FUNCTIONAL LIMIT THEOREMS FOR VON MISES STATISTICS OF A MEASURE PRESERVING TRANSFORMATION

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Limit theorems
for dependent data
and applications

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Outline

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INTRODUCTION: V-STATISTICS OF A MEASURE PRESERVING TRANSFORMATION

(after a joint work with
Herold Dehling and Manfred Denker)

V-statistics

Let T be a measure preserving transformation of a probability space $(\Omega, \mathcal{F}, P)$. Choose a point $\omega \in \Omega$ and consider its $n-$**orbit**

$$\omega, T\omega, \ldots, T^{n-1}\omega.$$

From statistician's point of view this is a **sample** of size n. Let us consider, for a certain measurable symmetric function $h : \Omega^d \to \mathbb{R}$, the expression

$$\sum_{1 \le i_1 < n, \ldots, 1 \le i_d \le n} h(T^{i_1}\omega, \ldots, T^{i_d}\omega). \tag{1}$$

Such a functional will be called a $V-$**statistic** (or **von Mises statistic**) **of degree** d with the **kernel** h.

BACK TO CLASSICAL DEFINITION

Let $X = (X_n)_{n\in\mathbb{Z}}$ be a strictly stationary real-valued sequence. Every such X admits a representation of the form

$$X_n = f \circ T^n, n \in Z,$$

where T is a measure preserving invertible transformation of a certain probability space and f is a measurable function. Let $H : \mathbb{R}^d \mapsto \mathbb{R}$ be a Borel measurable function. If we set

$$h(\omega_1, \ldots, \omega_d) = H(f(\omega_1), \ldots, f(\omega_d)),$$

we arrive from (1) at the standard expressions for a V-statistic:

$$\sum_{1\le i_1<n,\ldots,1\le i_d\le n} H(X_{i_1}, \ldots, X_{i_d}). \qquad (2)$$

GENERATION BY DYNAMICS

Dynamics can be used as follows to generate the function

$$\omega \mapsto h(T^{i_1}\omega, \dots, T^{i_d}\omega).$$

First, we consider an action of d commuting copies $T_1, \dots, T_d$ of T on some set $Y \subset \Omega^d$ to produce terms of the form

$$h(T_1^{i_1}\omega_1, \dots, T_d^{i_d}\omega_d).$$

Second, we restrict the constructed function to the **principal diagonal** $D = \{(\omega, \dots, \omega) : \omega \in \Omega\} \subset \Omega^d$ and obtain the desired term. The requirements which Y must satisfy are:
i) $T_k Y \subset Y, k = 1, \dots, d$;   ii) $D \subset Y$.
We choose as Y the **entire space** Ω^d with the **product measure** P^d and the **componentwise action** of copies of T.

RESTRICTION PROBLEM

Let $h : \Omega^d \to \mathbb{R}$ be (an equivalence class) of a certain measurable function on Ω^d. Consider the set

$$\bigcup_{(n_1,\dots,n_d)\in\mathbb{Z}^d} \{(T_1^{n_1}\omega, \dots, T_d^{n_d}\omega), \omega \in \Omega\}$$

of measure zero. For $d = 2$ this is the graph of the orbital equivalence relation of T.
What is the correct restriction of h to subsets of this set ?
In general, no idea.
However, the restriction problem is easily solvable for kernels h which are products of functions in one variable, or can be nicely approximated by sums of such functions.
We will use such an approximation.

SOME REFERENCES ON *V*- AND *U*-STATISTICS

Seminal papers
Hoeffding (1948): U-statistics for i.i.d. variables; Hoeffding's decomposition
von Mises (1949): *V*-statistics for i.i.d. variables

Books (i.i.d. variables):
Borovskikh and *Korolyuk* (1989)
Giné and *de la Peña* (1999)

SOME REFERENCES, CONTINUED

Dependent stationary case:
Kanagawa and *Yoshihara* (1994): a. s. invariance principle for completely degenerate (canonical) *U*-statistics of degree two
Aaronson, Burton, Dehling, Gilat, Hill and *Weiss* (1996): strong law of large numbers
Two papers by *Borovkova, Burton* and *Dehling* (2001): a version of the FCLT (along with other results)
Borisov, Volod'ko (2008): the CLT for power series' in a weakly dependent sequence
Mixing conditions, in particular, **absolute regularity** are assumed; the **coupling method**, the **method of moments** e.t.c. are employed

