Existence and Structure Results on Almost Periodic Solutions of Difference Equations

Joël BLOT* and Denis PENNEQUIN*

Abstract

We study the almost periodic solutions of Euler equations and of some more general Difference Equations. We consider two different notions of almost periodic sequences, and we establish some relations between them. We build suitable sequences spaces and we prove some properties of these spaces. We also prove properties of Nemytskii operators on these spaces. We build a variational approach to establish existence of almost periodic solutions as critical points. We obtain existence theorems for nonautonomous linear equations and for an Euler equation with a concave and coercive lagrangian. We also use a Fixed Point approach to obtain existence results for quasi-linear Difference Equations.

A.M.S. Classification: 39A10, 47H30, 43A60, 93C55

Keywords: Difference Equations, Almost Periodic Oscillations, Variational Methods, Fixed Point Methods.

Introduction

In this paper, we study the almost periodic (a.p.) solutions of two kinds of Differences Equations:

$$\sum_{j=0}^{p} D_{j+1} L_{t-j}(x_{t-j}, \cdots, x_{t+p-j}) = 0 \tag{1}$$

*CERMSEM, M.S.E., Université Paris 1 Panthéon-Sorbonne, 106-112 Bd de l'Hôpital, 75647 Paris cedex 13, France.

when $L_t : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, D_j denotes partial differential, and the dependance of L_t upon t is a.p. (precisely Section 2) and:

$$\sum_{\tau \in I} a_{t,\tau} x_{t+\tau} - \phi_t(x_{t+\tau_1}, \cdots, x_{t+\tau_p}) = 0 \quad (t \in \mathbb{Z}) \tag{2}$$

when I is a finite subset of $\mathbb{Z}$, $\tau_1, \cdots, \tau_p$ are distincts integers and and the dependance of ϕ_t upon t is a.p.

Euler Equations (1) arise in numerous theories: in Mechanics (cf. [2] and references therein), in Macroeconomics (cf. [11], [12] and references therein) and in others fields of Physics (cf. [16] and references therein). Some nonlinear Euler equations are particular cases of (2).

To study these problems, we use tools from Nonlinear Functional Analysis; Variational technics for (1), Fixed Point technics for (2). And so we build a Calculus of Variations in mean in discrete time in the spirit of this one of the continuous time ([3], [4] and references therein). First, by using the more common notion of a.p. sequence (Definition 1.1), when the Lagrangian is concave we can obtain results on the structure of the set of the a.p. solutions of (1). In a second time, by using the notion of Besicovitch a.p. sequence, the better topological properties of the space of such a.p. sequences permit us to prove some existence results. To work on (2), we use the theory of Unbounded Linear Operators and a Fixed Point Theorem.

Now, we describe the contents of the paper. In Section 1, we describe the different notions of a.p. sequences that we use. We give properties of the spaces of such a.p. sequences. In Section 2, we study the Nemytskii Operators defined on the spaces of a.p. sequences, since they are the fundamental tool to our functional analytic approach. In Section 3, we establish Variational Principles to rely the a.p. solutions of (1) and critical points of functionals defined on spaces of a.p. sequences. In Section 4, we establish a result about the structure of the set of a.p. solutions of (1) when the Lagrangian is concave. In Section 5, we establish existence results of a.p. solutions of (1) by using technics in the spirit of the direct methods of Calculus of Variations for (1) and a Fixed Point method for (2).

1 Spaces of Almost Periodic Sequences

In this section, we successively describe several notions of almost periodic sequences, and we show the links beetween these notions. We denote by $\mathbb{L}$ one of the sets $\mathbb{N}$, $\mathbb{N}^*$ or $\mathbb{Z}$ and by $\mathbb{E}$ a Banach space.

Definition 1.1 [[6] p.45] *A sequence $\underline{x} := (x_t)_t \in \mathbb{E}^{\mathbb{L}}$ is called almost periodic (a.p.) when the following assertion holds:*

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N}^*,\ \forall m \in \mathbb{L},\ \exists p \in \{m, \cdots, m+N\},\ \forall t \in \mathbb{L},$$

$$\|x_{t+p} - x_t\| \leq \varepsilon.$$

Remark that $\mathbb{Z}$ is a topological group, and when $\mathbb{L} = \mathbb{Z}$, this definition corresponds to this one of almost periodicity on a group ([15] p.133) which is equivalent to the Von Neumann one.

$AP(\mathbb{L}; \mathbb{E})$ stands for the space of almost periodic sequences from $\mathbb{L}$ in $\mathbb{E}$.

If $\underline{x} = (x_t)_t$, we denote by $f_{\underline{x}}$ the function $f_{\underline{x}} : \mathrm{conv}(\mathbb{L}) \to \mathbb{E}$ (where conv denotes the convex hull in $\mathbb{R}$) defined by:

$$\forall t \in \mathbb{L},\ \forall u \in [0; 1],\quad f_{\underline{x}}(t+u) := x_t + u.(x_{t+1} - x_t). \tag{3}$$

A sequence $\underline{x} \in \mathbb{E}^{\mathbb{L}}$ is a.p. if and only if there exists $f \in AP^0(\mathbb{R}; \mathbb{E})$ (the space of Bohr a.p. functions from $\mathbb{R}$ to $\mathbb{E}$, [1]) such that $f\mid_{\mathrm{conv}(\mathbb{L})}=$

$f_{\underline{x}}$ (for $\mathbb{L} = \mathbb{Z}$, see [6] p.47.)
Recall that an almost periodic function is completely determinated by its restriction on $[1; +\infty)$. Consequently, it is equivalent to consider a.p. sequences indexated by $\mathbb{N}$, $\mathbb{N}^*$ or $\mathbb{Z}$. In the following, we just consider a.p. sequences with $\mathbb{L} = \mathbb{Z}$. Endowed with the supremum norm $\|\underline{x}\|_{l^\infty(\mathbb{Z})} := \sup_{t\in\mathbb{Z}} \|x_t\|$, $AP(\mathbb{Z}; \mathbb{E})$ is a Banach space.
We can prove that $AP(\mathbb{Z}; \mathbb{E})$ is isometrically isomorphic to $AP^0(\mathbb{Z}; \mathbb{E})/\sim$, where $(f \sim g) \Leftrightarrow ((f - g) \mid_{\mathbb{L}} = 0)$.
If we denote by e_α the function $t \mapsto e^{2i\pi\alpha t}$ from $\mathbb{R}$ into $\mathbb{C}$, where $\alpha \in \mathbb{R}$, one has $e_\alpha \sim e_{\{\alpha\}}$, where $\{\alpha\}$ is the fractional part of α. So, to study a.p. sequences, it is sufficient to consider all the sequences $\{\hat{e}_\alpha \ : \ \alpha \in [0; 1)\}$, where $\hat{e}_\alpha := (e_\alpha(t))_t$.

Each $\underline{x} \in AP(\mathbb{Z}; \mathbb{E})$ possesses a mean value (see [6] p.48):

$$\mathcal{M}\{\underline{x}\} := \mathcal{M}\{x_t\}_t := \lim_{T\to+\infty} \frac{1}{2T+1} \sum_{t=-T}^{T} x_t$$

which equals the mean value of the a.p. function $f_{\underline{x}}$ defined in (3).

