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MORE GENERAL CONSTRUCTIONS OF WAVELETS ON THE INTERVAL

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Abstract

In this paper we present general constructions of orthogonal and biorthogonal multiresolution analysis on the interval. In the first one, we describe a direct method to define an orthonormal multiresolution analysis. In the second one, we use the integration and derivation method for constructing a biorthogonal multiresolution analysis. As applications, we prove that these analyses are adapted to study regular functions on the interval.

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1 Introduction

The search for wavelet bases on a bounded domain has been an active field for many years, since the beginning of the 1990's. All these constructions use either the basis of I. Daubechies or the spline basis. In his fundamental paper on wavelets on the interval [14], Y. Meyer proved that one can take restrictions of the orthonormal multiresolution analysis of I. Daubechies to the interval $[0,1]$ and then we can study functions known only on the interval. More precisely, he proves that the restrictions of Daubechies scaling functions on the interval are linearly independent but the restrictions of associated wavelets on the interval are not linearly independent.

In 1992, we have constructed multiresolution analysis on the interval by using Daubechies wavelets [9]. The associated bases have compact support and allow also the study of divergence-free vector functions on $[0,1]^n$.

There are related constructions as well by A. Canuto and coworkers [1] and by A. Jouini and P. G. Lemarié ([8] and [10]).

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In this paper we aime to generalize the result for every orthonormal multiresolution analysis. Next, we present orthogonal and biorthogonal systems on $[0,1]$ which are constructed by means of dyadic translations and dilatations from a finite number of basic functions and are well-adapted to study Sobolev spaces $H^s([0,1])$ and $H_0^s([0,1])$ $(s \in \mathbb{Z})$.

The contents of this paper is the following.

In Section 2, we at first define and construct new orthogonal multiresolution analysis on the interval $[0,1]$. Next, we prove the Meyer's lemma [14] for the general case of an orthonormal multiresolution analysis with compact support. Then,we construct the associated wavelet bases which are more technical. In section 3, we study biorthogonal multiresolution analysis (V_j, V_j^*) $(j \in \mathbb{Z})$ on the interval $[0,1]$. By a derivation on V_j and an integration on V_j^*, we get a new biorthogonal multiresolution analysis $(\widetilde{V}_j, \widetilde{V}_j^{\,*})$ of the space $L^2([0,1])$. If we denote P_j the projector from $L^2([0,1])$ on V_j parallel to $(V_j^*)^{\perp}$and $\widetilde{P}_j$ be the projector in $\widetilde{V}_j$ parallel to $(\widetilde{V}_j^{\,*})^{\perp}$, then we have the following commutation property

$$\frac{d}{dx} o P_j = \widetilde{P}_j o \frac{d}{dx}.$$

The section 4 is devoted to applications. We prove that the biorthogonal multiresolution analysis constructed in section 3 is adapted to study Sobolev spaces $H^s([0,1])$ and $H_0^s([0,1])$ for $s \in \mathbb{Z}$.

2 Orthogonal multiresolution analysis on the inerval $[0,1]$

It is clear that if we consider an orthogonal multiresolution analysis, and if we take its restriction to $[0,1]$, we do not get an orthogonal multiresolution analysis of $L^2([0,1])$. Moreover, for the orthogonal multiresolution analysis $V_j(\mathbb{R})$ of I. Daubechies, if we consider the associated scaling functions $\varphi_{j,k}(x)_{[0,1]}$, we have an independent system which is not orthogonal. However, if we consider the associated wavelets $\psi_{j,k}(x)_{[0,1]}$, we get a dependent system (see [14]) and the support of the wavelet ψ is very important in this case. Then, the construction of an orthogonal multiresolution analysis in $[0,1]$ (or biorthogonal) is technical especially near the boundaries 0 and 1.

In this section, we shall prove the precedent result for any orthogonal multiresolution analysis with compact support and regular (see definition 3 in [9]). More precisely, we use a direct method based on the result described in [14] to construct orthogonal multiresolution analysis on the interval $[0,1]$ which are generated by a finite number of basic functions. These analyses are regular and have compact support.

For this purpose, we consider an orthogonal multiresolution analysis $V_j(\mathbb{R})$ of $L^2(\mathbb{R})$ where the scaling function φ have a compact support $[N_1, N_2]$. We recall first the scaling equations for this analysis. The inclusion $V_0 \subset V_1$ gives the two following equations

$$\varphi(\frac{x}{2}) = \sum_{k=N_1}^{N_2} a_k \varphi(x-k) \text{ where } a_{N_1} a_{N_2} \neq 0 \tag{2.1}$$

and

$$\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi) \text{ where } m_0(\xi) = \frac{1}{2}\sum_{k=N_1}^{N_2} a_k e^{-ik\xi}. \tag{2.2}$$

We assume that the associated wavelet ψ have the same support (by a simple translation) and then is defined by

$$\psi(\frac{x}{2}) = \sum_{k=N_1}^{N_2} b_k \varphi(x-k) \text{ where } b_{N_1} b_{N_2} \neq 0. \tag{2.3}$$

