$.*

Proof. To prove this, we use the link between $f_{\underline{x}}$ and $\underline{x}$. If $\underline{x}$ is a.p., so is $f_{\underline{x}}$ and thus there exists a subsequence $(h_{\phi(n)})_n$ such that $(\tau_{h_{\phi(n)}} f_{\underline{x}})_n$ converges uniformly on $\mathbf{R}$. But if $t \in \mathbf{Z}$, $f_{\underline{x}}(t + h_{\phi(n)}) = x_{t+h_{\phi(n)}}$ since $h_{\phi(n)} \in \mathbf{Z}$ and so we obtain the result. Now, we prove the controverse. First, we see that $\underline{x}$ is bounded: if not, there would exist for any n, h_n such that $|x_{h_n}| \geq n$ for any n, and so $(\tau_{h_n} x_0)_n$ has no convergent subsequence. So $\underline{x}$ is bounded, let us show that $f_{\underline{x}}$ is a.p. Given a sequence $(k_n)_n \in \mathbf{R}^{\mathbf{N}}$, we set $k_n = h_n + \zeta_n$ with $h_n \in \mathbf{Z}$ and $\zeta_n \in [0; 1)$. By taking two times a subsequence if necessary, we may assume that:

$$\sup_{t \in \mathbf{Z}} |f_{\underline{x}}(t + h_n) - f_{\underline{x}}(t)| \leq 1/n$$

and that:

$$|\zeta_n - \zeta| \leq \frac{1}{2n\|\underline{x}\|_\infty}, \quad \zeta \in [0; 1].$$

From this, we obtain that

$$\sup_{x \in \mathbf{R}} |f_{\underline{x}}(x + h_n) - f_{\underline{x}}(x)| \leq 1/n$$

and since $f_{\underline{x}}$ is $2\|x\|_\infty$-Lipschitzian:

$$\sup_{x \in \mathbf{R}} |f_{\underline{x}}(x + \zeta_n) - f_{\underline{x}}(x + \zeta)| \leq 2\|\underline{x}\|_\infty |\zeta_n - \zeta| \leq 1/n.$$

Thus, we have for any $x \in \mathbf{R}$:

$$|f_{\underline{x}}(x+k_n) - f_{\underline{x}}(x+\zeta)| \leq |f_{\underline{x}}(x+k_n) - f_{\underline{x}}(x+h_n+\zeta)| + |f_{\underline{x}}(x+h_n+\zeta) - f_{\underline{x}}(x+\zeta)| \leq 2/n$$

thus $(\tau_{k_n} f_{\underline{x}})_n$ converges uniformly on $\mathbf{R}$ to $\tau_\zeta f_{\underline{x}}$, and so $f_{\underline{x}}$ is a.p., and we can conclude that $\underline{x}$ is a.p. ■

Now, we extend it to the parametric case. Let us consider $A : \mathbf{Z} \times P \to \mathbf{R}^N$. We can assert:

Proposition 1.2. *A is a.p. if and only if for any $(h_n)_n \in \mathbf{Z}^{\mathbf{N}}$, there exists a subsequence $(h_{\phi(n)})_n$ such that for all $K \subset P$ compact, $(A(. + h_{\phi(n)}, .))_n$ converges uniformly on $\mathbf{Z} \times K$.*

Proof. Since $APU(\mathbf{Z}, P, \mathbf{R}^N)$ is isomorphic to $AP(\mathbf{Z}, C^0(P, \mathbf{R}^N))$, the result is clear if P is compact. Now, we consider the case when $P = \cup_p K_p$ with K_p compact for any p and $K_p \subset Int(K_{p+1})$. Assume first that $A \in APU(\mathbf{Z}, P, \mathbf{R}^N)$. By induction, we can find for any p a strictly increasing function $\phi_p : \mathbf{N} \to \mathbf{N}$ such that for any $n \geq 1$ and any p: $(A(. + h_{(\phi_0 \circ ... \circ \phi_p)(n)}, .))_n$ is uniformly convergent on K_p. If we set $\psi(n) := (\phi_0 \circ ... \circ \phi_n)(n)$ we thus have that $(A(. + h_{\psi(n)}, .))_n$ is uniformly convergent on any K_p. Since any compact is a subset of some K_p, we have the result. For the other implication, note that the compact case gives that $A \in APU(\mathbf{Z}, K_p, \mathbf{R}^N)$ for any p, and so we have the result. ■