Now, we recall some facts about the Mauclaire almost periodic (m.a.p.) sequences. First we denote by $b\mathbb{Z}$ the Bohr compactification of $\mathbb{Z}$ ([9] p.1, [10] p.30). $b\mathbb{Z}$ is a compact topological group such that these exists an one-to-one continuous homomorphism $in : \mathbb{Z} \to b\mathbb{Z}$ with $in(\mathbb{Z})$ dense in $b\mathbb{Z}$.

Definition 1.2 *A sequence $\underline{x} = (x_t)_t$ is a m.a.p. sequence when there exists $\varphi \in C^0(b\mathbb{Z}; \mathbb{E})$ such that $\varphi \circ in(t) = x_t$.*

Such a function is unique and will be denoted by $\varphi^{\underline{x}}$. $AP_M(\mathbb{Z}; \mathbb{E})$ stands for the space of m.a.p. sequences from $\mathbb{Z}$ in $\mathbb{E}$. Endowed with

the supremum norm, it is a Banach space. An element of $AP_M(\mathbb{Z}; \mathbb{E})$ is completely determinated by its values on $\mathbb{N}$ or $\mathbb{N}^*$ ([9] p.4).
Each $\underline{x} \in AP_M(\mathbb{Z}; \mathbb{E})$ possesses a mean value defined as follows:

$$\mathcal{M}_M\{\underline{x}\} := \mathcal{M}_M\{x_t\}_t := \int_{b\mathbb{Z}} \varphi^{\underline{x}}(\theta) d\mu(\theta)$$

where $\varphi^{\underline{x}}$ is defined in (1.2) and μ is the normalized Haar measure of $b\mathbb{Z}$.

Proposition 1.3 *There exists an isometrical isomorphism of Banach spaces:*

$$\Theta : AP_M(\mathbb{Z}; \mathbb{E}) \longrightarrow AP(\mathbb{Z}; \mathbb{E}).$$

Proof. First, we see that $\Phi : AP_M(\mathbb{Z}; \mathbb{E}) \to C^0(b\mathbb{Z}; E)$ such that $\Phi(\underline{x}) = \varphi^{\underline{x}}$ is an isometrical isomorphism of Banach spaces. So, it is sufficient to have one beetween $C^0(b\mathbb{Z}; \mathbb{E})$ and $AP(\mathbb{Z}; \mathbb{E})$.
If χ is a character of $b\mathbb{Z}$, we have for some $\alpha \in [0; 1)$, $\chi \circ in = \chi_\alpha$, where $\chi_\alpha(t) := e^{2i\pi\alpha t}$ for all $t \in \mathbb{Z}$.
Set $P(b\mathbb{Z}; E) := \{\sum_{j=1}^p a_j \chi^{(j)}; a_i \in \mathbb{E}; \chi^{(j)} \in (b\mathbb{Z})'\}$. By using the Weierstrass theorem and the paracompactedness of $\mathbb{E}$, we know that $P(b\mathbb{Z}; \mathbb{E})$ is dense in $C^0(b\mathbb{Z}; \mathbb{E})$. We set, for $\sum_{j=1}^p a_j \chi_{\alpha_j} \in P(b\mathbb{Z}; E)$,

$$\Xi\left(\sum_{j=1}^p a_j \chi_{\alpha_j}\right) = \sum_{j=1}^p a_j \hat{e}_\alpha.$$

Ξ is an isometry, so it is one-to-one and continuous, and can uniquely be extended in an isometrical homomorphism Ξ beetween $C^0(b\mathbb{Z}; \mathbb{E})$ and $AP(\mathbb{Z}; \mathbb{E})$. We have next to prove that Ξ is onto. Let us consider $\underline{x} \in AP(\mathbb{Z}; \mathbb{E})$. There exists a sequence $(P_n)_n$ of trigonometric polynomials on $AP^0(\mathbb{R}; \mathbb{E})$ such that $\lim_{n\to+\infty} \|f_{\underline{x}} - P_n\|_\infty = 0$. So $(P_n)_n$ is a Cauchy sequence, and since we have:

$$\left\|(P_{n+p}(t))_t - (P_n(t))_t\right\|_{l^\infty(\mathbb{Z})} \leq \|P_{n+p} - P_n\|$$

the sequence $((P_n(t))_t)_n$ is a Cauchy sequence in $AP(\mathbb{Z};\mathbb{E})$. If $P_n = \sum_{j\in I_n} a_j^n e_{\alpha_j}$, set $\Pi_n := \sum_{j\in I_n} a_j^n \chi_{\{\alpha_j\}}$. One has $\Xi(\Pi_n) = (P_n(t))_t$ and since Ξ is an isometry, we see that $(\Pi_n)_n$ is a Cauchy sequence on $C^0(b\mathbb{Z};\mathbb{E})$, so it has a limit f. By continuity of Ξ, one has $\Xi(f) = \underline{x}$. One set $\Theta := \Xi \circ \Phi^{-1}$ to conclude. ■

In [16] (and references therein), Zaslavski uses a different notion of a.p. sequence. Zaslavski a.p. sequences are also a.p. sequences in the before described sense, but for instance $(\sin(t))_t$ is not Zaslavski almost periodic (since for all $m \geq 1$, the set $\{\sin(mt); t \in \mathbb{Z}\}$ is dense on $[-1;1]$).

We now describe Hilbert spaces such as Besicovitch spaces. Thourough this section, $\mathbb{E}$ will be an Hilbert space, whose scalar product will be denoted by $\cdot$ for simplicity.

$AP(\mathbb{Z};\mathbb{E})$ can be endowed with the following scalar product:

$$\langle \underline{x} \mid \underline{y} \rangle_2 := \mathcal{M}\{x_t \cdot \overline{y_t}\}_t.$$

The associated norm will be denoted by $\|.\|_2$. The $AP(\mathbb{Z};\mathbb{E})$ Hilbert completion with respect to this scalar product will be denoted by $B^2(\mathbb{Z};\mathbb{E})$. A representation of this completion can be seen as follows. We note that $AP(\mathbb{Z};\mathbb{E}) \subset \mathcal{M}^2(\mathbb{Z};\mathbb{E})$, where $\mathcal{M}^2(\mathbb{Z};\mathbb{E})$ is the Marcinkiewicz space ([13], [14]):

$$\mathcal{M}^2(\mathbb{Z};\mathbb{E}) := \left\{ (x_t)_t \quad : \quad \limsup_{T\to+\infty} \frac{1}{2T+1} \sum_{t=-T}^{T} \|x_t\|^2 < +\infty \right\}.$$