Note that we cannot define in the same manner as classical wavelet theory the notion of multiresolution analysis in the interval because we do not have the invariance and dilatation properties in a bounded domain. Then, we present differently this notion. Let j_0 be an integer such that $2^{j_0} \geq 2(N_2 - N_1 - 1)$ (we can separate the boundaries functions). We denote by

$$V_j([0,1]) = Vect\{\varphi_{j,k[0,1]}, \varphi_{j,k} \in V_j(\mathbb{R})\}, \tag{2.4}$$

and

$$v_j([0,1]) = Vect\{\varphi_{j,k}, supp\varphi_{j,k} \subset [0,1]\}. \tag{2.5}$$

Definition 2.1. A sequence $\{V_j\}_{j \geq j_0}$ of closed subspaces of $L^2([0,1])$ is called a multiresolution analysis on $L^2([0,1])$ associated with $V_j(\mathbb{R})$ if

i) $\forall j \geq j_0$, $v_j([0,1]) \subset V_j \subset V_j([0,1])$

ii) $\forall j \geq j_0$, $V_j \subset V_{j+1}$.

It is clear that these spaces contain a finite number of functions due to compacity of the support and then the Gram-Schmidt method gives orthonormal systems if these systems are linearly independent. We now proceed to prove an elementary lemma which will be useful in analysis for functions defined on the interval $[0,1]$. We begin by the case of the interval $]-\infty,0]$. In fact, we prove that only the functions $\varphi_{j,k}$ whose support intersects the interval $]-\infty,0[$ occur in the analysis of an arbitrary function in $V_0(\mathbb{R})$ and with support in $]-\infty,0]$.

Lemma 2.1. *If* $f(x) = \sum_{-\infty}^{+\infty} c_k \varphi(x-k)$ *is a function of* $V_0(\mathbb{R})$ *such that* $f(x) = 0$ *for* $x \leq 0$. *Then* $c_k = 0$ *for* $k \leq -N_1 - 1$.

Proof. The support of the function $\varphi(x-k)$ is $[k+N_1, k+N_2]$ and then is included in $]-\infty,0]$ for $k \leq -N_2$. We have $c_k = \int_{-\infty}^{+\infty} f(x)\overline{\varphi(x-k)}dx = 0$ for $k \leq -N_2$.

Let p be the smallest integer of k such that $c_k \neq 0$. If $p \geq -N_1$, then we have the result. If $p < -N_1$, then $f(x) = 0$ on the interval $[p+N_1, p+N_1+1]$ Because the support of the scaling function φ is equal to $[N_1,N_2]$. Using the hypothesis that f is a function of $V_0(\mathbb{R})$, we obtain $f(x) = c_p\varphi(x-p)$. Then, we have a contradiction. □

The following result generalizes the result of Y. Meyer [14] and gives an other multiresolution analysis of $L^2([0,1])$.

Theorem 2.1. *Let* $j \geq j_0$ *and* $f(x) = \sum_{-\infty}^{+\infty} c_k \varphi(2^j x - k)$ *be a function of* $V_j(\mathbb{R})$ *such that* $f(x) = 0$ *for* $0 \leq x \leq 1$. *Then* $c_k = 0$ *for* $-N_2 + 1 \leq k \leq 2^j - N_1 - 1$.

Proof. Let $j \geq j_0$ and $2^{j_0} \geq 2(N_2 - N_1 - 1)$, we can consider three cases.
1) If $-N_2 + 1 \leq k \leq -N_1 - 1$, the support of the scaling functions $\varphi_{j,k}$ is included in $]-\infty, N_2 - N_1 - 1] \subset]-\infty, \frac{1}{2}]$.
2) If $-N_1 \leq k \leq 2^j - N_2$, the support of the scaling functions $\varphi_{j,k}$ is included in [0,1].
3) If $2^j - N_2 + 1 \leq k \leq 2^j - N_1 - 1$, the support of the scaling functions $\varphi_{j,k}$ is included in $[N_2 - N_1 - 1, +\infty[\subset [\frac{1}{2}, +\infty[$.
We see that in the case 2), we have

$$c_k = \int_{-\infty}^{+\infty} f(x)\overline{\varphi(x-k)}dx = 0.$$

Applying Lemma 2.1 to the first case and the third case, we get $c_k = 0$. This yields $c_k = 0$ for $-N_2 + 1 \leq k \leq 2^j - N_1 - 1$. □

Theorem 2.1 is the basis for our strategy: to get the bases on the interval. As a consequence, we have the following result.

Corollary 2.1. *For $j \geq j_0$, the functions $\varphi_{j,k/[0,1]}$, $-N_2 + 1 \leq k \leq 2^j - N_1 - 1$, form a Riesz basis of the space $V_j([0,1])$.*

Remark 2.1. The results described above are true for every integer j by using an iteration and Lemma 2 in [14].