2. **Bounded solutions and Almost periodicity.**

A reason for looking for a.p. solutions by studying bounded solutions is given by the basic linear case. In this section, we look for solutions in $AP(\mathbf{Z}, \mathbf{R}^N)$.

2.1. **The basic linear case.**

We begin with the case of sums:

Lemma 2.1. *Let $\underline{x}$ be a.p. Then the sequence $(\sum_{j=0}^{t} x_j)_t$ is a.p. iff it is bounded.*

It is known that for a.p. functions, their indefinite integrals are a.p. iff they are bounded. An easy computation shows that:

$$\int_0^{t+1} f_{\underline{x}}(u)du = \frac{x_0 + x_{t+1}}{2} + \sum_{j=0}^{t} x_j.$$

Proof. Since $f_{\underline{x}}$ is a.p., the indefinite integral of $f_{\underline{x}}$ is a.p. if and only if it is bounded: the lemma is proven. ■
Now, we study the basic linear case. As in continuous time, we have:

Proposition 2.2. *Let A be a $N \times N$ matrix, and $\underline{b}$ an a.p. sequence with values in $\mathbf{R}^N$. Then a solution of:*

$$x_{t+1} = Ax_t + b_t$$

is almost periodic if and only if it is bounded. Moreover, if A has no eigenvalue with modulus 1, there exists a unique a.p. solution.

Proof. An a.p. solution is bounded, just the controverse need to be shown. There exist a triangular matrix T and an invertible matrix T such that $A = PTP^{-1}$. Let us consider $\underline{y}$ defined by $y_t := P^{-1}x_t$. $\underline{y}$ is a.p. (resp. bounded) iff $\underline{x}$ is a.p. (resp. bounded) and $\underline{y}$ is solution of the following triangular system:

$$y_{t+1} = Ty_t.$$

So we see that if Proposition (2.2) is true for $N = 1$, then it is always true. So we now just consider the scalar case. Let $a \in \mathbf{C}$ be given and $\underline{b}$ a.p. We can assume that $a \neq 0$, since if $a = 0$ the result is clear. We have three cases.
Case 1. If $|a| > 1$, any solution can be written

$$x_t = a^t \left[x_0 + \sum_{j=0}^{t-1} \frac{b_j}{a^{j+1}} \right].$$

Since $|a^t|$ goes to $+\infty$ when $t \to +\infty$, a bounded solution satisfies necessarly:

$$x_0 = -\sum_{j=0}^{+\infty} \frac{b_j}{a^{j+1}}.$$

Note that the sum is convergent since $(b_j)_j$ is bounded and $|a| > 1$. So the only possibility for a solution to be bounded is to have the form:

$$x_t = a^t \sum_{j=t}^{+\infty} \frac{b_j}{a^{j+1}}.$$

From this, it is clear that this solution is bounded and it is also a.p. since:

$$|x_{t+p} - x_t| \leq \frac{\sup_\tau |b_{\tau+p} - b_\tau|}{|a| - 1}$$

so the only bounded solution in this case is a.p.
Case 2. If $|a| < 1$ the reasoning is similar.
Case 3. We now consider the case $|a| = 1$. The general solution is written as:

$$x_t = a^t \left[x_0 + \sum_{j=0}^{t-1} \frac{b_j}{a^{j+1}} \right].$$

Since $(a^t)_t$ is bounded (and a.p.) in this case, we see that there is bounded (resp. a.p.) solutions iff the sequence $(\sum_{j=0}^{t-1} \frac{b_j}{a^{j+1}})_t$ is bounded (resp. a.p.). But in view of the preceeding lemma, since $(\frac{b_t}{a^{t+1}})_t$ is a.p., these facts are equivalent, and the proof is complete. Note that in this case there exists an a.p. solution if and only if all solutions are a.p.■