On $\mathcal{M}^2(\mathbb{Z};\mathbb{E})$,

$$p(\underline{x}) := \left(\limsup_{T\to+\infty} \frac{1}{2T+1} \sum_{t=-T}^{T} \|x_t\|^2 \right)^{1/2}$$

is a semi-norm, and if $\mathcal{B}^2(\mathbb{Z};\mathbb{E})$ is the closure of $AP(\mathbb{Z};\mathbb{E})$ into $\mathcal{M}^2(\mathbb{Z};\mathbb{E})$ with respect to this semi-norm, one has $B^2(\mathbb{Z};\mathbb{E}) = \mathcal{B}^2(\mathbb{Z};\mathbb{E})/p$. And so, $B^2(\mathbb{Z};\mathbb{E})$ is a set of equivalent classes of sequences. Moreover, another representation of $B^2(\mathbb{Z};\mathbb{E})$ can be obtained by using Harmonic Analysis:

$$B^2(\mathbb{Z};\mathbb{E}) = \left\{ \underline{x} \quad : \quad \exists(\lambda_\alpha)_\alpha \in \ell^2([0;1);\mathbb{E}), x \sim_2 \sum_{\alpha\in[0;1)} \lambda_\alpha \hat{e}_\alpha \right\},$$

where $x \sim_2 \sum_{\alpha\in[0;1)} \lambda_\alpha \hat{e}_\alpha$ stands for:

$$\mathcal{M}\left\{ \left\| \underline{x} - \sum_{\alpha\in[0;1)} \lambda_\alpha \hat{e}_\alpha \right\|^2 \right\} = 0.$$

For the m.a.p. sequences, we consider:

$$B^2_M(\mathbb{Z};\mathbb{E}) := \left\{ \underline{x} \quad : \quad \exists\varphi \in L^2(b\mathbb{Z};\mathbb{E})\ ,\ \forall t \in \mathbb{Z}, \varphi \circ in(t) = x_t \right\}.$$

Recall that $L^2(b\mathbb{Z};\mathbb{E})$ is the completion of $C^0(b\mathbb{Z};\mathbb{E})$ with respect to L^2 norm. So, since for all $\underline{x} \in AP_M(\mathbb{Z};\mathbb{E})$ we have:

$$\|\varphi^{\underline{x}}\|_{L^2(b\mathbb{Z};\mathbb{E})} = \|\Theta(\underline{x})\|_2,$$

the spaces $B^2(\mathbb{Z};\mathbb{E})$ and $B^2_M(\mathbb{Z};\mathbb{E})$ are isometrically isomorphic as Hilbert spaces.

2 A.p. Sequences Depending on Parameters and Nemytskii Operators

In the following, P is a compact subset or an open subset of $\mathbb{R}^k$, for $k \geq 1$.

If P is open, let us consider the family:

$$K_n := \{x \in P \quad : \quad \|x\| \leq n \text{ and } d(x; P^c) \geq 1/n\}.$$

All K_n are compact in $\mathbb{R}^k$, $P = \cup_n K_n$ and $K_n \subset \mathrm{Int}(K_{n+1})$ for all n.

Definition 2.1 *A sequence $\underline{x}(.) = [\alpha \mapsto (x_t(\alpha))_t] \in \mathbb{E}^{\mathbb{Z}\times P}$ is said to be a.p. in $t \in \mathbb{Z}$, uniformly in $\alpha \in P$ if for all K compact subset of $\mathbb{R}^k$ such that $K \subset P$ and for all $\varepsilon > 0$, one has:*

$$\exists N \geq 1,\ \forall m \in \mathbb{Z},\ \exists \tau \in \{m, \cdots, m+N\},\ \sup_{t\in\mathbb{Z}} \sup_{\alpha\in K} \|x_{t+\tau}(\alpha) - x_t(\alpha)\| \leq \varepsilon.$$

We denote by $APU(\mathbb{Z}; P; \mathbb{E})$ the subset of all these sequences. If P is compact, $APU(\mathbb{Z}; P; \mathbb{E})$ is a Banach space with the norm:

$$\|\underline{x}(.)\|_{APU} := \sup_{(t,\alpha)\in\mathbb{Z}\times P} \|x_t(\alpha)\|$$

and if P is open, $APU(\mathbb{Z}; \mathbb{E})$ is a Fréchet space with the family of semi-norms $(p_n)_n$, where:

$$p_n(\underline{x}(.)) := \sup_{(t,\alpha)\in\mathbb{Z}\times K_n} \|x_t(\alpha)\|.$$

The space $C^0(b\mathbb{Z} \times P; \mathbb{E})$ is endowed with the norm

$$\|f\|_{C^0(\mathbb{Z}\times P;\mathbb{E})} := \sup_{(t,\alpha)\in\mathbb{Z}\times P} \|f(t,\alpha)\|$$

if P is compact, and with the family $(\pi_n)_n$, where

$$\pi_n(f) := \sup_{(t,\alpha)\in\mathbb{Z}\times K_n} \|f(t,\alpha)\|$$

if P is open.

Proposition 2.2 *There exists an isometrical isomorphism of Fréchet spaces beetween $APU(\mathbb{Z}; P; \mathbb{E})$ and $C^0(b\mathbb{Z} \times P; \mathbb{E})$.*

Proof. It is sufficient to prove it when P is compact. We will set the proof into three steps.

1. The mapping $\underline{x} \mapsto (x_t(.))_t$ is an isometrical isomorphism of Fréchet spaces beetween $APU(\mathbb{Z}; P; \mathbb{E})$ and $AP(\mathbb{Z}; C^0(P; E))$.

Indeed, it is clearly an homomorphism which is isometric, so one-to-one. It is also onto: if we consider $\phi \in AP(\mathbb{Z}; C^0(P; E))$, and if we set $\underline{x}$ such that $x_t(\alpha) := \phi(t)(\alpha)$. One has $\underline{x} \in APU(\mathbb{Z}; P; \mathbb{E})$ and $\underline{x}$ is solution of the problem.

2. There exists an isometrical isomorphism of Fréchet spaces beetween $AP(\mathbb{Z}; C^0(P; \mathbb{E}))$ and $C^0(b\mathbb{Z}; C^0(P; \mathbb{E}))$ (see Proposition 1.3).

3. There exists an isometrical isomorphism of Fréchet spaces beetween $C^0(b\mathbb{Z} \times P; \mathbb{E}))$ and $C^0(b\mathbb{Z}; C^0(P; \mathbb{E}))$. Let us consider $f \longmapsto [t \mapsto f(t, .)]$. It is clearly well defined and an isometrical homomorphism. We next prove that this homomorphism is onto. Consider $\lambda \in C^0(b\mathbb{Z}; C^0(P; \mathbb{E}))$ and $(t_0; \alpha_0) \in b\mathbb{Z} \times P$. If we put $f(t, \alpha) := (\lambda(t))(\alpha)$, one has:

$$\|f(t, \alpha) - f(t_0, \alpha_0)\| \leq \|f(t, \alpha) - f(t_0, \alpha)\| + \|f(t_0, \alpha) - f(t_0, \alpha_0)\| \leq$$

$$\leq \|\lambda(t) - \lambda(t_0)\|_{C^0(P;\mathbb{E})} + \|\lambda(t_0)(\alpha) - \lambda(t_0)(\alpha_0)\|.$$

Consider $\varepsilon > 0$. Since $\lambda(t_0) \in C^0(P; \mathbb{E})$, there exists a neighbourhood V_2 of α_0 in P such that if $\alpha \in V_2$, one has: $\|\lambda(t_0)(\alpha) - \lambda(t_0)(\alpha_0)\| \leq \varepsilon/2$. Since $\lambda \in C^0(b\mathbb{Z}; C^0(P; \mathbb{E}))$, there exists a neighbourhood V_1 of t_0 in P such that if $t \in V_1$, one has $\|\lambda(t) - \lambda(t_0)\|_{C^0(P;\mathbb{E})} \leq \varepsilon/2$. If $(t, \alpha) \in V_1 \times V_2$, one has: $\|f(t, \alpha) - f(t_0, \alpha_0)\| \leq \varepsilon$, so $f \in C^0(\mathbb{Z} \times P; \mathbb{E})$. ■

Given $L \in APU(\mathbb{Z}; \mathbb{R}^k; \mathbb{R})$, we consider the Nemystkii operator $\mathcal{N}_L : AP(\mathbb{Z}; \mathbb{R}^k) \longrightarrow AP(\mathbb{Z}; \mathbb{R})$ such that $\mathcal{N}_L(\underline{x}) := (L(t, x_t))_t$.