Corollary 2.2. *For $j \geq j_0$,*

i) there exist $(N_2 - N_1 - 1)$ functions φ_i^α $(1 \leq i \leq N_2 - N_1 - 1)$ and $(N_2 - N_1 - 1)$ functions φ_i^β $(1 \leq i \leq N_2 - N_1 - 1)$ such that the functions $\varphi_{i,j}^\alpha = 2^{j/2}\varphi_i^\alpha(2^j x)|_{[0,1]}$, $(1 \leq i \leq N_2 - N_1 - 1)$, $\varphi_{j,k} = 2^{j/2}\varphi(2^j x - k)$, $(-N_1 \leq k \leq 2^j - N_2)$, and $\varphi_{i,j}^\beta = 2^{j/2}\varphi_i^\beta(2^j x - 2^j)|_{[0,1]}$, $(1 \leq i \leq N_2 - N_1 - 1)$, form an orthonormal basis of $V_j([0,1])$.

ii) Let $V_j, j \geq j_0$ (for large value j), be a multiresolution analysis of $L^2([0,1])$ associated with $V_j(\mathbb{R})$ and satisfying separation condition, then there exist N_α functions φ_i^α $(1 \leq i \leq N_\alpha)$ and N_β functions φ_i^β $(1 \leq i \leq N_\beta)$ such that the functions $\varphi_{i,j}^\alpha = 2_i^{j/2}\varphi_i^\beta(2^j x)|_{[0,1]}, (1 \leq i \leq N_\alpha), \varphi_{j,k} = 2^{j/2}\varphi(2^j x - k)$, $(-N_1 \leq k \leq 2^j - N_2)$, and $\varphi_{i,j}^\beta = 2^{j/2}\varphi_i^\beta(2^j x - 2^j)|_{[0,1]}, (1 \leq i \leq N_\beta)$, form an orthonormal basis of V_j.

Proof. It is clear now how to get an orthogonal basis of $V_j([0,1])$. It is enough to apply Gram-Schmidt to functions $\varphi_{j,k/[0,1]}$, $-N_2 + 1 \leq k \leq -N_1 - 1$ (near the boundary 0) and then to functions $\varphi_{j,k/[0,1]}$, $2^j - N_2 + 1 \leq k \leq 2^j - N_1 - 1$ (near the boundary 1). In every case, we have $(N_2 - N_1 - 1)$ functions. We obtain new functions $\varphi_{i,j}^\alpha = 2^{j/2}\varphi_i^\alpha(2^j x)|_{[0,1]}$, $(1 \leq i \leq N_2 - N_1 - 1)$ near the boundary 0 and in the same way new functions $\varphi_{i,j}^\beta = 2^{j/2}\varphi_i^\beta(2^j x - 2^j)|_{[0,1]}$, $(1 \leq i \leq N_2 - N_1 - 1)$ near the boundary 1. To prove ii), we apply the method described above to every multiresolution analysis on the interval defined as Definition 2.1. □

Remark 2.2. It is easy to see that the space V_j contains the orthonormal system $\varphi_{j,k} = 2^{j/2}\varphi(2^j x-k)$,$(-N_1 \le k \le 2^j - N_2)$, and we add boundaries functions near 0 and 1 from the collections $\varphi^{\alpha}_{i,j}$ and $\varphi^{\beta}_{i,j}$.

We define

$$V_j^T([0,1]) = \{f \in V_j([0,1])/f|_T = 0\}, \tag{2.6}$$

where $T \subset \{0,1\}$ and $j \ge j_0$. We obviously have

$$V_j^T([0,1]) \subset V_{j+1}^T([0,1]). \tag{2.7}$$

The corresponding spaces $V_j^T([0,1]$ are generated by the functions $(\varphi_{j,k})|_{[0,1]}$, $k \in D_j^T$ where the set D_j^T is defined by

* $D_j^T = \{k| -N_1 \le k \le 2^j - N_2\}$ if $T = \{0,1\}$.
* $D_j^T = \{k| -N_1 \le k \le 2^j - N_1 - 1\}$ if $T = \{0\}$.
* $D_j^T = \{k| -N_2 + 1 \le k \le 2^j - N_2\}$ if $T = \{1\}$.
* $D_j^T = \{k| -N_2 + 1 \le k \le 2^j - N_1 - 1\}$ if $T = \emptyset$.

Using Corollary 2.2, we obtain an orthonormal basis of $V_j^T([0,1])$.

Theorem 2.2. *The space $V_j^T([0,1])$ has orthonormal basis $(\varphi^T_{j,k})$, $k \in D_j^T$ where*

i) $\varphi^T_{j,k} = \varphi_{j,k} = 2^{j/2}\varphi(2^j x-k)$,$(-N_1 \le k \le 2^j - N_2)$ *if* $T = \{0,1\}$.

ii) $\varphi^T_{j,k} = \varphi_{j,k} = 2^{j/2}\varphi(2^j x-k)$,$(-N_1 \le k \le 2^j - N_2)$, $\varphi^T_{j,k} = \varphi^{\beta}_{j,k-2^j+N_2}$, $(2^j - N_2 + 1 \le k \le 2^j - N_1 - 1)$ *if* $T = \{0\}$.

iii) $\varphi^T_{j,k} = \varphi^{\alpha}_{j,k+N_2}$, $(-N_2 + 1 \le k \le -N_1 - 1)$, $\varphi^T_{j,k} = \varphi_{j,k} = 2^{j/2}\varphi(2^j x-k)$, $(-N_1 \le k \le 2^j - N_2)$ *if* $T = \{1\}$.

iv) $\varphi^T_{j,k} = \varphi^{\alpha}_{j,k+N_2}$,$(-N_2 + 1 \le k \le -N_1 - 1)$,$\varphi^T_{j,k} = \varphi_{j,k} = 2^{j/2}\varphi(2^j x-k)$, $(-N_1 \le k \le 2^j - N_2)$, $\varphi^T_{j,k} = \varphi^{\beta}_{j,k-2^j+N_2}(2^j - N_2 + 1 \le k \le 2^j - N_1 - 1)$ *if* $T = \emptyset$.

