Inverse problems for regular variation of linear filters, a cancellation property for σ-finite measures, and identification of stable laws

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A random variable Z is said to have a *regularly varying* (*right*) *tail with exponent* $\alpha > 0$ if its law μ satisfies the relation

$$\mu(x,\infty)\big(= P(Z > x)\big) = x^{-\alpha} L(x) \ \text{ for } x > 0,$$

where L is a slowly varying (at infinity) function.

A similar definition applies to infinite measures μ as well, as long as

$$\mu(x,\infty) \ \text{ is finite for } x \text{ large enough.}$$

If the above holds with $\alpha = 0$, we speak of *slow variation* of the tail.

Regular variation it is preserved under various operations common in probability theory.

Example 1 Weighted sums Let $Z_1, Z_2, \ldots$, be iid random variables, and $\psi_1, \psi_2, \ldots$, non-negative weights. If a generic element Z of the sequence (Z_j) is regularly varying with exponent $\alpha > 0$, then under appropriate conditions on the coefficients, the infinite series $X = \sum_{j=1}^{\infty} \psi_j Z_j$ converges with probability 1, and

$$\lim_{x \to \infty} \frac{P(X > x)}{P(Z > x)} = \sum_{j=0}^{\infty} \psi_j^{\alpha} .$$

Mikosch and Samorodnitsky (2000) provide the most general conditions for that.

Example 2 Products

Let Z be a random variable that is regularly varying with exponent $\alpha \geq 0$, independent of another random variable $Y > 0$, and write $X = YZ$.

If the tail of Y is light enough, then the tail of X is also regularly varying with exponent α. If, for example, $EY^{\alpha+\varepsilon} < \infty$ for some $\varepsilon > 0$, then the tail equivalence

$$\lim_{x \to \infty} \frac{P(X > x)}{P(Z > x)} = EY^{\alpha}$$

holds; see Breiman (1965).

Example 3 Stochastic integrals Let $(M(s))_{s\in\mathbb{R}}$ be a Lévy process with Lévy measure η, and $g:\mathbb{R}\to\mathbb{R}_+$ a measurable function. Under certain integrability assumptions on g, the random variable

$$X=\int_{\mathbb{R}} g(s)\,M(ds)$$

is well defined, see Rajput and Rosinski (1989). If the Lévy measure η has a regularly varying tail with exponent $\alpha\geq 0$ then the integral X itself is regularly varying and

$$\lim_{x\to\infty}\frac{P(X>x)}{\eta(x,\infty)}=\int_{\mathbb{R}}[g(s)]^{\alpha}\,ds\,,$$

see Rosinski and Samorodnitsky (1993).

- Linear filters preserve regular variation.
- We are interested in the inverse problem: suppose that the output from a linear filter is regularly varying with index $\alpha \geq 0$. When may we conclude that the input to the filter is with regularly varying the same index?
- For $\alpha = 0$ the answer is "**always**".
- For $\alpha > 0$ this question is connected to the **cancellation property** of certain σ-finite measures.
- The latter is related to the existence of real zeros of certain Fourier transforms.

Let ν and ρ be two σ-finite measures on $(0, \infty)$. We define a new measure on $(0, \infty)$, the multiplicative convolution of the measures ν and ρ, by

$$(\nu \circledast \rho)(B) = \int_0^\infty \nu\big(x^{-1}B\big)\,\rho(dx), \quad B \text{ a Borel subset of } (0, \infty).$$

We say that a σ-finite measure ρ has the *cancellation property* with respect to a family $\mathcal{N}$ of σ-finite measures on $(0, \infty)$ if for any σ-finite measures $\nu, \overline{\nu}$ on $(0, \infty)$ with $\overline{\nu} \in \mathcal{N}$,

$$\nu \circledast \rho = \overline{\nu} \circledast \rho \quad \Longrightarrow \quad \nu = \overline{\nu}.$$

If $\mathcal{N} = \{\delta_1\}$, then the problem is known as the Choquet-Deny equation in the multiplicative form. The class of measures having the cancellation property with respect to $\{\delta_1\}$ can be determined by the well studied Choquet-Deny theory, cf. Rao and Shanbhag (1994).

We are interested in the case when $\mathcal{N}$ consists of a single measure ν_α with power density function, i.e., ν_α is a σ-finite measure on $(0, \infty)$ with density

$$\frac{\nu_\alpha(dx)}{dx} = \begin{cases} |\alpha|\, x^{-(\alpha+1)} & \alpha \neq 0, \\ x^{-1} & \alpha = 0. \end{cases}$$

Theorem Let $\alpha \in \mathbb{R}$ and ρ a non-zero σ-finite measure such that

$$\int_0^\infty y^{\alpha-\delta} \vee y^{\alpha+\delta} \, \rho(dy) < \infty \quad \text{for some } \delta > 0.$$

Then the measure ρ has the cancellation property with respect to $\mathcal{N} = \{\nu_\alpha\}$ if and only if

$$\int_0^\infty y^{\alpha+i\theta} \, \rho(dy) \neq 0 \quad \text{for all } \theta \in \mathbb{R}\,.$$

If

$$\int_0^\infty y^{\alpha+i\theta_0}\,\rho(dy) = 0$$

for some $\theta_0 \in \mathbb{R}$, then for any real a, b with $0 < a^2 + b^2 \leq 1$, the σ-finite measure

$$\nu(dx) := g(x)\,\nu_\alpha(dx)$$

with

$$g(x) := 1 + a\cos(\theta_0 \log x) + b\sin(\theta_0 \log x), \quad x > 0,$$

satisfies the equation $\nu \circledast \rho = \nu_\alpha \circledast \rho$.