Proposition 2.3 $\mathcal{N}_L$ *is well defined, and*

$$\mathcal{N}_L \in C^0(AP(\mathbb{Z};\mathbb{R}^k); AP(\mathbb{Z};\mathbb{R})).$$

Proof. By Proposition 2.2, we see that L is associated to $\tilde{L} \in C^0(b\mathbb{Z} \times \mathbb{R}^k; \mathbb{R})$ and $\varphi^{\underline{x}} \in C^0(b\mathbb{Z}; \mathbb{R}^k)$. So, $[t \mapsto \tilde{L}(t, \varphi^{\underline{x}}(t))] \in C^0(b\mathbb{Z}; \mathbb{R})$ and we conclude that $\mathcal{N}_L$ is well defined.
Now, we prove that $\mathcal{N}_L$ is continuous. Fix $\underline{x} \in AP(\mathbb{Z};\mathbb{R}^k)$ and set K the closed ball in $\mathbb{R}^k$ of center 0 and radius $\|\underline{x}\|_{\ell^\infty(\mathbb{Z})} + 1$. Since $b\mathbb{Z} \times K$ is compact, it can be endowed with a uniform structure ([5] p. T.G. II.28, Corollaire 1.). $\tilde{L}$ is uniformly continuous (on $b\mathbb{Z} \times K$, so given $\varepsilon > 0$, there exists U vicinity in $b\mathbb{Z}$ and $\eta \in (0;1)$ such that:

$$\Big[(p,q) \in U, (x,y) \in K^2, \mid x - y \mid \leq \eta\Big] \Longrightarrow \Big[\mid \tilde{L}(p,x) - \tilde{L}(q,y) \mid \leq \varepsilon\Big].$$

Set $x := x(p)$, $p = q$ and taking the sup, we have:

$$\Big[\|\underline{x} - \underline{y}\| \leq \eta\Big] \Longrightarrow \Big[\|\mathcal{N}_L(\underline{x}) - \mathcal{N}_L(\underline{y})\| \leq \varepsilon\Big]. \blacksquare$$

In the following, $L_t := L(t,.)$ and we shall assimilate $APU(\mathbb{Z};\mathbb{R}^k;\mathbb{R})$ and $C^0(b\mathbb{Z} \times \mathbb{R}^k; \mathbb{R})$.
Let us consider the following assumption:

(H1) for all $t \in \mathbb{Z}$, $L_t \in C^1(\mathbb{R}^k;\mathbb{R})$ and $D_2L \in C^0(b\mathbb{Z} \times \mathbb{R}^k; \mathbb{R})$.

Proposition 2.4 *Under* **(H1)** , $\mathcal{N}_L$ *is of class* C^1 *and:*

$$\mathcal{N}_L'(\underline{x}) \cdot \underline{h} = (D_2L(t,x_t).h_t)_t \,.$$

Proof. Fix $\underline{x}$. By using the mean value inequality, we have:

$$\mid L(t, x_t + h_t) - L(t,x_t) - D_2L(t,x_t).h_t \mid \leq$$

$$\leq \sup_{\theta\in(0;1)} |D_2L(t, x_t + \theta h_t) - D_2L(t, x_t)| \mid h_t \mid .$$

Let K be the same compact as in the previous proof. For all $\varepsilon > 0$, there exists $\eta \in (0; 1)$ and an vicinity U in $b\mathbb{Z}$ such that:

$$\Big[(t, s) \in U, (x, y) \in K^2, \mid x - y \mid\leq \eta\Big] \Longrightarrow [\mid D_2L(t, x) - D_2L(s, y) \mid\leq \varepsilon] .$$

Set $x := x_t$, $y := x_t + \theta h_t$, $s = t$ and taking the sup, we have:

$$[\|\underline{h}\|_{\ell^\infty} \leq \eta] \Longrightarrow \left[\sup_{\theta\in[0;1]} |D_2L(t, x_t + \theta h_t) - D_2L(t, x_t)|\right] \leq \varepsilon$$

and so we have:

$$[\|\underline{h}\|_{\ell^\infty} \leq \eta] \Longrightarrow [|\mathcal{N}_L(\underline{x} + \underline{h}) - \mathcal{N}_L(\underline{x}) - (D_2L(t, x_t).h_t)_t| \leq \varepsilon\|\underline{h}\|_{\ell^\infty}] .$$

Continuity of $\mathcal{N}'_L$ follows from the continuity of $\mathcal{N}_{D_2L}$.■

Now we consider the case of $B^2(\mathbb{Z}; \mathbb{E})$ (when $\mathbb{E}$ is an Hilbert space), by the identification of this space with $L^2(b\mathbb{Z}; \mathbb{E})$. We consider a Caratheodory function (see [7]) $L : b\mathbb{Z} \times \mathbb{R}^k \to \mathbb{R}$ such that:

(H2) $\quad \forall(t, x) \in b\mathbb{Z} \times \mathbb{R}^k, \quad \mid L(t, x) \mid\leq a \mid x \mid^2 + b(t)$

where $a > 0$ and $b \in L^1(b\mathbb{Z}; \mathbb{R})$.

We define a Nemytskii operator $\mathcal{N}_L : L^2(b\mathbb{Z}; \mathbb{R}^k) \to L^1(b\mathbb{Z}; \mathbb{R})$ as follows:

$$\mathcal{N}_L(\varphi) := [\theta \mapsto L(\theta; \varphi(\theta))].$$

Proposition 2.5 *Under* **(H2)**, $\mathcal{N}_L$ *is well defined, bounded and continuous.*

Proof. $\|\mathcal{N}_L(\varphi)\|_{L^1(b\mathbb{Z};\mathbb{R})} \leq a\|\varphi\|^2_{L^2(b\mathbb{Z};\mathbb{R}^k)} + \|b\|_{L^1(b\mathbb{Z};\mathbb{R})} < +\infty$ and so $\mathcal{N}_L$ is well defined and bounded. If $\mathcal{N}_L$ is not continuous, there exists $\varphi, (\varphi_n)_n$ such that $\varphi_n \to \varphi$ in $L^2(b\mathbb{Z}; \mathbb{R}^k)$ and $(\mathcal{N}_L(\varphi_n))_n$ has not $\mathcal{N}_L(\varphi)$ as adherence value. Since $\mathcal{N}_L$ is bounded, there exists a dominated subsequence $\mathcal{N}_L(\varphi_{n_k})_k$. The Lebesgue Theorem shows that $\mathcal{N}_L(\varphi_{n_k}) \to \mathcal{N}_L(\varphi)$ in L^1 as $k \to +\infty$, which leads a contradiction.$\blacksquare$

Remark 2.6 *As in [7], it can be shown that if $\mathcal{N}_L$ maps L^2 in L^1 such an estimation must hold.*

We now also do the following assumption:

(H3) : for all t, L'_t exists, is Caratheodory, and satisfies:

$$| D_2L(t,\alpha) | \leq c \mid \alpha \mid + d(t)$$

where $c > 0$ and $d \in L^1(b\mathbb{Z}; \mathbb{R})$.