We conclude that the orthogonal projector P_j^T from $L^2([0,1])$ into $V_j^T([0,1])$ is given by

$$P_j^T f = \sum_{k \in D_j^T} \langle f|\varphi^T_{(j,k)}\rangle \varphi^T_{(j,k)}, \tag{2.8}$$

and satisfies $P_j^T o P_{j+1}^T = P_{j+1}^T o P_j^T = P_j^T$.

In the following, we establish the second goal of this paper. In fact, we should construct a wavelet basis of the space $W_j([0,1]) = V_{j+1}([0,1]) \cap (V_j([0,1]))^{\perp}$. We denote by

$$V_j([N_1,+\infty[) = Vect\{\varphi_{j,k/[N_1,+\infty[}, \varphi_{j,k} \in V_j(\mathbb{R})\}. \tag{2.9}$$

Recall that the QMF condition gives that the mask of an orthonormal scaling function must have an even number of coefficients. This means that $N_2 - N_1$ is odd. We have the first important result.

Lemma 2.2. *The functions* $\psi(x-k)_{/[N_1,+\infty[}$, $N_1-N_2+1\le k\le -\frac{1}{2}(N_2-N_1+1)$, *belong to* $V_0([N_1,+\infty[)$.

Proof. The relations (2.1) and (2.3) gives

$$\varphi(2x)=\overline{a_{N_1}}\varphi(x+\frac{1}{2}N_1)+\overline{a_{N_1+2}}\varphi(x+\frac{1}{2}N_1+1)+..+\overline{a_{N_2-1}}\varphi(x+\frac{1}{2}N_2-\frac{1}{2}) \quad (2.10)$$

$+\overline{b_{N_1}}\psi(x+\frac{1}{2}N_1)+\overline{b_{N_1+2}}\psi(x+\frac{1}{2}N_1+1)+...+\overline{b_{N_2-1}}\psi(x+\frac{1}{2}N_2-\frac{1}{2})$,
and

$$\varphi(2x-1)=\overline{a_{N_1+1}}\varphi(x+\frac{1}{2}N_1)+\overline{a_{N_1+3}}\varphi(x+\frac{1}{2}N_1+1)+...+\overline{a_{N_2-1}}\varphi(x+\frac{1}{2}N_2-\frac{1}{2}) \quad (2.11)$$

$+\overline{b_{N_1+1}}\psi(x+\frac{1}{2}N_1)+\overline{b_{N_1+3}}\psi(x+\frac{1}{2}N_1+1)+...+\overline{b_{N_2-1}}\psi(x+\frac{1}{2}N_2-\frac{1}{2})$.

We replace now x by $x+N_2-\frac{3}{2}N_1-1$ in (2.10), then by $x+N_2-\frac{3}{2}N_1-2,\ldots$ and finally, by $x+\frac{1}{2}(N_2-2N_1+2)$. Recall that support of φ and ψ is $[N_1,N_2]$, then, we obtain $\overline{a_{N_1}}\varphi(x+N_2-N_1-1)+\overline{b_{N_1}}\psi(x+N_2-N_1-1)=0$. Next $\overline{a_{N_1}}\varphi(x+N_2-N_1-2)+\overline{a_{N_1+1}}\varphi(x+N_2-N_1-1)+\overline{b_{N_1}}\psi(x+N_2-N_1-2)+\overline{b_{N_1+1}}\psi(x+N_2-N_1-1)=0$ until the last equation. We conclude that the functions $\psi(x-k)_{[N_1,+\infty[}, N_1-N_2+1\le k\le -\frac{1}{2}(N_2-N_1+1)$, belong to $V_0([N_1,+\infty[)$. □

Lemma 2.3. *The functions* $\psi(2^jx-k)_{[0,1]}$, $-N_2+1\le k\le -\frac{1}{2}(N_2+N_1+1)$, *belong to* $V_j([0,1])$.

Proof. By replacing x by $2^j(x-N_1)$ and using Lemma 2.2, we obtain the result. □

We reach the main result of this section

Theorem 2.3. *For* $j\ge j_0$, *the functions* $\varphi_{j,k/[0,1]}$, $-N_2+1\le k\le 2^j-N_1-1$ *and* $2^{j/2}\psi(2^jx-k)_{/[0,1]}$, $-\frac{1}{2}(N_2+N_1-1)\le k\le 2^j-\frac{1}{2}(N_2+N_1+1)$ *form a Riesz basis of the space* $V_{j+1}([0,1])$.

Proof. Lemmas 2.2 and 2.3 immediately imply the main result of this section. □

Remark 2.3. If we apply the results described above to the orthogonal multiresolution of I. Daubechies, we obtain the Meyer's lemma in [14].