Corollary 2.3. *We consider the equation:*

$$\sum_{j=0}^{p} a_j x_{t+j} = b_t$$

with $a_j \in \mathbf{R}$, $\underline{b} \in AP(\mathbf{Z}, \mathbf{R})$, *and we assume that there exists* $s \in \{0, ...p\}$ *such that:*

$$|a_s| > \sum_{j \neq s} |a_j|.$$

Then there exists a unique solution in $AP(\mathbf{Z}, \mathbf{R})$ *to the equation.*

Proof. The characteristic polynomial is $P(z) := \sum_{j=0}^{p} a_j z^j$. It is sufficient to prove that P has no zero with modulus 1. If there exists such a zero, z_0, we have $a_s z_0^s = -\sum_{j \neq s} a_j z_0^j$ and so $|a_s| \leq \sum_{j \neq s} |a_j|$, which is impossible. So, P has no zero with modulus 1, and the lemma is proven.■

Remark 2.4. *When A has an eigenvalue whose modulus is 1, the equation may have no a.p. solution. For instance:*

$$x_{t+1} + x_t = (-1)^t$$

has no a.p. solution.

2.2. **Some nonlinear systems.** Now, we consider some nonlinear systems. We shall establish a criterion closed to the Amerio one for almost periodicity of bounded solutions.
We consider a system:

$$x_{t+1} = A_t(x_t) \tag{2.1}$$

with $A \in APU(\mathbf{Z}, \mathbf{R}^N, \mathbf{R}^N)$. Let $\mathcal{F}$ be the hull of A, i.e. $B \in \mathcal{F}$ if and only if there exists $(h_n)_n \in \mathbf{Z}^{\mathbf{N}}$ such that $A(.+h_n,.) \to B$ uniformly on $\mathbf{R} \times K$ for any compact K. To $B \in \mathcal{F}$, we associate the following system:

$$y_{t+1} = B_t(y_t) \tag{2.2}$$

We now fix a connected compact set K, and we note $D = \mathbf{R} \times K$. For a system as (2.1), we say that $\underline{x}$ is a separated solution on D if either it is the only one solution such that $Gr(\underline{x}) \subset D$ or if for any such solution $\underline{y}$, there exists $\rho = \rho(\underline{y}) > 0$ such that for any t, $\|x_t - y_t\| \geq \rho$. Note that since $f_{\underline{x}} - f_{\underline{y}} = f_{\underline{x-y}}$, we also have $\|f_{\underline{x}}(t) - f_{\underline{y}}(t)\| \geq \rho$ for any $t \in \mathbf{R}$.

Lemma 2.5. *The following assertions hold.*

1. *If Φ is a subset of all $\underline{x}$ a.p. such that $Gr(\underline{x}) \subset D$, then the set $\{f_{\underline{x}}, \underline{x} \in \Phi\}$ is relatively compact in any normed vector space $C^0([a,b];\mathbf{R}^N)$, with $-\infty < a < b < +\infty$.*
2. *If (2.1) has only separated solutions, there is a finite number of solutions.*
3. *If (2.1) has a solution $\underline{x}$ such that $Gr(\underline{x}) \subset D$, then (2.2) has a solution $\underline{y}$ such that $Gr(\underline{y}) \subset D$.*
4. *If (2.1) has a solution $\underline{x}$ such that $x_t \in K$ for $t \geq t_0$, then (2.2) has a solution $\underline{y}$ such that $y_t \in K$ for any t.*
5. *If any system (2.2) a separated solutions, then there exists $\sigma > 0$ such that for any $B \in \mathcal{F}$, for any $\underline{y^i}$, $i = 1, 2$ solutions of the system (2.2), we have for any t, $\|y_t^1 - y_t^2\| \geq \sigma$.*