TENSOR PRODUCTS

Let for every $1 \leq p \leq \infty$ $\hat{L}_p(P^d)$ denote the **projective** (or **maximal**) tensor product

$$L_p(\Omega_1, \mathcal{F}_1, P_1) \hat{\otimes} \cdots \hat{\otimes} L_p(\Omega_d, \mathcal{F}_d, P_d).$$

Since the projective norm is stronger than the norm of $L_p(P^d)$, $\hat{L}_p(P^d)$ can be embedded into $L_p(P^d)$.
Example. For $p = 2$ and $d = 2$ the space $\hat{L}_2(P^2)$ can be identified with the space of (the kernels of) the **trace class operators** mapping $L_2(P)^*$ to $L_2(P)$.
The space $\hat{L}_p(P^d)$ is preserved by the operators $(U^{\mathbf{n}}, U^{*\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}_+^d}$. We will use the denotation $(U^{\mathbf{n}}, U^{*\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}_+^d}$ for the restrictions of $(U^{\mathbf{n}}, U^{*\mathbf{n}})$ to $\hat{L}_p(P^d)$ as well.

RESTRICTION TO THE DIAGONAL

Proposition

Let $p_1, \ldots, p_d, r \in [1, \infty]$ *satisfy* $\sum_{i=1}^{d} 1/p_i = 1/r$. *Then the map sending every function*

$$(\omega_1, \ldots, \omega_d) \mapsto f_1(\omega_1) \cdots f_l(\omega_d)$$

with $f_1 \in L_{p_1}, \ldots, f_d \in L_{p_d}$ *to the function*

$$\omega \mapsto f_1(\omega) \cdots f_d(\omega)$$

extends in a unique way to a linear operator of norm 1

$$D_d : L_{p_1} \hat{\otimes} \cdots \hat{\otimes} L_{p_d} \to L_r.$$

APPROXIMATING RESTRICTION

Remark

Let $(\mathcal{A}_n)_{n\geq 1}$ be a refining sequence of finite measurable partitions $\mathcal{A}_n = \{A_{1,n}, \ldots, A_{m_n,n}\}$ such that $\mathcal{F}$ is the smallest $\sigma-$field containing all $\mathcal{A}_n$, $n \geq 1$. Then the operator D_d can be represented as a strong limit of the sequence of operators $(D_{d,n})_{n\geq 1}$, where

$$D_{d,n}f = \sum_{i=1}^{m_n} \frac{I_{A_{i,n}}}{P(A_{i,n})^d} \int_{A_{i,n}^d} f(\omega_1, \ldots, \omega_d) P(d\omega_1) \cdots P(d\omega_d).$$

COMMUTING COPIES OF T

Let $T_1, \ldots, T_d$ be copies of the transformation T which act on Ω^d via

$$T_i(\omega_1, \ldots, \omega_i, \ldots, \omega_d) = (\omega_1, \ldots, T_i\omega_i, \ldots, \omega_d), i = 1, \ldots, d.$$

Let $\mathbb{Z}_+^d$ be the additive semigroup of $d-$tuples of nonnegative integers. The transformations $T_1, \ldots, T_d$ pairwise commute and give rise to the measure preserving action $\mathbf{n} = (n_1, \ldots, n_d) \mapsto T^{\mathbf{n}} = T_1^{n_1} \cdots T_d^{n_d}$, of $\mathbb{Z}_+^d$ on $(\Omega, \mathcal{F}, P)^d$. Set $U_k f = f \circ T_k$ for $f \in L_p$. Let U_k^* be the adjoint of U_k and I denote the identity operator. Clearly, $U_1, \ldots, U_d$ pairwise commute, and so are $U_1^*, \ldots, U_d^*$.

V-STATISTICS: DEFINITION

From now on by a V-statistics of a measure preserving transformation T acting on a probability space $(\Omega, \mathcal{F}, P)$ we mean the function of the form

$$\frac{1}{N^d} \sum_{1 \le n_k \le N,\, k=1,\ldots,d} D_d\left(h \circ T^{(n_1,\ldots,n_d)}\right). \tag{3}$$

The function h is called the **kernel** of the corresponding V-statistics.

