Numerical Method for Reflected Backward Stochastic Differential Equations

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Outline

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- Discretization of BSDES and Conditional Expectation.
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- The penalization method in the continuous setting.
- Decomposition of the error.
- Main Result and Ideas of the proof.

Backward Stochastic Differential Equations (BSDEs in short)

The unique solution of a BSDE consists of a pair of adapted process (Y, Z) satisfying:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T.$$

- Interest : PDEs, Stochastic Control, Mathematical Finance.
- Existence and Uniqueness results.

BSDE: Briand, Delyon and Mémin, (2001)

Discretization of BSDEs and Conditional Expectation

Time step $h := \frac{1}{n}$, $t_k := \frac{k}{n}$

- The symmetric random walk $W_t^n := \frac{1}{\sqrt{n}} \sum_{k=0}^{c_n(t)} \zeta_k^n$.
- $Y_T^n := \xi^n$ and (Y^n, Z^n) is the unique solution of the **Discrete** BSDE:

$$Y_t^n \;:=\; Y_{t_i}^n + \int_t^{t_i} f(s, Y_s^n; Z_s^n)ds - \int_t^{t_i} Z_s^n dW_s^n$$

- Moreover, if f depends only on y then (Ma, Protter, San Marín, Torres (2002).

$$Y_{t_k}^n = \mathbb{E}\left[Y_{t_{k+1}}^n + hf(Y_{t_k}^n) \,|\, \mathcal{F}_{t_k}^n\right], Z_{t_k}^n := \sqrt{n}\mathbb{E}\left(Y_{t_{k+1}}^n \zeta_{k+1}^n / \mathcal{F}_{t_k}^n\right).$$

Introduction

The case of RBSDEs Martínez, San Martín, Torres (2007)
The solution of a RBSDE with obstacle (S_t) and coefficient f consists of a triple of progressive measurable processes (Y, Z, K) satisfying

$$Y_t = \xi + \int_t^T f(s, Y_s; Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t. \qquad (1)$$

- $Y_t > S_t, \qquad 0 \le t \le T$, ($Y$ stays above the barrier S)
- $\mathbb{E}\left[\int_0^T |Z_t|^2 dt\right] < \infty$,
- (K_t) is a continuous increasing process such that $K_0 = 0$ and $\int_0^T (Y_t - S_t)dK_t = 0$.

Hypothesis: RBSDE

The symmetric random walk $W_t^n := \frac{1}{\sqrt{n}} \sum_{k=0}^{c_n(t)} \zeta_k^n$, ζ_k^n is an i.i.d. Bernoulli symmetric sequence.

- (A1) the function f is bounded ;
- (A2) the function f is uniformly Lipschitz with respect to variables (y, z) ;
- (A3) the barrier S is assumed to be almost surely constant.

- Additional Hypothesis : (H)

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{s\in[0,T]} \left| \mathbb{E}[\xi \mid \mathcal{F}_s] - \mathbb{E}[\xi^n \mid \mathcal{F}^n_{c_n(s)}] \right| \right] = 0.$$

Hypothesis

The strongest assumption is **(A3)**. For the general case, we consider $S_t = S_0 + \int_0^t J_s ds + \int_0^t H_s dB_s$, then $R_t = Y_t - S_t$ satisfies the following RBSDE:

$$R_t = \hat{\xi} + \int_t^T \hat{f}(s, R_s, \Gamma_s)ds - \int_t^T \Gamma_s dB_s + K_T - K_t \quad 0 \le t \le T,$$

$$R_t \ge 0, \quad 0 \le t \le T, \quad \text{and} \quad \int_0^T R_t dK_t = 0,$$

where $\hat{\xi} = \xi + S_T$, $\Gamma_t = Z_t + H_t$, and $\hat{f}(s, r, \gamma) = f(s, r + S_s, \gamma - H_s) + J_s$.

Numerical Method for RBSDE

The method is based in two steps:

- Step I: The penalization term and Picard's ieration procedure in the continuous case. In this case we follow with the main ideas given in N. El Karoui et al.
- Step II: The penalization term and Picard's ieration procedure in the discrete case. In this step we will follow the ideas given in Briand et al.