Proof.
1. Define $M := \sup_{\zeta \in K} \|\zeta\|$. Since for any t, $\|x_t\| \leq M$ and $\|x_{t+1} - x_t\| \leq 2M$, we have, by construction of $f_{\underline{x}}$, $\|f_{\underline{x}}(t)\| \leq M$ for any $t \in \mathbf{R}$ and $\|f_{\underline{x}}(u) - f_{\underline{x}}(v)\| \leq 2M\|u - v\|$. Thus, this familly is equicontinuous and equibounded, and the Ascoli theorem gives the result.
2. Consider the familly of all separated solutions. If it is infinite, we can choose a infinite familly $(f_{\underline{x^{(n)}}})_n$ with distincts terms which converges uniformly on any compact sets (cf. 1.). Thus the limit is a solution of the equation, but is not separated, which leads a contradiction.
3. Consider any $B \in \mathcal{F}$. Let $(h_n)_n \in \mathbf{Z}^{\mathbf{N}}$ such that $A(.+h_n,.) \to B$. By the point 1. of this Lemma, the familly $f_n := f_{\underline{x}}(.+h_n)$ has a subsequent which converges uniformly on any compacts. The limit is a solution to the problem.
4. Let us consider $\underline{x}^{(n)} := (x_{t-n})_t$. Let $\alpha \in \mathbf{Z}$ and $N(\alpha) := t_0 - \alpha$. The familly $\{f_{\underline{x}^{(n)}}; n \geq N(\alpha)\}$ is equibounded and equicontinuous, thus there exists a sequence $(k_n)_n \in \mathbf{Z}^{\mathbf{N}}$ such that uniformly on any compacts we have $f_{\underline{x}}(.+k_n) \to f_{\underline{y}}$ and $A(.+k_n, Y) \to C(., Y)$. Then $y_{t+1} = C(t, y_t)$, $t \geq \alpha$ for any α and we can apply the point 3. of this Lemma to this equation.
5. Let $\sigma > 0$ the term of the separation condition for (2.1). Given two solutions $\underline{z}^i$ $(i = 1, 2)$ of (2.1), there exists a common sequence $(h_n)_n$ such that $(z^i(.+h_n))_n$ converges uniformly to a sequence $(y_t^i)_t$. We have:

$$\sigma \leq \inf_{t \in \mathbf{R}} \|f_{\underline{z}^1}(t) - f_{\underline{z}^2}(t)\| \leq \|y_t^1 - y_t^2\|$$

thus $\underline{y}^1$ and $\underline{y}^2$ are distincts. So, for any B, equations (2.1) and (2.2) have the same number of solutions and the σ for equation (2.1) is valid for (2.2). ∎

Theorem 2.6. *If all systems (2.2) possess separated solutions, then all solutions are a.p.*