STRONG LAW OF LARGE NUMBERS

Let $T_1, \ldots, T_d$ be copies (acting on the cartesian product) of a transformation T. Remind that $\hat{L}_{p,\pi}(P^d) = L_p^{\hat{\otimes} d}$.

Theorem

Let $d \geq 2$, $p \geq d$ and $r = p/d$. Let T be an **ergodic** *P-preserving transformation of the space $(\Omega, \mathcal{F}, P)$. Assume also that $f \in \hat{L}_{p,\pi}(P^d)$. Then, as $N \to \infty$, the sequence*

$$\frac{1}{N^d} \sum_{1 \leq n_k \leq N,\, k=1,\ldots,d} D_d\left(f \circ T^{(n_1,\ldots,n_d)}\right) \tag{4}$$

converges with probability 1 and in $L_r(P)$ to the limit

$$\int_{\Omega^d} f(\omega_1, \ldots, \omega_d) P(d\omega_1) \cdots P(d\omega_d).$$

CASE $p = d$

Corollary

If $p = d$, the above Theorem applies and asserts the convergence with probability 1 and in L_1.

HOEFFDING'S AND MARTINGALE-COBOUNDARY DECOMPOSITIONS

HOEFFDING'S DECOMPOSITION: DEFINITION

Let $(\Omega, \mathcal{F}, P)$ be a probability space and

$$\Omega^d = \prod_{i=1}^{d} \Omega_i, \mathcal{F}^d = \prod_{i=1}^{d} \mathcal{F}_i, P^d = \prod_{i=1}^{d} P_i,$$

where $\Omega_1, \ldots, \Omega_d$, $\mathcal{F}_1, \ldots, \mathcal{F}_d$, $P_1, \ldots, P_d$ are copies of Ω, $\mathcal{F}$ and P, respectively. Denoting by π_i the projection from Ω^d onto Ω_i ($i = 1, \ldots, d$), we set for every $S \in \mathcal{S}_d$

$$\mathcal{F}^S = \bigvee_{i \in S} \pi_i^{-1}(\mathcal{F}_i), \;\; E^S = E^{\mathcal{F}_S}, \;\; \hat{E}^i = E^{\{1,\ldots,d\} \setminus \{i\}}.$$

In other terms, $\hat{E}^i$ integrates out the $i-$th variable.
The identity I in $L_p\,(P^d)$ decomposes as

$$I = \prod_{i=1}^{d} (\hat{E}^i + (I - \hat{E}^i)) = \sum_{k=0}^{d} \sum_{S \in \mathcal{S}_d^k} \prod_{i \notin S} \hat{E}^i \prod_{i \in S} (I - \hat{E}^i)$$

CANONICAL KERNELS: DEFINITION

For every $S \in \mathcal{S}_d^k$ the function

$$\prod_{i \notin S} \hat{E}^i \prod_{i \in S} (I - \hat{E}^i) f$$

can be thought of as a function f_S of k variables $\omega_m, m \in S$, with the property

$$\int_\Omega f_S(\cdots, \omega_i, \cdots) P(d\omega_i) = 0$$

for every $i \in S$. Functions of k variables with this property are called **completely degenerate** or **canonical**. Observe, that for f symmetric we obtain a symmetric function of k variables.

NON-INVERTIBILITY AND EXACTNESS ASSUMPTIONS

The second order (compared to the SLLN) asymptotics for V-statistics can be studied by means of a T-invariant filtration and martingale approximation. We consider a (non-invertible) transformation T and its canonical decreasing filtration $(T^{-n}\mathcal{F})_{n\geq 0}$. This is equivalent, up to time reversal, to considering invertible transformations, decreasing filtrations and adapted random sequences. For simplicity we assume that the transformation T is **exact**. This means that $\bigcap_{k=0}^{\infty} T^{-k}\mathcal{F} = \mathcal{N}$, where $\mathcal{N}$ is the trivial sub $\sigma-$field of $\mathcal{F}$.