Proposition 2.7 *Under* **(H3)**, *$\mathcal{N}_L$ is of class C^1, and:*

$$\mathcal{N}'_L(\varphi).h = [\theta \mapsto D_2L(\theta; \varphi(\theta))h(\theta)].$$

Proof. $[\theta \mapsto D_2L(\theta; \varphi(\theta))h(\theta)]$ is L^1 as product as two L^2 functions. Set:

$$R(h) := \mathcal{N}_L(\varphi + h) - \mathcal{N}_L(\varphi) - D_2L(., \varphi).h.$$

Since:

$$\mathcal{N}_L(\varphi + h) - \mathcal{N}_L(\varphi) = \int_0^1 D_2L(\cdot, \varphi + th).hdt$$

one has:

$$\int_{b\mathbb{Z}} \mid R(h) \mid d\mu \leq$$

$$\leq \int_{b\mathbb{Z}} \left| \int_0^1 \left(D_2L(\theta; \varphi(\theta) + th(\theta)) - D_2L(\theta; \varphi(\theta))\right) h(\theta)dt \right| d\mu(\theta) \leq$$

$$\|h\|_{L^2(b\mathbb{Z};\mathbb{R}^k)} \left[\int_0^1 \int_{b\mathbb{Z}} \left(D_2L(\theta; \varphi(\theta) + th(\theta)) - D_2L(\theta; \varphi(\theta))\right)^2 dtd\mu(\theta) \right]^{1/2}$$

Since $\mathcal{N}_{D_2L}$ is continuous, the integrand goes to 0 when $h \to 0$ and we conclude by using the Lebesgue theorem. $\blacksquare$

3 Variational Principles

Consider a positive integer N.

First, we consider a Lagrangian L which satisfies **(H1)** with $k = (p+1)N$. From L, we define a functional $J : AP(\mathbb{Z}; \mathbb{R}^N) \to \mathbb{R}$ as follows:

$$J(\underline{x}) := \mathcal{M}\{L_t(x_t, \cdots, x_{t+p})\}_t.$$

Lemma 3.1 *J is of class C^1, and:*

$$J'(\underline{x}) \cdot \underline{h} = \mathcal{M}\left\{\left(\sum_{j=0}^{p} D_{j+1}L_t(x_{t-j}, \cdots, x_{t+p-j})\right).h_t\right\}_t.$$

Proof. Consider the bounded linear operator $T : AP(\mathbb{Z}; \mathbb{R}^N) \to AP(\mathbb{Z}; \mathbb{R}^N)^{p+1}$ such that: $T((x_t)_t) := (x_t, \cdots, x_{t+p})_t$. We have:

$$J := \mathcal{M} \circ \mathcal{N}_L \circ T$$

and J is C^1 a superposition of C^1 operators, and from the Chain Rule we have:

$$J'(\underline{x}) \cdot \underline{h} = \mathcal{M}\left\{\sum_{j=0}^{p} D_{j+1}L_t(x_t, \cdots, x_{t+p}).h_{t+j}\right\}$$

and since the mean is invariant under translations, we have:

$$\mathcal{M}\{D_{j+1}L_t(x_t, \cdots, x_{t+p}).h_{t+j}\}_t = \mathcal{M}\{D_{j+1}L_{t-j}(x_{t-j}, \cdots, x_{t-j+p}).h_t\}_t$$

and the lemma is proven.■

Proposition 3.2 *The following assertions are equivalent:*

1. *$\underline{x}$ is a $AP(\mathbb{Z}; \mathbb{R}^N)$ a.p. solution of (1).*

2. *$\underline{x}$ is a critical point of J on $AP(\mathbb{Z}; \mathbb{R}^N)$.*

Proof. By using Lemma 3.1, it is sufficient to prove that **(1)** implies **(2)**. Put $(h_t)_t := (\sum_{j=0}^{p} D_{j+1}L_t(x_{t-j}, \cdots, x_{t+p-j}))_t$ which is possible since $(\sum_{j=0}^{p} D_{j+1}L_t(x_{t-j}, \cdots, x_{t+p-j}))_t \in AP(\mathbb{Z}; \mathbb{R}^N)$. We have:

$$\mathcal{M}\{| h_t |^2\}_t = 0 \quad \text{i.e. :} \quad \int_{b\mathbb{Z}} \left(\varphi^{\underline{h}}(\theta)\right)^2 d\mu(\theta) = 0.$$

Since $\left(\varphi^{\underline{h}}(.)\right)^2$ is continuous nonnegative, we have $\left(\varphi^{\underline{h}}(\theta)\right)^2 = 0$ for all θ. And so **(2)** is proven. ■

Now, we consider L which satisfies **(H2)** and **(H3)** with $k = (p+1)N$. From L, we define a functional $J : B^2(\mathbb{Z}; \mathbb{R}^N) \to \mathbb{R}$ as follows:

$$J(\underline{x}) := \mathcal{M}\{L_t(x_t, \cdots, x_{t+p})\}_t.$$

Lemma 3.3 *J is of class C^1, and:*

$$J'(\underline{x}) \cdot \underline{h} = \mathcal{M}\left\{\left(\sum_{j=0}^{p} D_{j+1}L_t(x_{t-j}, \cdots, x_{t+p-j})\right).h_t\right\}_t .$$

The proof is the same as the proof of Lemma 3.1.

Proposition 3.4 *The following assertions are equivalent:*

1. *$\underline{x}$ is a $B^2(\mathbb{Z}; \mathbb{R}^N)$ a.p. solution of (1).*
2. *$\underline{x}$ is a critical point of J on $B^2(\mathbb{Z}; \mathbb{R}^N)$.*

Proof. As in Proposition 3.2, we just have to prove that **(1)** implies **(2)**. Put $(h_t)_t := (\sum_{j=0}^{p} D_{j+1}L_t(x_{t-j}, \cdots, x_{t+p-j}))_t$ which is possible since $(\sum_{j=0}^{p} D_{j+1}L_t(x_{t-j}, \cdots, x_{t+p-j}))_t \in B^2(\mathbb{Z}; \mathbb{R}^N)$. We have:

$$\mathcal{M}\{| h_t |^2\}_t = 0 \quad \text{i.e. :} \quad \int_{b\mathbb{Z}} \left(\varphi^{\underline{h}}(\theta)\right)^2 d\mu(\theta) = 0$$

i.e. $\left(\varphi^{\underline{h}}(.)\right)^2 = 0$ in $L^2(b\mathbb{Z}; \mathbb{R})$. And so **(2)** is proven. ■

4 Structure Results on $AP(\mathbb{Z};\mathbb{R}^N)$

In this section, we give some structure results on the set of a.p. solutions in $AP(\mathbb{Z};\mathbb{R}^N)$ of the Euler equation (1). The main tool will be the variational structure of the problem. When the Lagrangian is concave, this concavity is not simply usable on (1) but it becomes a simple and powerfull tool in the variational viewpoint.