Proof. Let us consider $\underline{x}$ a solution of (2.1) and a sequence $(h_n)_n \in \mathbf{Z}^{\mathbf{N}}$. We have to prove that a subsequence of $(f_{\underline{x}}(.+h_n))_n$ is uniformly convergent on the whole real line. Following the proof of Lemma 2.5, 3., we can assume that $f_{\underline{x}}(.+h_n) \to f_{\underline{z}}$ uniformly on any compacts subsets. We assume that the conclusion does not hold. We have uniformly $A(.+h_n,.) \to B$, thus $z_{t+1} = B(t, z_t)$. By Lemma 2.5, 4., there exists $\rho > 0$ such that if $(y_t^i)_t$ are two solutions of (2.1), $\inf_t \|z_t^1 - z_t^2\| \geq 2\rho$. For $n < p$, consider $\phi_{n,p}(t) := \|f_{\underline{x}}(t+h_n) - f_{\underline{x}}(t+h_p)\|$ and $I_{n,p} = \phi_{n,p}^{-1}(\mathrm{cl}(B(0;\rho)))$, where cl stands for closure. $\phi_{n,p}$ is continuous, $I_{n,p}$ is closed, nonempty for sufficiently large (n,p) (since $0 \in I_{n,p}$ for large n, p). Let us consider $\delta_{n,p} := \sup_{t \in I_{n,p}} \phi_{n,p}(t)$. We have $\delta_{n,p} \leq \rho$ and $\lim_{n,p \to +\infty} \delta_{n,p} \neq 0$ (if not, $(f_{\underline{x}}(.+h_n))_n$ would be uniformly convergent). So $\limsup \delta_{n,p} =: 2\alpha > 0$. Thus, we have monotones increasing sequences $(n_r)_r, (p_r)_r$ such that $\delta_{n_r,p_r} \geq 3\alpha/2$ i.e. there exists for any r, t_r such that: $\phi_{n_r,p_r}(t_r) \geq \alpha$. Thus:

$$\alpha \leq \|f_{\underline{x}}(t_r + h_{n_r}) - f_{\underline{x}}(t_r + h_{p_r})\| \leq \rho$$

and passing to subsequences if necessary we can find two vectors U, V such that $f_{\underline{x}}(t_r + h_{n_r}) \to U$ and $f_{\underline{x}}(t_r + h_{p_r}) \to V$, and thus $\alpha \leq \|U - V\| \leq \rho$.
Passing to subsequences if necessary, we may assume that $f_{\underline{x}}(.+t_r+h_{n_r})_r \to \underline{z}^1$ and $f_{\underline{x}}(.+t_r+h_{p_r}) \to \underline{z}^2$ solutions of equations such that $z_{t+1}^i = B_t^i(z_t^i)$ with $\alpha \leq \|z_0^2 - z_0^1\| \leq \rho$. Immediatly, we see that $B^2 = B^1$, and since $z_0^2 \neq z_0^1$, we must have for all t, $\|z_t^2 - z_t^1\| \geq 2\rho$, which gives the contradiction. ■

As a consequence, we obtain the following result which is closed to Favard's one in continuous time:

Corollary 2.7. *We consider a linear system*

$$x_{t+1} = A_t x_t + b_t \tag{2.3}$$

and we assume that A and b are a.p. We assume that for any Ξ in the Hull of A, the system:

$$x_{t+1} = \Xi_t x_t$$

has just the trivial solution as bounded solution. Then, any solution bounded solution of (2.3) is a.p.

Proof. We see that there exists at most one bounded solution, since the difference between 2 solutions is solution of: $x_{t+1} = A_t x_t$. Let $(h_n)_n \in \mathbf{Z}^{\mathbf{N}}$ be such that $\tau_{h_n} A \to \Xi$ and $\underline{C} := \lim \tau_{h_n} b$. Each system

$$x_{t+1} = \Xi_t x_t + C_t$$

has a bounded solution (cf 2.5, 3) which is unique so separated. Thus, any solution of (2.3) is a.p.■

3. **A quite general case with growth conditions.** We now consider the equation:

$$A_t(x_t, ..., x_{t+p}) = 0 \tag{3.4}$$

where $A : b\mathbf{Z} \times (\mathbf{R}^N)^{p+1} \to \mathbf{R}^N$ is supposed to satisfy:

(H1) $D_2 A$ exists, A and $D_2 A$ are Caratheodory.
(H2) $(A_t(0))_t \in L^2(b\mathbf{Z}; \mathbf{R}^N)$ and $(DA_t(0))_t \in L^2(b\mathbf{Z}; \mathbf{R}^N \times \mathbf{R}^N)$.