We can obtain an orthogonal basis of $W_j([0,1])$. In fact, first we do corrections to the functions $\psi_{j,k/[0,1]}$, $-\frac{1}{2}(N_2+N_1-1)\le k\le -N_1-1$ to get orthogonalilty to $\varphi^{\alpha}_{i,j}$, $(1\le i\le N_2-N_1-1)$. Then, by using Gram-Schmidt for new functions, we get wavelets near 0. We do the same thing for the functions $\psi_{j,k/[0,1]}$ $2^j-N_2+1\le k\le 2^j-\frac{1}{2}(N_2+N_1+1)$ to get wavelets near 1. Moreover, the result of A. Jouini and P. G. Lemarié given in [9] allows to construct the basis for every space W_j (orthogonal complement of V_j in V_{j+1}).

We will now construct an orthonormal basis of the space $W_j^T([0,1])$. We remark first that $\dim W_j^T([0,1])=2^j$. We denote by $\Delta_j^T=\{d\in D_{j+1}^T/d\notin D_j^T\}$. The space $W_j^T([0,1])$ contains the functions $\psi_{j,k},-N_1\le k\le 2^j-N_2$. We have $(2^j-(N_2-N_1-1))$ functions in $W_j^T([0,1])$. Then, we must construct (N_2-N_1-1) functions. We denote by $A_j^T(I)=V_j^T(([0,1])\oplus Vect\{\psi_{j,k}\ -N_1\le k\le 2^j-N_2\}$. We see that, for $-N_1\le k\le N_2-2N_1-2$,

$\varphi_{j+1,k} \in A_j^T(I)$. We have the same treatment for $\varphi_{j+1,2^{j+1}-N_2}$. We conclude that $\varphi_{j+1,-N_1+2k}$ and $\varphi_{j+1,2^{j+1}-N_2-2k}$, $0 \leq k \leq \frac{N_2-N_1-1}{2} - 1$ form a generating system of a supplement of $A_j^T(I)$ in $V_{j+1}^T(I)$. Using Gram-Schmidt, we obtain an orthonormal basis $\psi_{j,k}^T$, $k \in \Delta_j^T$.

We conclude that the orthogonal projector Q_j^T from $L^2(I)$ into $W_j^T([0,1])$ is given by

$$Q_j^T f = \sum_{k \in \Delta_j^T} \langle f | \psi_{(j,k)}^T \rangle \psi_{(j,k)}^T, \tag{2.12}$$

and satisfies $Q_j^T o Q_{j+1}^T = Q_{j+1}^T \, o \, Q_j^T = Q_j^T$.

3 Biorthogonal multiresolution analysis on the interval $[0,1]$

First we give some definitions of biorthogonal multiresolution analysis on the interval $[0,1]$, and then we describe constructions on this interval.

Definition 3.1. A sequence (V_j, V_j^*) of closed subspaces of $L^2([0,1])$ associated with a biorthogonal multiresolution analysis $(V_j(\mathbb{R}), V_j^*(\mathbb{R}))$ of $L^2(\mathbb{R})$ is called a biorthogonal multiresolution analysis of $L^2([0,1])$ if

i) $v_j([0,1]) \subset V_j \subset V_j([0,1])$ and $v_j^*([0,1]) \subset V_j^* \subset V_j^*([0,1])$.

ii) $V_j \subset V_{j+1}$ and $V_j^* \subset V_{j+1}^*$.

iii) $L^2([0,1]) = V_j \oplus (V_j^*)^\perp$.

Let $(V_j(\mathbb{R}), V_j^*(\mathbb{R}))$ be a biorthogonal multiresolution analysis of $L^2(\mathbb{R})$ with multiscale functions (g, g^*). We assume that suppg=$[N_1, N_2]$, and we denote by

$$P_i^\alpha(x) = \sum_{k \leq -N_1-1} k^i g(x-k), \tag{3.1}$$

and

$$P_i^\beta(x) = \sum_{k \geq -N_2+1} k^i g(x-k). \tag{3.2}$$

Our construction is based on the following result:

Theorem 3.1. *We consider a biorthogonal multiresolution analysis* $(V_j(\mathbb{R}), V_j^*(\mathbb{R}))$ *of* $L^2(\mathbb{R})$, (g, g^*) *are the multiscale functions with compact support and* (V_j, V_j^*) *the associated biorthogonal multiresolution analysis of* $L^2([0,1])$. *We assume that*

i) g *is differentiable and* $g'(x) = \widetilde{g}(x) - \widetilde{g}(x-1)$.

ii) V_j *contains the functions* $P_{0,j}^\alpha(x) = P_0^\alpha(2^j x)_{[0,1]}$ *and* $P_{0,j}^\beta(x) = P_0^\beta(2^j x - 2^j)_{[0,1]}$.