Theorem 4.1 *The two following assertions hold:*

1. *If L_t is concave (convex) for all t, then the set of solutions is a convex closed subset of $AP(\mathbb{Z};\mathbb{R}^N)$.*

2. *If L_t is strictly concave (convex) for all t, then there is at most one solution in $AP(\mathbb{Z};\mathbb{R}^N)$ of (1).*

Proof. Assume for instance that L_t is concave for all t. By using Lemma 3.1, it is equivalent to study the set of critical points of the functional J, denoted by $\mathcal{C}(J)$.
Assumption **1** is a consequence of the fact that the concavity of L_t and the linearity of the mean imply the concavity of J.
2. Consider $\underline{x}, \underline{y} \in \mathcal{C}(J)$. We have to prove that $\underline{x} = \underline{y}$. Set:

$$z_t := L_t\left(\frac{x_t + y_t}{2}, \cdots, \frac{x_{t+p} + y_{t+p}}{2}\right) - \frac{L_t(x_t, \cdots, x_{t+p}) + L_t(y_t, \cdots, y_{t+p})}{2}$$

We have for all t, $z_t \geq 0$.If $\mathcal{M}\{\underline{z}\} = 0$, we have $\varphi^{\underline{z}} = 0$ since this function is nonnegative (by **1.**) and continuous. So, we have $z_t = 0$ for all t, and by strict concavity of L_t for all t, $\underline{x} = \underline{y}$. By contraposing, it follows that if $\underline{x} \neq \underline{y}$, $\mathcal{M}\{\underline{z}\} > 0$, and:

$$J\left(\frac{\underline{x} + \underline{y}}{2}\right) - \frac{J(\underline{x}) + J(\underline{y})}{2} = \mathcal{M}\{\underline{z}\} > 0$$

which is not possible.■

5 Results in $B^2(\mathbb{Z}; \mathbb{R}^N)$

In this Section, we study the a.p. solutions (in $B^2(\mathbb{Z}; \mathbb{R}^k)$) of the Euler equation (1). Like in the previous Section, we use a variational viewpoint but here the Hilbert structure of $B^2(\mathbb{Z}; \mathbb{R}^N)$ permits us to obtain existence theorem by using direct methods of Calculus of Variations.

Theorem 5.1 (A linear case) *We consider the following Euler equation:*

$$M_t x_{t+1} + \Lambda_t x_t + M_{t-1} x_{t-1} = N_t.$$

We assume that $(\Lambda_t)_t, (M_t)_t \in AP(\mathbb{Z}; \mathbb{R})$ and $(N_t)_t \in AP(\mathbb{Z}; \mathbb{R}^N)$ and that there exists $\epsilon \in \{-1; 1\}$ and $\alpha > 0$ such that for all $t \in \mathbb{Z}$, we have: $\epsilon Lambda_t \geq \alpha$. Then (1) possesses a unique a.p. solution on $B^2(\mathbb{Z}; \mathbb{R}^N)$ when one of the following condition is fulfilled:

1. $\epsilon = 1$ *and* $\|(M_t)_t\|_{\ell^\infty(\mathbb{Z})} < \inf_{t \in \mathbb{Z}} \Lambda_t$.

2. $\epsilon = -1$ *and* $\|(M_t)_t\|_{\ell^\infty(\mathbb{Z})} < -\sup_{t \in \mathbb{Z}} \Lambda_t$.

Proof. This Euler equation is associated to the Lagrangian:

$$L_t(x, y) := \Lambda_t \mid x \mid^2 + 2M_t x \cdot y + 2N_t \cdot x.$$

By changing L on $-L$, it is sufficient so assume that $\epsilon = 1$. We choose $\alpha := \inf_t \Lambda_t$. $\underline{x}$ is a critical point when it satisfies:

$$(\forall \underline{y} \in B^2(\mathbb{Z}; \mathbb{R}^N)) \qquad a(\underline{x}, \underline{y}) = \ell(\underline{y})$$

where:

$$a(\underline{x}, \underline{y}) := \mathcal{M}\{\Lambda_t x_t \cdot y_t + M_t(x_t \cdot y_{t+1} + x_{t+1} \cdot y_t)\}_t$$

and:

$$\ell(\underline{y}) := -2\mathcal{M}\{N_t \cdot y_y\}_t.$$

By using the Cauhy-Schwarz inequality, we see that a (resp. ℓ) is a bilinear (resp. linear) continuous form. The Lax-Milgram theorem prove us that such an equation has solutions when a is elliptic.
We have :

$$a(\underline{x}, \underline{x}) \geq \alpha\mathcal{M}\{| x_t |^2\}_t - 2\|(M_t)_t\|_{\ell^\infty(\mathbb{Z})}\mathcal{M}\{| x_t \cdot x_{t+1} |\}_t$$

and since the Cauchy-Schwarz inequality shows :

$$\mathcal{M}\{| x_t \cdot x_{t+1} |\}_t \leq \|\underline{x}\|_2^2$$

we obtain :

$$a(\underline{x}, \underline{x}) \geq (\alpha - \|(M_t)_t\|_{\ell^\infty(\mathbb{Z})})\|\underline{x}\|_2^2$$

which gives ellipticity, since $\alpha - \|(M_t)_t\|_{\ell^\infty(\mathbb{Z})} > 0$ by assumption.■

We now consider coercive concave problems.

Theorem 5.2 (The concave-coercive case) *If the Lagrangian L satisfies assumption* **(H2)** *, and the two following conditions:*

1. *For all $t \in \mathbb{Z}$, L_t is concave.*

2. *There exists $(\alpha_i)_i \in \mathbb{R}^{p+1}$ with $\sum_i \alpha_i > 0$ and $\gamma \in L^1(b\mathbb{Z}; \mathbb{R})$ such that for all $t \in \mathbb{Z}$:*

$$L_t(x_1, \cdots, x_p) \leq -\left(\sum_{i=1}^{p} \alpha_i \mid x_i \mid^2\right) + \gamma(t).$$

Then there exists a $B^2(\mathbb{Z}; \mathbb{R}^N)$ a.p. solution of (1).