(H3) There exists $c > 0$ such that for all t, A_t and DA_t are $c-$Lipschitzian, i.e.:
- $\forall (t,x,y) \in \mathbf{Z} \times \mathbf{R}^N \times \mathbf{R}^N$, $|A(t,x) - A(t,y)| \le c|x-y|$
- $\forall (t,x,y) \in \mathbf{Z} \times \mathbf{R}^N \times \mathbf{R}^N$, $\|DA_t(x) - DA_t(y)\|_{\mathcal{L}} \le c|x-y|$ where $\|.\|_{\mathcal{L}}$ is the operator norm of $\mathrm{End}(\mathbf{R}^N)$ associated to $|.|$.

(H4) $\exists \gamma > 0$, $\exists s \in \{0, ..., p\}$, $\exists \epsilon \in \{-1; 1\}$, $\forall v \in \mathbf{R}^N$, $\forall (t, \alpha) \in \mathbf{Z} \times (\mathbf{R}^N)^{p+1}$:

$$\epsilon \, v^T D_{s+1} A_t(\alpha) v \ge \left(\gamma + \sum_{j \ne s} \sup_{(\tau,\beta)} \|D_{j+1} A_\tau(\beta)\|_{\mathcal{L}} \right) |v|^2$$

where v^T is the transpose of v.

Remark 3.1. *From* **(H2)** *and* **(H3)** *we deduce in particular that there exists* $d \in L^2(b\mathbf{Z}, \mathbf{R})$ *such that:*

$$\forall (t,x) \in \mathbf{Z} \times \mathbf{R}^N, \quad \max\{|A(t,x)|; \|DA_t(x)\|_{\mathcal{L}}\} \le c|x| + d(t).$$

The aim of this section is to prove:

Theorem 3.2. *Under (3), equation (3.4) possesses a solution in* $B^2(\mathbf{Z}, \mathbf{R}^N)$.

By changing A into $-A$ if necessary, we may assume that $\epsilon = 1$. We just consider this case in the proof. We consider any $\underline{b} \in AP(\mathbf{Z}, \mathbf{R}^N)$ and we consider the equation:

$$A_t(x_t, ..., x_{t+p}) = b_t. \tag{3.5}$$

We define $\phi_{\underline{b}} : L^2(b\mathbf{Z}, \mathbf{R}^N) \to (L^2(b\mathbf{Z}, \mathbf{R}^N))'$ by setting

$$\phi_{\underline{b}}(\underline{x}) := \left[\underline{v} \mapsto \int_{b\mathbf{Z}} (A_t(x_t, ..., x_{t+p}) - b_t).v_{t+s} d\mu_{b\mathbf{Z}}(t) \right],$$

thus (3.4) is $\phi_0(\underline{x}) = 0$. We shall first prove:

Proposition 3.3. *There exists* $C > 0$ *depending only on* A *such that if* $\phi_{\underline{b}}(\underline{x}) = 0$ *possesses a solution, then for all* $\underline{b'}$ *such that* $\|\underline{b'} - \underline{b}\| \le C$, *the equation* $\phi_{\underline{b'}}(\underline{x}) = 0$ *possesses a solution.*

Note that if this proposition is valid, the theorem is also true. Indeed, we consider a chain $(\underline{b_j})_{0 \le j \le p}$ such that $\underline{b_0} := (A_t(0))_t$, $\|\underline{b_{j+1}} - \underline{b_j}\| \le C$ and $\underline{b_p} := 0$. By induction, any equation $\phi_{\underline{b_j}}(\underline{x}) = 0$ has a solution, so the last one has a solution.

First, to prove Proposition 3.12, we need the following lemmas.