COMPLETE COMMUTATION OF COPIES OF T

For every $k = 1, \ldots, d$, $n \geq 0$ we have

$$U_k^{*n} U_k^n = I \text{ and } U_k^n U_k^{*n} = E^{T_k^{-n} \mathcal{F}^{\times d}}.$$

Observe that for every $1 \leq i, j \leq d$, $i \neq j$, we have

$$U_i U_j^* = U_j^* U_i.$$

Transformations $T_1, \ldots, T_d$ are **completely commuting** which means that they commute and enjoy the above property. The complete commutativity implies that the conditional expectations $\left(E^{T_k^{-n} \mathcal{F}^{\times d}}\right)_{n \geq 0,\, k=1,\ldots,d}$ commute.

FILTRATION FOR $\mathbb{Z}_+^d$-ACTION ON $(\Omega, \mathcal{F}, P)^d$

For every $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$ we set

$$\mathcal{F}^{\mathbf{n}} = T^{-\mathbf{n}} \mathcal{F}^{\times d}, \; E^{\mathbf{n}} = E^{\mathcal{F}^{\mathbf{n}}}.$$

Let $\overline{\mathbb{Z}_+^d} = \{0, 1, \ldots, \infty\}^d$ be a **completion** of $\mathbb{Z}_+^d$ endowed with the natural partial order $\le$ which extends that of $\mathbb{Z}_+^d$. Let us extend by continuity the families $\left(\mathcal{F}^{\mathbf{n}}\right)_{\mathbb{Z}_+^d}$ and $\left(E^{\mathbf{n}}\right)_{\mathbb{Z}_+^d}$ to $\overline{\mathbb{Z}_+^d}$. Thus, $\left(\mathcal{F}^{\mathbf{n}}\right)_{\mathbf{n} \in \overline{\mathbb{Z}_+^d}}$ is a **decreasing filtration** parameterized by the partially ordered set $\overline{\mathbb{Z}_+^d}$.

Let $(\mathbf{l}, \mathbf{m}) \mapsto \mathbf{l} \bigvee \mathbf{m}$ be the operation of taking the coordinatewise maximum in $\overline{\mathbb{Z}_+^d}$.

We have $E^{\mathbf{l}} E^{\mathbf{m}} = E^{\mathbf{m}} E^{\mathbf{l}} = E^{\mathbf{l} \bigvee \mathbf{m}}$ for all $\mathbf{l}, \mathbf{m} \in \overline{\mathbb{Z}_+^d}$, that is **the $\sigma-$fields $\mathcal{F}^{\mathbf{l}}$ and $\mathcal{F}^{\mathbf{m}}$ are conditionally independent given $\mathcal{F}^{\mathbf{l} \bigvee \mathbf{m}}$**.

MULTIPARAMETER MARTINGALE DIFFERENCES

Definition

Let $(X_{\mathbf{n}}, \mathcal{F}^{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}_+^d}$ be a family of random variables defined on $(\Omega, \mathcal{F}, P)$ and sub-σ-fields of $\mathcal{F}$. $(X_{\mathbf{n}}, \mathcal{F}^{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}_+^d}$ is said to be a family of *reversed martingale differences* if

1. the map $\mathbb{Z}_+^d \ni \mathbf{n} \mapsto \mathcal{F}^{\mathbf{n}}$ is decreasing ($\mathbb{Z}_+^d$ is taken with its natural partial order, the σ-fields are ordered by inclusion);
2. for every $\mathbf{n} \in \mathbb{Z}_+^d$ the random variable $X_{\mathbf{n}}$ is measurable with respect to $\mathcal{F}^{\mathbf{n}}$;
3. $E^{\mathcal{F}^{\mathbf{m}}} X_{\mathbf{n}} = 0$ whenever $\mathbf{m} \not\leq \mathbf{n}$.

Variants of this definition can be found in the literature.

SOLVABILITY OF THE POISSON EQ-ON IMPLIES MARTINGAL -COBOUNDARY DECOMPOSITION

Let $\mathcal{S}_d$ denote the set of all subsets of $\{1, \ldots, d\}$.