If we denote by

$$\widetilde{V}_j = \{ f \in L^2([0,1]) / \exists g \in V_j, f = g' \}, \tag{3.3}$$

and

$$\widetilde{V}_j^* = \{f \in L^2([0,1]) \; / \; f' \in V_j^*, \quad \text{and } f(0) = f(1) = 0\}. \tag{3.4}$$

Then $(\widetilde{V}_j, \widetilde{V}_j^*)$ *is a biorthogonal multiresolution analysis of* $L^2([0,1])$*. Moreover, if we denote by* P_j *(resp* $\widetilde{P}_j$*) the projector from* $L^2([0,1])$ *into* V_j *(resp.* $\widetilde{V}_j$*) parallel to* $(V_j^*)^\perp$ *(resp* $(\widetilde{V}_j^*)^\perp$*), then we have the following commutation property*

$$\frac{d}{dx} o P_j = \widetilde{P}_j o \frac{d}{dx}. \tag{3.5}$$

Proof. We clearly have $\widetilde{g}(x-k) = (\sum_{p=0}^{\infty} g(x-k-p))'$ and

$$(\widetilde{g}^*(x-k))' = g^*(x-k+1) - g^*(x-k).$$

Then $\widetilde{v}_j \subset \widetilde{V}_j([0,1])$ and $\widetilde{v}_j^* \subset \widetilde{V}_j^*([0,1])$. Moreover, since V_j contains the functions $P^{\beta}_{0,j}(x)$, we have $\widetilde{V}_j([0,1]) \subset \widetilde{V}_j$ and $\widetilde{V}_j^*([0,1]) \subset \widetilde{V}_j^*$. In the same way, we have

$$\widetilde{V}_j \subset \widetilde{V}_{j+1}$$

and

$$\widetilde{V}_j^* \subset \widetilde{V}_{j+1}^*.$$

To see the duality between $\widetilde{V}_j$ and $\widetilde{V}_j^*$, we consider a basis $(\alpha_0 = 1, \alpha_1, ..., \alpha_n)$ of V_j with $\dim V_j = n+1$ and a dual basis $(\beta_0, \beta_1, ..., \beta_n)$ of V_j^*. Then the derivation is an isomorphism from $\widetilde{V}_j^*$ onto Vect$(\beta_1, ..., \beta_n)$ and from Vect$(\alpha_1, ..., \alpha_n)$ onto $\widetilde{V}_j$. If we define

$$\widetilde{\alpha}_i = \frac{d}{dx}\alpha_i \quad \text{and} \quad \widetilde{\beta}_i = -\int_0^x \beta_i(t)dt,$$

then, by integration, we conclude that the bases $(\widetilde{\alpha}_i)$ and $(\widetilde{\beta}_i)$ are biorthogonal and we have a duality between $\widetilde{V}_j$ and $\widetilde{V}_j^*$. Finally, the commutation property is satisfied. In fact, we have

$$\begin{aligned} \frac{d}{dx} o (P_j f) &= \frac{d}{dx}\langle f, \beta_0\rangle 1 + \sum_{i=1}^{n} \langle f, \beta_i\rangle \frac{d}{dx}\alpha_i \,, \\ &= \sum_{i=1}^{n} \langle f, \beta_i\rangle \widetilde{\alpha}_i \,, \end{aligned}$$

and

$$\begin{aligned} \widetilde{P}_j o (\frac{d}{dx} f) &= \sum_{i=1}^{n} \langle \frac{d}{dx} f, \widetilde{\beta}_i\rangle \widetilde{\alpha}_i \,, \\ &= \sum_{i=1}^{n} (\Big[f\widetilde{\beta}_i\Big]_0^1 + \langle f, \beta_i\rangle) \widetilde{\alpha}_i \,, \\ &= \sum_{i=1}^{n} \langle f, \beta_i\rangle \widetilde{\alpha}_i \,. \end{aligned}$$

□

Corollary 3.1. *Let $V_j(\mathbb{R})$ be the orthogonal multiresolution analysis of $L^2(\mathbb{R})$ with the scaling function φ of class C^m ($m \in \mathbb{N}^*$). We denote by $V_j^{(m)}(\mathbb{R})$ and $V_j^{*(m)}(\mathbb{R})$ the multiresolution analysis constructed by m derivations and m integrations. Then $V_j^{(m)}([0,1])$ and $V_j^{*(m)}([0,1]) \cap H_0^m([0,1])$ form a biorthogonal multiresolution analysis of $L^2([0,1])$. Moreover, if we denote by $P_j^{(m)}$ the projector on $V_j^{(m)}([0,1])$ parallel to $[V_j^{*(m)}([0,1]) \cap H_0^m([0,1])]^{\perp}$, we have*

$$\frac{d}{dx} o P_j^{(m)} = P_j^{(m+1)} o \frac{d}{dx}. \tag{3.6}$$

We can apply the method described in section 2 to construct Riesz bases of the spaces $V_j^{(m)}([0,1])$ and $V_j^{*(m)}([0,1]) \cap H_0^m([0,1])$. In fact, we define g and g^* by

$$(1-e^{-i\xi})^m \hat{g}(\xi) = (i\xi)^m \hat{\varphi}(\xi), \tag{3.7}$$

and

$$(i\xi)^m \hat{g^*}(\xi) = (e^{i\xi}-1)^m \hat{\varphi}(\xi). \tag{3.8}$$

The functions $g_{j,k[0,1]}$ form a basis of $V_j^{(m)}([0,1])$. To construct a basis of $V_j^{*(m)}([0,1]) \cap H_0^m([0,1])$ we take the functions $g_{j,k}$ with support in $[0,1]$ and the boundaries functions defined by