Lemma 3.4. $\phi_{\underline{b}}$ *is well defined, continuous and Gâteaux-differentiable.*

Proof. Since $(A_t)_t$ are uniformly Lipchizian, the Nemytskii operator $\mathcal{N}_A$ satisfies $\mathcal{N}_A(L^2(b\mathbf{Z}, \mathbf{R}^N)^{(p+1)}) \subset L^2(b\mathbf{Z}, \mathbf{R}^N)$ and is continuous (the results of [9], chap 2. are also true with $L^p(b\mathbf{Z}, \mathbf{R}^q)$), and so $\phi_{\underline{b}}$ is well defined. By using the Cauchy-Schwarz-Buniakovski inegality, we obtain:

$$\|\phi_{\underline{b}}(\underline{x}) - \phi_{\underline{b}}(\underline{y})\|_{L^{2\prime}}^2 \le \int_{b\mathbf{Z}} |A_t(x_t, ...x_{t+p}) - A_t(y_t, ...y_{t+p})|^2 \, d\mu_{b\mathbf{Z}}(t)$$

and since the Nemytskii operator on A is continuous, the term in right goes to 0 when $\underline{y} \to \underline{x}$.

We now look at the derivative. Fix $\underline{x}$ and consider the bilinear form

$$\beta(\underline{h}, \underline{v}) := \int_{b\mathbf{Z}} \sum_{j=0}^{p} (D_{j+1} A_t(x_t, ..., x_{t+p}) h_{t+j}).v_{t+s} d\mu_{b\mathbf{Z}}(t).$$

A natural candidate to be the Gâteaux-differential is $\beta(\underline{h},.)$. Let us consider:

$$\xi(\theta) := \left\| \frac{\phi_{\underline{b}}(\underline{x}+\theta\underline{h}) - \phi_{\underline{b}}(\underline{x})}{\theta} - \beta(\underline{h},.) \right\|_{L^2(b\mathbf{Z};\mathbf{R}^N)'}.$$

By the Cauchy-Schwarz-Buniakovski inequality we obtain:

$$\xi(\theta)^2 \leq \int_{b\mathbf{Z}} \psi(\theta,t) d\mu_{b\mathbf{Z}}(t)$$

where:

$$\psi(\theta,t) := \left| \frac{A_t(x_t+\theta h_t) - A_t(x_t)}{\theta} - \sum_{j=0}^{p} D_{j+1}A_t(x_t,...,x_{t+p})h_{t+j} \right|^2$$

goes to 0 when $\theta \to 0$, and $\psi(\theta,t)$ is dominated by $8c^2|h_t|^2 \in L^1(b\mathbf{Z},\mathbf{R})$ in view of the mean value theorem: the Lebesgue Theorem gives the results. ■

Lemma 3.5. *The Gâteaux-differential $D_G\phi_{\underline{b}}(\underline{x})$ is invertible for any $\underline{x}$, and there exists a constant M depending only on A such that for any $\underline{b}$, $\underline{x}$, we have $\|D_G\phi_{\underline{b}}^{-1}(\underline{x})\| \leq M$.*

Proof. Given $L \in (L^2(b\mathbf{Z};\mathbf{R}^N))'$, we have to search a $\underline{h}$ such that for any $\underline{v}$, we have $\beta(\underline{h},\underline{v}) = L(\underline{v})$. Since L is linear continuous and β is bilinear continuous, and we use the Lax-Milgram theorem. Let us note for $j \neq s$, $M_j := \sup_{(\tau,\beta)} \|D_{j+1}A_\tau(\beta)\|_{\mathcal{L}}$. We note that:

$$\beta(\underline{v},\underline{v}) = \int_{b\mathbf{Z}} \sum_{j=0}^{p} (D_{j+1}L_t(x_t,...,x_{t+p})v_{t+j}).v_{t+s} d\mu_{b\mathbf{Z}}(t) \geq$$

$$\int_{b\mathbf{Z}} v_{t+s}^T D_{s+1}A_t(x_t,...,x_{t+p})v_{t+s} d\mu_{b\mathbf{Z}}(t) - \sum_{j\neq s} M_j \int_{b\mathbf{Z}} |v_{t+j}|.|v_{t+s}| d\mu_{b\mathbf{Z}}(t)$$