Proposition

Let for some $1 \le p \le \infty$ *and* $f, g \in L_p$
$f = \left(\prod_{k=1}^{d}(I - U_k^*)\right)g$.
Then f *can be represented in the form*

$$f = \sum_{S \in \mathcal{S}_d} \left(\prod_{k \in S}(U_k - I) \prod_{l \notin S}(I - U_l U_l^*)\right) h_S, \tag{5}$$

where for every $S \in \mathcal{S}_d$ *the function* $h_S \in L_p$ *is defined by*

$$h_S = \left(\prod_{m \in S} U_m^*\right) g\,. \tag{6}$$

POTENTIAL SERIES PRESENTS SOLUTION OF POISSON EQUATION

Let for some function f the **potential series**

$$\sum_{\mathbf{n}\in\mathbb{Z}_+^d} U^{*\mathbf{n}} f, \tag{7}$$

converges in the L_p-norm (or in $\hat{L}_p$ norm), where summation is performed over coordinate rectangles with growing edges. Then its sum presents a solution of the Poisson equation.

CONVERGENCE AND CANONICITY

For a kernel h of degree d the following properties are equivalent:

$$E^{(n_1,\dots,n_d)}h \underset{\max(n_1,\dots,n_d)\to\infty}{\to} 0,$$

$$U^{*(n_1,\dots,n_d)}h \underset{\max(n_1,\dots,n_d)\to\infty}{\to} 0,$$

and

$$E^{(n_1,\dots,n_d)}h = 0$$

whenever at least one of n_k equals ∞. The latter property is means the canonicity of h.

Canonical kernels of degree d with convergent potential series form a dense subspace (among all canonical kernels of degree d).

EXAMPLE

Let $\Omega = \{z \in \mathbb{C} : |z| = 1\}$, P be the probability Haar measure on Ω, $Tz = z^2, z \in \Omega$, $d = 2$. Clearly,

$$(Uf)(x) = f(x^2), (U^* f)(x) = 1/2 \sum_{\{u : u^2 = x\}} f(u).$$

If $f \in L_2(P)$ and $\int_\Omega f(x)P(dx) = 0$ then the series $\sum_{k \geq 0} U^{*k} f$ converges in L_2 under very mild conditions. The condition $\sum_{k \geq 0} w^{(2)}(f, 2^{-k}) < \infty$ is sufficient. Here $w^{(2)}(f, \delta)$ is the continuity modulus of f in $L_2(P)$.

EXAMPLE (CONTINUED 1)

Let now $f_2 \in L_2(\mu^2)$ be of the form

$$f_2(x_1, x_2) = g(x_1 x_2^{-1})$$

with

$$g(x) = \sum_{k \in \mathbb{Z}} g_k x^k \in L^2(\mu).$$

Assume that $f_2 \in \hat{L}^2_{sym}$ and is canonic. This implies

$$g_0 = 0, g_{-k} = g_k, \text{and} \sum_{k \in \mathbb{Z}} |g_k| < \infty.$$

EXAMPLE (CONTINUED 2)

Let A_2 be the Banach space of double absolutely converging Fourier series

$$a : (x_1, x_2) \mapsto \sum_{(k_1,k_2)\in\mathbb{Z}^2} a_{k_1,k_2} x_1^{k_1} x_2^{k_2}$$

furnished with the norm $|\cdot|_{A_2} : a \mapsto \sum_{(k_1,k_2)\in\mathbb{Z}^2} |a_{k_1,k_2}|$. The projective tensor norm of the space $\hat{L}_{2,\pi} \cong l_2 \hat{\otimes}_\pi l_2$ does not exceed the norm of $A_2 \cong l_1 \hat{\otimes}_\pi l_1$. Hence, the series

$$\sum_{(i_1,i_2)\in\mathbb{Z}_+^2} U^{*(i_1,i_2)} f_2 \tag{8}$$

converges in $\hat{L}_{2,\pi}(\mu^2)$ if it converges in A_2.

EXAMPLE (CONTINUED 3)

Every $U^{*(i,j)}$ is a contraction in A_2. Furthermore, $|U^{*(k,0)}f_2|_{A_2} = |U^{*(0,k)}f_2|_{A_2} = |U^{*k}g|_{A_1}$, where A_1 is the space of one-dimensional absolutely convergent trigonometric series $a : x \mapsto \sum_{k\in\mathbb{Z}} a_k x^k$ with the norm $|a|_{A_1} = \sum_{k\in\mathbb{Z}} |a_k|$. Thus we have

$$\sum_{(k_1,k_2)\in\mathbb{Z}_+^2} |U^{*(k_1,k_2)}f_2|_{A_2}$$

$$\leq \sum_{0\leq k_1\leq k_2<\infty} |U^{*(k_1,k_2)}f_2|_{A_2} + \sum_{0\leq k_2\leq k_1<\infty} |U^{*(k_1,k_2)}f_2|_{A_2}$$

$$= \sum_{k\in\mathbb{Z}_+} (k+1)\big(|U^{*(k,0)}f_2|_{A_2} + |U^{*(0,k)}f_2|_{A_2}\big) \leq 2\sum_{k=0}^{\infty}(k+1)|U^{*k}g|_{A_1}, \tag{9}$$