$$g_{j,k}^{\alpha *} = \sum_{p=-N_2+1}^{-N_1+m-1} \alpha_{i,j,p} g^*_{j,p/[0,1]}, 1 \le i \le N_2 - N_1 - 1, \tag{3.9}$$

and

$$g_{j,k}^{\beta *} = \sum_{p=2^j-N_2+1}^{2^j-N_1+m-1} \alpha_{i,j,p} g^*_{j,p/[0,1]}, 1 \le i \le N_2 - N_1 - 1. \tag{3.10}$$

The real constants $\alpha_{i,j,p}$ are determined by the following conditions: for $1 \le i \le N_2 - N_1 - 2)$,
$\int_0^{+\infty} (\sum_{p=-N_2+1}^{-N_1+m-1} \alpha_{i,j,p} 2^{\frac{j}{2}} g^*(2^j x - p) 2^{\frac{j}{2}} g(2^j x + N_2 - N_1 - m - q) dx = \delta_{i,q}$.

We define

$$V_j^T([0,1]) = \{f \in V_j([0,1]) / f|_T = 0\}$$

where $T \subset \{0,1\}$ and $j \ge j_0$. We obviously have

$$V_j^T([0,1]) \subset V_{j+1}^T([0,1]).$$

We shall construct a subspace $V_j^{*T}([0,1])$ of $V_j^*([0,1])$ such that $V_j^{*T}([0,1]) \subset V_{j+1}^{*T}([0,1])$ and $V_j^T([0,1])$ and $V_j^{*T}([0,1])$ are in duality for the scalar product on $[0,1]$. A direct method as in the previous section gives the basis of $V_j^T([0,1])$. A basis of $V_j^{*T}([0,1])$ is given by the functions $\varphi^*_{j,k}$ with compact support in $[0,1]$ and we add boundaries functions in a way similar in [9]. Using the Gram-Schmidt orthogonalization, we obtain biorthogonal bases of $V_j^T([0,1])$ and $V_j^{*T}([0,1])$. More precisely, Theorem 3.1 and Corollary 3.1 give a biorthogonal multiresolution analysis $(V_j^{(m),T}([0,1]), V_j^{*(m),T}([0,1]))$ of $L^2([0,1])$ and furthermore a straightforward computation yields $\frac{d}{dx} o P_j^{(m),T} = P^{(m+1),T} o \frac{d}{dx}$.

A method similar to that used in the previous section shows that dual bases of $W_j^T([0,1])$ and $W_j^{*T}([0,1])$ are given by : $\psi^Z_{(j,k)}$ and $\psi^{*Z}_{(j,k)}$ for $k \in \Delta_j^T$.

4 The study of regular spaces of functions on the interval $[0,1]$

In this section, we give some applications of the multiresolution analysis on the interval $[0,1]$ described above. In fact, we study regular spaces of functions (Sobolev spaces) on the interval $[0,1]$.

We denote by

• $V_j(\mathbb{R})$: an orthogonal multiresolution analysis of $L^2(\mathbb{R})$ with the associated scaling function φ of class $C^{m+\varepsilon}$ on $\mathbb{R}$ ($m \in \mathbb{N}^*$).

• $V_j^{(m)}(\mathbb{R})$: the multiresolution analysis constructed by derivation and g the function in $V_0^{(m)}(\mathbb{R})$ defined by

$$(1-e^{-i\xi})^m \hat{g}(\xi) = (i\xi)^d \hat{\varphi}(\xi).$$

• $V_j^{*(m)}(\mathbb{R})$: the multiresolution analysis constructed by integration and g^* the function in $V_0^{*(m)}(\mathbb{R})$ defined by

$$(i\xi)^m \hat{g^*}(\xi) = (e^{i\xi}-1)^d \hat{\varphi}(\xi).$$

• $V_j^{(m)} = V_j^{(m)}([0,1])$ and $V_j^{*(m)} = V_j^{*(m)}([0,1]) \cap H_0^m([0,1])$.

• $(V_j^{(m)}, V_j^{*(m)})$ forms a biorthogonal multiresolution analysis of $L^2([0,1])$.

• $W_j^{(m)} = V_{j+1}^{(m)} \cap (V_j^{*(m)})^{\perp}$ and $W_j^{*(m)} = V_{j+1}^{*(m)} \cap (V_j^{(m)})^{\perp}$,

Proposition 4.1. *Let $P_j^{(m)}$ be the projector on $V_j^{(m)}$ parallel to $(V_j^{*(m)})^{\perp}$ and $P_j^{*(m)}$ its adjoint. We define $Q_j^{(m)} = P_j^{(m+1)} - P_j^{(m)}$, $Q_j^{*(m)} = P_{j+1}^{*(m)} - P_j^{*(m)}$ and let j_0 be an integer satisfying $2^{j_0} - 1 \geq 2N_2 - 2N_1 - 2 + 2m$. Then we have the following commutation properties*

$$\frac{d}{dx}(P_j^{(m)} f) = P_j^{(m+1)}(\frac{df}{dx}) \quad \text{if} \quad f \in H^1([0,1]), \tag{4.1}$$

and

$$\frac{d}{dx}(P^{*(m+1)} f) = P_j^{*(m)}(\frac{df}{dx}) \quad \text{if} \quad f \in H_0^1([0,1]). \tag{4.2}$$