Proof. By using Proposition 3.4, it is equivalent to search maxima of J, whose set is denoted by $Argmax(J)$. First, we see that $J(\underline{x}) \leq -(\sum_i \alpha_i)\|\underline{x}\|_2^2 + \mathcal{M}\{| \gamma_t |\}_t$, and so: $\lim_{\|\underline{x}\|_2 \to +\infty} J(\underline{x}) = -\infty$. Set $\sigma := \sup_{\underline{x} \in B^2(\mathbb{Z};\mathbb{R}^N)} J(\underline{x})$. Since $\lim_{\|\underline{x}\|_2 \to +\infty} J(\underline{x}) = -\infty$ and J is concave, we have $\sigma < +\infty$. For all positive integer n, there exists $\underline{x}^{(n)}$ s.t. $J(\underline{x}^{(n)}) \geq \sigma - 1/n$. Since $\lim_{\|\underline{x}\|_2 \to +\infty} J(\underline{x}) = -\infty$, $(\underline{x}^{(n)})_n$ is bounded in the Hilbert space $B^2(\mathbb{Z};\mathbb{R}^N)$, and so has a weakly convergent subsequence $(\underline{x}^{(n_k)})_k$. If $\underline{x}$ is the limit, since J is concave, we have:

$$J(\underline{x}) \geq \lim_{k \to +\infty} J\left((\underline{x}^{(n_k)})_k\right) = \sigma$$

and so $\underline{x} \in Argmax(J)$.■

Theorem 5.3 (A quasi-linear case) *Let I be a finite subset of $\mathbb{Z}$, for any $\tau \in I$, $(a_{t,\tau})_t \in \ell^\infty(\mathbb{Z};\mathbb{R})$, $\tau_1, \cdots, \tau_p \in \mathbb{Z}$ distincts integers, $\phi \in APU(\mathbb{Z};(\mathbb{R}^N)^p;\mathbb{R}^N)$. We assume that:*

(i) $\alpha := \sum_{\tau \in I} \inf_t(a_{t,\tau}^2) - \sum_{\tau \neq \tau'} \|a_{\cdot,\tau}\|_{\ell^\infty} \|a_{\cdot,\tau'}\|_{\ell^\infty} > 0.$

(ii) $\forall (t, x_1, \cdots, x_p, y_1, \cdots, y_p) \in \mathbb{Z} \times (\mathbb{R}^N)^{2p}$

$$\|\phi_t(x_1, \cdots, x_p) - \phi_t(y_1, \cdots, y_p)\| \leq \lambda \sum_{j=1}^{p} \|x_j - y_j\|$$

(iii) $p\lambda < \alpha^{1/2}$.

Then, the equation (2) as a solution in $B^2(\mathbb{Z};\mathbb{R}^N)$. There exists just one solution which satisfies:

$$\|\underline{x}\|_2 \leq \frac{\|(\phi_t(0))_t\|_2}{\alpha^{1/2} - p\lambda}.$$

Proof. We define a bounded linear operator A on $B^2(\mathbb{Z};\mathbb{R}^N)$ by the formula:

$$A\underline{x} := \left(\sum_{\tau\in I} a_{t,\tau}x_{t+\tau}\right)_t .$$

1. We prove that A satisfies $\forall \underline{x},\quad \|A\underline{x}\|_2 \geq \alpha^{1/2}\|\underline{x}\|_2$.

Indeed, we have:

$$\|A\underline{x}\|_2^2 = \mathcal{M}\left\{\left\|\sum_\tau a_{t,\tau}x_{t+\tau}\right\|^2\right\}_t =$$

$$= \mathcal{M}\left\{\sum_\tau a_{t,\tau}^2\|x_{t+\tau}\|^2 + \sum_{\tau\neq\tau'} a_{t,\tau}a_{t,\tau'}x_{t+\tau}\cdot x_{t+\tau'}\right\}_t$$

By using Cauchy-Schwarz inequality and the fact that mean is invariant under translation, we get:

$$\mathcal{M}\left\{\sum_\tau a_{t,\tau}^2\|x_{t+\tau}\|^2 + \sum_{\tau\neq\tau'} a_{t,\tau}a_{t,\tau'}x_{t+\tau}\cdot x_{t+\tau'}\right\}_t \geq \alpha\|\underline{x}\|^2.$$

This fact shows that A is into.

2. We prove that A is onto.

To prove this, let us first evaluate A^*. We have:

$$\langle A\underline{x} \mid \underline{y}\rangle_2 = \mathcal{M}\left\{\sum_{\tau\in I} a_{t,\tau}x_{t+\tau}\cdot y_t\right\} = \mathcal{M}\left\{\sum_{\tau\in I} a_{t-\tau,\tau}x_t\cdot y_{t-\tau}\right\} = \langle \underline{x} \mid A^*\underline{y}\rangle_2$$

and so:

$$A^*\underline{y} = \left(\sum_{\tau\in I} a_{t-\tau,\tau}y_{t-\tau}\right)_t .$$

Thus, we have:

$$\|A^*\underline{y}\|_2^2 = \mathcal{M}\left\{\left\|\sum_{\tau\in I} a_{t-\tau,\tau}y_{t-\tau}\right\|^2\right\}$$

and the same calculation as in **(i)** shows that for all $\underline{y}$:

$$\|A^*\underline{y}\|_2 \geq \alpha^{1/2}\|\underline{y}\|_2.$$

Thus, by [8] Theorem II.4.4. p.63, we know that A is onto.

3. We prove that $\|A^{-1}\| \leq \alpha^{-1/2}$.

We have in fact:

$$\|A^{-1}\| = \sup_{\underline{y}\neq 0} \frac{\|A^{-1}\underline{y}\|_2}{\|\underline{y}\|_2} = \sup_{\underline{x}\neq 0} \frac{\|\underline{x}\|_2}{\|A\underline{x}\|_2} = \left(\inf_{\underline{x}\neq 0} \frac{\|A\underline{x}\|_2}{\|\underline{x}\|_2} \right)^{-1}$$

and since $\inf_{\underline{x}\neq 0} \frac{\|A\underline{x}\|_2}{\|\underline{x}\|_2} \geq \alpha^{1/2}$, the result holds.

4. An estimation.

For any τ, we set S_τ the shift operator: $S_\tau(\underline{x}) := (x_{t+\tau})$. We show there that $\mathcal{N}_\phi \circ (S_{\tau_1}, \cdots, S_{\tau_p})$ is $p\lambda$-lipschitzian. We set $S := (S_{\tau_1}, \cdots, S_{\tau_p})$ for simplicity.

$$\|\mathcal{N}_\phi \circ S(\underline{x}) - \mathcal{N}_\phi \circ S(\underline{y})\|_2^2 =$$

$$\mathcal{M}\left\{\|\phi_t(x_{t+\tau_1}, \cdots, x_{t+\tau_p}) - \phi_t(y_{t+\tau_1}, \cdots, y_{t+\tau_p})\|^2\right\}_t \leq$$

$$\leq \lambda^2 \mathcal{M}\left\{\left(\sum_{j=1}^{p} \|x_{t+\tau_j} - y_{t+\tau_j}\|\right)^2\right\}_t \leq \lambda^2 p \mathcal{M}\left\{\sum_{j=1}^{p} \|x_{t+\tau_j} - y_{t+\tau_j}\|^2\right\}_t =$$

$$= (\lambda p)^2 \|\underline{x} - \underline{y}\|_2^2$$

thus the result holds.