EXAMPLE (CONTINUED 4)

Therefore, a sufficient condition for series (8) to converge in $\hat{L}_{2,\pi}(\mu^2)$ is

$$\sum_{n\in\mathbb{Z}}\sum_{k\geq 0}(k+1)|g_{2^k n}| < \infty,$$

which holds, for example, whenever for some $C > 0$ and $\delta > 0$

$$|g_m| \leq \frac{C}{|m|(\log|m|)^{1+\delta}},\ m \in \mathbb{Z}\setminus\{0\}.$$

FCLT

MARTINGALE–COBOUNDARY DECOMPOSITION OF A CANONICAL KERNEL

Proposition

Let $h \in \hat{L}_2(P^2)$ be a canonical kernel of degree 2. Assume that the limit

$$\lim_{n_1,n_2\to\infty} \sum_{0\le i_1\le n_1-1, 0\le i_2\le n_2-1} U^{*(i_1,i_2)}h \tag{10}$$

exists in $\hat{L}_2(P^2)$. Then h admits a unique representation in the form
$h = g + (U^{(1,0)} - I)g_1 + (U^{(0,1)} - I)g_2 + (U^{(1,0)} - I)(U^{(0,1)} - I)g_{1,2}$,
where $g \in \hat{L}^2(P^2)$, $g_1, g_2, g_{1,2} \in \hat{L}^2(P^2)$ and
$E(g|T^{-(1,0)}\mathcal{F}^2) = 0$, $E(g|T^{-(0,1)}\mathcal{F}^2) = 0$,
$E(g_1|T^{-(1,0)}\mathcal{F}^2) = 0$, $E(g_2|T^{-(0,1)}\mathcal{F}^2) = 0$.
Moreover, if h is a symmetric function, so is g.

Assume $d = 2$. Holds for every $d \geq 1$.

Theorem

Let $f \in \hat{L}_2(P^2)$ *be a symmetric kernel with Hoeffding's decomposition*

$$f(x_1, x_2) = f_0 + f_1(x_1) + f_1(x_2) + f_2(x_1, x_2),$$

where

$$\int_X f_1(z)p(dz) = 0,$$

$f_2 \in L^p_{sym}(\mu^2)$ *and*

$$\int_X f_2(z_1, x_2)\mu(dz_1) = 0.$$

Assume that the series $\sum_{k=0}^{\infty} U^{*k} f_1 = g_1$ converges in L_2 and the limit

$$\lim_{N_1, N_2 \to \infty} \sum_{(n_1, n_2) = \mathbf{0}}^{(N_1 - 1, N_2 - 1)} U^{(n_1, n_2)} f_2$$

exists in $\hat{L}_2(P^2)$. Then the distributions of random variables

$$t \mapsto N^{-3/2} \sum_{n_1, n_2 = 0}^{[Nt]} (f(T^{n_1} \dot{;} T^{n_2}) - f_0), t \in [0, 1]$$

weakly converge to the distribution of $2\sigma_f^2 w(\dot{})$, where w is the standard Brownian motion and

$$\sigma_f^2 = |g|_2^2 - |U^* g|_2^2.$$

MAXIMAL INEQUALITY

Lemma

There exists an absolute constant C such that

$$\left| \max_{0 \le n_1 \le N_1 - 1, 0 \le n_2 \le N_2} D_2 \sum_{(n_1,n_2)=\mathbf{0}}^{(N_1-1,N_2-1)} U^{(n_1,n_2)} f_2 \right|_1 \le C |f_2|_{2,\pi} \sqrt{N_1 N_2} \tag{11}$$

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  Martingale-coboundary representation for a class of random fields.
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