Proof. To prove this Proposition, it is enough to remark that if $f \in H^1([0,1])$ and $g \in H_0^1([0,1])$, then we have

$$\langle P_j f, g \rangle_{L^2([0,1])} = \langle f, P_j^* g \rangle$$

and

$$\langle \frac{df}{dx}, g \rangle = -\langle f, \frac{dg}{dx} \rangle.$$

We can now establish the main result of this section. □

Theorem 4.1. *Assume that φ is a $C^{p+\varepsilon}$ -function, $p \in N^*$, $p \geq m$, $\varepsilon > 0$ and let j_0 be an integer satisfying $2^{j_0} - 1 \geq 2N_2 - 2N_1 - 2 + 2p$. Then we have*

i) for $f \in L^2([0,1])$, $\|f\|_2 \approx \|P_{j_0}^{(m)} f\|_2 + (\sum_{j \geq j_0} \|Q_j^{(m)} f\|_2^2)^{\frac{1}{2}}$.

ii) For $f \in L^2([0,1])$, $\|f\|_2 \approx \|P_{j_0}^{*(m)} f\|_2 + (\sum_{j \geq j_0} \|Q_j^{*(m)} f\|_2^2)^{\frac{1}{2}}$.

iii) For $s \in \mathbb{Z}$ such that $-m \leq s \leq p-m$, we have

- $f \in H^s([0,1]) \Leftrightarrow P_{j_0}^{(m)} f \in L^2([0,1])$ and $\sum_{j \geq j_0} 4^{js} \|Q_j^{(m)} f\|_2^2 < +\infty$.
- $f \in H_0^{-s}([0,1]) \Leftrightarrow P_{j_0}^{*(m)} f \in L^2([0,1])$ and $\sum_{j \geq j_0} 4^{js} \|Q_j^{*(m)} f\|_2^2 < +\infty$.

Proof. The proof of this Theorem is classical in the wavelet theory. We obtain the direct inequalities in i) and ii) from the vaguelette Lemma [10] and the inverse inequalities by duality. The equivalences in iii) are immediate because if $f \in H^s([0,1])$ then its norm is equal to $||f||_2 + ||f^{(s)}||_2$. We set

$$f = P_{j_0}^{(m)} f + \sum_{j=j_0}^{\infty} Q_j^{(m)} f,$$

then, we have

$$||f||_2 \approx ||P_{j_0}^{(m)} f||_2 + (\sum_{j=j_0}^{\infty} ||Q_j^{(m)} f||_2^2)^{frac12},$$

$$f^{(s)} = (\frac{d}{dx})^s (P_{j_0}^{(m)} f + \sum_{j=j_0}^{\infty} Q_j^{(m)} f) = P_{j_0}^{(m+s)} f^{(s)} + \sum_{j=j_0}^{\infty} Q_j^{(m+s)} f^{(s)},$$

and

$$||f^{(s)}||_2 \approx ||p_{j_0}^{(m+s)}{}_{j_0} f^{(k)}||_2 + (\sum_{j=j_0}^{\infty} ||Q_j^{(m+s)} f^{(s)}||_2^2)^{\frac{1}{2}}.$$

thus, we obtain

$$||P_{j_0}^{(m+s)} f^{(s)}||_2 = ||(\frac{d}{dx})^s (P_{j_0}^{(m)} f)||_2 \leq C ||P_{j_0}^{(m)} f||_2,$$

and

$$\|Q_j^{(m+s)} f^{(s)}\|_2 = \|(\frac{d}{dx})^s (Q_j^{(m)} f)||_2 \approx 2^{js} ||Q_j^{(m)} f||_2.$$

Then the characterization of $H^s([0,1])$ is immediate. We characterize in the same way the spaces $H_0^s([0,1])$.

If we apply the same method described above for the biorthogonal multiresolution analysis $(V_j^{(m),T}([0,1]), V_j^{*(m),T}([0,1]))$, then corollary 3.1 and classical wavelet theory give the same result for the space $H^{s,T}([0,1]) = \{f \in H^s([0,1]) \ / \ f^{(p)}|_T = 0, 0 \leq p \leq s-1\}$.

We have the equivalent results for Besov spaces. □

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5 Conclusion

In this paper, we have described more general constructions of compact wavelet bases on the interval. More precisely, we have constructed orthogonal and biorthogonal systems on $[0,1]$ which are provided by dyadic translations and dilatations of a finite number of basic functions. By derivation and integration, we obtain new regular multiresolution analyses on the interval $[0,1]$ which satisfy the commutation properties (4.1) and (4.2). We then deduced that these analyses are well adapted to study Sobolev spaces $H^s([0,1])$ and $H^{s,T}([0,1])$ $(s \in \mathbb{Z})$.

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