5. Conclusion.

The considered equation can be written:

$$A\underline{x} - \mathcal{N}_\phi \circ S(\underline{x}) = 0$$

and since A is bijective, it is equivalent to:

$$\underline{x} - A^{-1} \circ \mathcal{N}_\phi \circ S(\underline{x}) = 0.$$

We consider now the continuous nonlinear operator $T := A^{-1} \circ \mathcal{N}_\phi \circ S$.

We see that T is a contraction, since:

$$\|T\underline{x} - T\underline{y}\|_2 = \|A^{-1}(\mathcal{N}_\phi \circ S(\underline{x}) - \mathcal{N}_\phi \circ S(\underline{y}))\|_2 \leq \frac{\|\mathcal{N}_\phi \circ S(\underline{x}) - \mathcal{N}_\phi \circ S(\underline{y})\|_2}{\alpha^{1/2}} \leq$$

$$\leq \frac{p\lambda}{\alpha^{1/2}}\|\underline{x}-\underline{y}\|_2.$$

Let us set:

$$r := \frac{\|(\phi_t(0))_t\|_2}{\alpha^{1/2}-p\lambda}$$

and let be B the closed ball of center 0 and radius r in $B^2(\mathbb{Z};\mathbb{R}^N)$. We show that T maps B into B. Indeed, if $\|\underline{x}\|_2 \leq r$, we have:

$$\|T(\underline{x})\|_2 \leq \|T(\underline{x})-T(0)\|_2+\|T(0)\|_2 \leq \frac{p\lambda}{\alpha^{1/2}}\|\underline{x}\|_2+\frac{\|(\phi_t(0))_t\|_2}{\alpha^{1/2}} \leq r.$$

The Banach-Picard Contraction Theorem shows that T has a unique fixed point on B.■

The following corollary shows that assumption **(iii)** can be relaxed in some cases.

Corollary 5.4 *With the notations of the previous theorem, we assume condition* **(ii)** *of the previous theorem and these conditions:*

(0') $\exists \tau_0 \in I \cap \{\tau_1,\cdots,\tau_p\}, \quad a_0 := \inf_t a_{t,\tau_0} > 0.$

(i') $\alpha+\rho^2+2\rho\left(\sum_{\tau\neq\tau_0}\|a_{.,\tau_0}\|_{\ell^\infty}-a_0\right) > 0$ *with* $\rho \in [0;a_0]$.

(iii') $\alpha-p^2\lambda^2+2\rho\left(\sum_{\tau\neq\tau_0}\|a_{.,\tau_0}\|_{\ell^\infty}-a_0-p^2\lambda\right)-(p^2-1)\rho^2 > 0.$

Then equation (2) has a solution.

Remark 5.5 *When condition* **(0')** *is satisfied, condition* **(iii')** *can be less restrictive than* **(iii)**. *For instance, when* $a_0 > \sum_{\tau\neq\tau_0}\|a_{.,\tau_0}\|_{\ell^\infty}-p^2\lambda$, *a small* $\rho > 0$ *give a better condition.*

Proof. Let $k \in \{1, \cdots, p\}$ be the index such that $\tau_0 = \tau_k$, $\hat{A} := A - \rho S_{\tau_0}$, $\hat{\phi} := [(x_1, \cdots, x_p) \mapsto \phi(x_1, \cdots, x_p) - \rho x_k]$. Our equation is equivalent to:

$$\hat{A}(\underline{x}) - \left(\mathcal{N}_{\hat{\phi}} \circ S(\underline{x})\right) = 0.$$

We apply the previous theorem to this equation. Since $a_0 - \rho \geq 0$, we see that assumption **(i')** gives condition **(i)** for the operator $\hat{A}$. The function $\hat{\phi}$ is $\lambda + \rho$ lipschitzian, then the condition to apply the previous theorem is $\alpha' - p^2(\lambda + \rho)^2 > 0$, where α' is the α associated to $\hat{A}$. We have:

$$\alpha' - p^2(\lambda+\rho)^2 = \alpha + 2\rho \left(\sum_{\tau \neq \tau_0} \|a_{\cdot,\tau_0}\|_{\ell^\infty} - a_0 \right) + \rho^2 - p^2(\lambda^2 - 2\lambda\rho + \rho^2) > 0$$

by assumption **(iii')**.■

Conclusion

The paper introduces several sequences spaces and nonlinear functional analytic viewpoints (notably a variational approach) in order to study the almost periodic oscillations in Difference Equations. These new tools and methods are open to be developped.

References

[1] L. Amerio, G. Prouse, *Almost Periodic Functions and Functional Equations*, Von Nostrand Reinhold Company, New York, 1971.

[2] V. Bangert, Mather Sets for Twist Maps and Geodesics on Tori, in *Dynamics Reported*, Vol 1, U. Kirchgraber and H.O. Walther

(Editors), J.Wiley and Sons Ltd and B.G. Teubner, Stuttgart, 1988, pp.1-56.

[3] J. Blot, Calculus of Variations in Mean and Convex Lagrangians IV, *Reserche di Mathematica*, XL (1991), 3-18.

[4] J. Blot, Oscillations presque-périodiques forcées d'équations d'Euler-Lagrange, *Bulletin de la Soc. Math. France*, 122 (1994), 285-304.

[5] N. Bourbaki, *Topologie Générale*, Chapitres 1 à 4, Hermann, Paris, 1971.

[6] C. Corduneanu, *Almost Periodic Functions*, Second Edition, Chelsea Publ. Comp., New York, 1989.

[7] D.G. De Figueiredo, *Lectures on the Ekeland Variational Principles with Applications and Detours*, Tata Institute of Fundamental Research, Bombay, 1989.

[8] S. Goldberg, *Unbounded Linear Operators, Theory and Applications*, Mc Graw-Hill Book Comp. Inc., New York, 1966.

[9] J.L. Mauclaire, *Intégration et Théorie des Nombres*, Hermann, Paris, 1986.

[10] W. Rudin, *Fourier Analysis on Groups*, Interscience Publishers, New York, 1962.

[11] T. Sargent, *Dynamic Economic Theory*, Harvard University Press, Harvard, 1987.

[12] N.L. Stokey and R.E. Lucas, *Recursive Methods in Economics and Dynamics*, Harvard University Press, Harvard, 1989.

[13] K. Vo-Khac, *Etude des fonctions quasi-stationnaires et de leurs applications aux équations différentielles opérationnelles*, Bull. Soc. Math. France, Mémoire 6, 1966.

[14] K. Vo-Khac, Fonctions et distributions stationnaires, applications à l'étude de solutions stationnaires d'E.D.P., in *Espaces de Marcinkiewicz, Corrélations, Mesures, Systèmes Dynamiques*, J. Bass (Editor), Masson, 1985.

[15] A. Weil, *L'intégration dans les Groupes Topologiques*, Hermann, Paris, 1965.

[16] A.J. Zaslavski A.J., Existence and Structure of Optimal Solutions of Variational Problems, in *Recent Developments in Optimization and Nonlinear Analysis*, Contemporary Mathematics, Vol 204, Am. Math Soc., Providence, Rhode Island, 1997, pp.247-278