by using the Cauchy-Schwarz-Buniakovski inegality in $\mathbf{R}^N$ and the definition of the operator norm. By using **(H4)** and the Cauchy-Schwarz-Buniakovski inequality in $L^2(b\mathbf{Z};\mathbf{R}^N)$, we obtain:

$$\beta(\underline{v},\underline{v}) \geq \left(\gamma + \sum_{j\neq s} M_j\right) \int_{b\mathbf{Z}} |v_{t+s}|^2 d\mu_{b\mathbf{Z}}(t) - \sum_{j\neq s} \left[M_j \left(\int_{b\mathbf{Z}} |v_{t+j}|^2 d\mu_{b\mathbf{Z}}(t) \right)^{1/2} \right.$$

$$\left. \left(\int_{b\mathbf{Z}} |v_{t+s}|^2 d\mu_{b\mathbf{Z}}(t) \right)^{1/2} \right] = \gamma \|\underline{v}\|_{\mathcal{L}}^2$$

and since $\gamma > 0$, we have the ellipticity. Thus, $D_G\phi_{\underline{b}}(\underline{x})$ is invertible. We have now, if $\underline{h} = D_G\phi_{\underline{b}}^{-1}(\underline{x})(L)$:

$$\gamma\|\underline{h}\|^2 \leq \beta(\underline{h},\underline{h}) = L(\underline{h}) \leq \|L\| \ \ \|\underline{h}\|$$

and so we can take $M := \gamma^{-1}$. ■

Now we can complete the proof of Proposition 3.12. We shall apply Newton's method. The first remark to do is that in the Newton Theorem ([7], 7.5.1 p.161), just the continuity of the function and the mean value theorem are invoqued, so the theorem is also valid with a function which is just continuous and Gâteaux-differentiable. We take for $\underline{x}^{(0)}$ a solution of $\phi_{\underline{b}}(\underline{x}) = 0$, and let us set $\phi_0 := \|\phi_{\underline{b'}}(\underline{x}^{(0)})\|$. To apply the theorem we have to find $r > 0$ and $\beta \in (0;1)$ such that

$2cr \leq \beta/M$ and $M\phi_0/r \leq 1-\beta$. By a simple computation we see that it is possible if and only if:

$$\phi_0 \leq C := \frac{\gamma^2}{8c}$$

where C depends only on A. But, $\phi_0 \leq \|\underline{b}-\underline{b'}\|$, so if $\|\underline{b}-\underline{b'}\| \leq C$, we can find a solution to $\phi_{\underline{b'}}(\underline{x}) = 0$.■

Remark 3.6. *If we consider the basic linear case with $N = 1$, i.e. we consider the equation:*

$$\sum_{j=0}^{p} a_j x_{t+j} = b_t$$

then **(H1)**-**(H3)** *are true and* **(H4)** *is exactly the condition given in 2.3. In particular, we see that the condition can not be strenghtened by assuming that $\gamma \geq 0$ (cf. Remark 2.4).*

Even if the equation is an Euler equation, **(H4)** does not imply and is not implied by concavity or convexity of the Lagrangian. For instance, consider the equation with $N = 1$:

$$bx_{t-1} + (a+c)x_t + bx_{t+1} = d_t$$

where $a, b, c \in \mathbf{R}$ and $\underline{d}$ is a.p., which is Euler equation for the Lagrangean:

$$L_t(x,y) := \frac{ax^2 + 2bxy + cy^2}{2} + d_t x.$$

(H1)-(H3) are true and an easy computation show for instance that when $a > 0$, $c > 0$ and $|b| \in (\sqrt{ac}; \frac{a+c}{2})$, the condition **(H4)** is valid but the functional L_t is not convex nor concave.

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