Connection with inverse problems

Theorem Let ρ be a non-zero σ-finite measure such that for some σ-finite measure ν on $(0, \infty)$, the measure $\nu \circledast \rho$ has a regularly varying tail with exponent $\alpha > 0$ (and a technical assumption). If

$$\int_0^\infty y^{\alpha+i\theta} \rho(dy) \neq 0 \quad \text{for all } \theta \in \mathbb{R},$$

then the measure ν has regularly varying tail with exponent α, equivalent to the tail of $\nu \circledast \rho$. Conversely, if the Fourier transform vanishes at some point, then there exists a σ-finite measure ν on $(0, \infty)$ without a regularly varying tail, such that the measure $\nu \circledast \rho$ has regularly varying tail with exponent α and the tail equivalence holds.

Theorem Assume that $\alpha > 0$ and (Z_j) is a sequence of iid random variables.
Suppose that (ψ_j) is a sequence of nonnegative coefficients satisfying $\psi_1 = 1$ and

$$\sum_{j=1}^{\infty} \psi_j^{\alpha-\delta} < \infty \quad \text{for some } 0 < \delta < \alpha.$$

If $\sum_{j=1}^{\infty} \psi_j = \infty$, assume additionally that

$$\limsup_{x\to\infty} \frac{P\big(Z < -x\big)}{P\big(Z > x\big)} < \infty\,.$$

Assume that the series $X = \sum_{j=1}^{\infty} \psi_j Z_j$ converges a.s., and that X is regularly varying with exponent α.

(i) If

$$\sum_{j=1}^{\infty} \psi_j^{\alpha+i\theta} \neq 0 \quad \text{for all } \theta \in \mathbb{R},$$

then a generic noise variable Z is regularly varying with exponent α as well, and tail equivalence holds.

(ii) Suppose that the Fourier transform vanishes at some point, then there exists a random variable Z that is not regularly varying, the series $X = \sum_{j=1}^{\infty} \psi_j Z_j$ converges a.s., and X is regularly varying with exponent α.

Consider the case of finite sums. We say that a set of $q \geq 2$ positive coefficients $1 = \psi_1, \ldots, \psi_q$ is *α-regular variation determining* if iid random variables $Z_1, \ldots, Z_q$ are regularly varying with exponent α if and only if

$$X_q = \sum_{j=1}^{q} \psi_j \, Z_j$$

is regularly varying with exponent α.

The corresponding notion in the slowly varying case, i.e., when $\alpha = 0$, is not of interest: any set of positive coefficients $\psi_1, \ldots, \psi_q$ is 0-regular variation determining.

For $\alpha > 0$ positive coefficients $\psi_1, \dots, \psi_q$ are α-regular variation determining if and only if

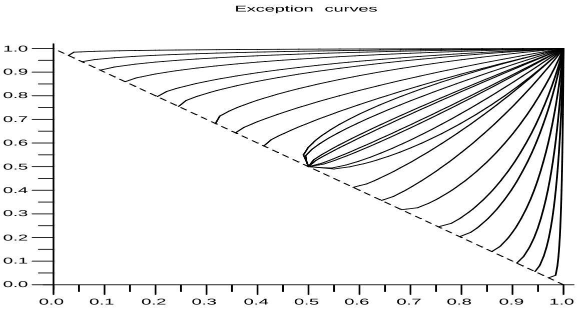
$$\sum_{j=1}^{q} \psi_j^{\alpha + i\theta} \neq 0 \quad \text{for all } \theta \in \mathbb{R}\,.$$

Example Any set of $q = 2$ positive coefficients $1 = \psi_1, \psi_2$ is α-regular variation determining because the relation

$$1 + \psi_2^{\alpha + i\theta} = 0$$

is not possible.

Example There exist $q = 3$ positive coefficients $1 = \psi_1, \psi_2, \psi_3$ that are NOT α-regular variation determining. These coefficients must lie on a a countable set of curves in the (ψ_2, ψ_3) plane. Some of these curves (for $\alpha = 1$):



Products of random variables

Let $\alpha > 0$ and Y a positive random variable satisfying $EY^{\alpha+\delta} < \infty$ for some $\delta > 0$. We will call Y and its distribution α-regular variation determining if the α-regular variation of $X = YZ$ for any random variable Z which is independent of Y, implies that Z itself has a regularly varying tail with exponent α.

The corresponding notion in the slowly varying case ($\alpha = 0$) is trivial.

Theorem A positive random variable Y with $EY^{\alpha+\delta} < \infty$ for some $\delta > 0$ is α-regular variation determining if and only if

$$E[Y^{\alpha+i\theta}] \neq 0 \quad \text{for all } \theta \in \mathbb{R}.$$

Corollary A sufficient condition for Y with $EY^{\alpha+\delta} < \infty$ for some $\delta > 0$ to be α-regular variation determining is

$$\log Y \quad \text{is an infinitely divisible random variable.}$$

Example The following random variables are α-regular variation determining:

- Gamma random variables, and the absolute value of a centered normal random variable to any positive power.
- Pareto random variables with exponent $p > \alpha$, and their reciprocals.
- Lognormal random variables.
- Cauchy random variables with $\alpha < 1$.

But: there exist non α-regular variation determining random variables.