Step I: Penalization CP

For each $\varepsilon > 0$, let $\{(Y_t^\varepsilon, Z_t^\varepsilon);\ 0 \le t \le 1\}$ denote the unique pair of progressively measurable $\mathcal{F}_t$ processes with values in $\mathbb{R} \times \mathbb{R}$ satisfying the following BSDE:

$$Y_t^\varepsilon = \xi + \int_t^1 f(s, Y_s^\varepsilon, Z_s^\varepsilon)ds - \int_t^1 Z_s^\varepsilon dB_s + \frac{1}{\varepsilon}\int_t^1 (S - Y_s^\varepsilon)^+ ds, \quad (2)$$

$$K_t^\varepsilon := \frac{1}{\varepsilon}\int_0^t (S - Y_s^\varepsilon)^+ ds, \quad 0 \le t \le 1. \quad (3)$$

$$\mathbb{E}\left[\int_0^1 |Y_t^\varepsilon - Y_t|^2\, dt + \int_0^1 |Z_t^\varepsilon - Z_t|^2 dt + \sup_{0\le t\le 1} |K_t^\varepsilon - K_t|^2\right] \to 0 \quad (4)$$

as $\varepsilon \to 0$. We follow the proof given in El Karoui et al.

Step I:Picard's iteration procedure CPI

We set $Y_t^{\varepsilon,0} = 0$, $Z_t^{\varepsilon,0} = 0$, $0 \leq t \leq T$.
For $p \in \mathbb{N}$, we define $(Y_t^{\varepsilon,p+1}, Z_t^{\varepsilon,p+1})$ by recurrence through Picard's Iteration:

$$\begin{aligned} Y_t^{\varepsilon,p+1} &= \xi + \int_t^T f(s, Y_s^{\varepsilon,p}, Z_s^{\varepsilon,p})ds - \int_t^T Z_s^{\varepsilon,p+1} dB_s \\ &\quad + K_T^{\varepsilon,p} - K_t^{\varepsilon,p}. \end{aligned}$$

where $K_t^{\varepsilon,p} = \frac{1}{\varepsilon}\int_0^t (S - Y_s^{\varepsilon,p})^+ ds$. Then

$$\|(Y^\varepsilon - Y^{\varepsilon,p}, Z^\varepsilon - Z^{\varepsilon,p})\|_\beta \to 0, \quad \text{as} \quad p \quad \text{tends to} \quad \infty. \tag{5}$$

Discrete Penalization term

For $t \in [t_{i-1}, t_i[$, and for each $\varepsilon > 0$, let $\{(Y_t^{\varepsilon,\infty,n}, Z_t^{\varepsilon,\infty,n}, K_t^{\infty,n});\ 0 \le t \le 1\}$ denote the unique pair of progressively measurable $\mathcal{F}_t$ processes with values in $\mathbb{R} \times \mathbb{R}$ satisfying the following discrete BSDE:

$$Y_t^{\varepsilon,\infty,n} = Y_{t_i}^{\varepsilon,\infty,n} + \int_t^{t_i} f(s, Y_s^{\varepsilon,\infty,n}, Z_s^{\varepsilon,\infty,n})ds - \int_t^{t_i} Z_s^{\varepsilon,\infty,n} dW_s^n + K_{t_i}^{\varepsilon,\infty,n} - K_t^{\varepsilon,\infty,n};$$

$$Y_1^{\varepsilon,\infty,n} = \xi^n.$$

where $K_0^{\varepsilon,\infty,n} = 0$ and for $t \in]t_{i-1}, t_i[$ and we define

$$K_t^{\varepsilon,\infty,n} := \frac{1}{n\varepsilon} \sum_{j=1}^{i} \left(S - Y_{t_{j-1}}^{\varepsilon,\infty,n}\right)^+ . \qquad (6)$$

Implicit Discrete Time BSDE

We introduce the following implicit discrete-time scheme BSDE :

$$\begin{aligned} Y_{t_i}^{\varepsilon,\infty,n} &= Y_{t_{i+1}}^{\varepsilon,\infty,n} + \frac{1}{n} f\left(t_i, Y_{t_i}^{\varepsilon,\infty,n}, Z_{t_i}^{\varepsilon,\infty,n}\right) + \frac{1}{\varepsilon}\left(S - Y_{t_i}^{\varepsilon,\infty,n}\right)^+ \\ &- \frac{1}{\sqrt{n}} Z_{t_i}^{\varepsilon,\infty,n} \zeta_{i+1}, \end{aligned} \tag{7}$$

for $i \in \{n-1, \ldots, 0\}$, with $Y_1^{\varepsilon,\infty,n} = \xi^n$.

Picard's iteration procedure

An explicit solution of (7) can be found using a discrete Picard's iteration method. Let us set $Y^{\varepsilon,0,n} \equiv 0$, $Z^{\varepsilon,0,n} \equiv 0$, we define $\left(Y^{\varepsilon,p+1,n}, Z^{\varepsilon,p+1,n}\right)$ by induction as the solution of the iterated discrete-time scheme BSDE :

$$\begin{aligned} Y_{t_i}^{\varepsilon,p+1,n} &= Y_{t_{i+1}}^{\varepsilon,p+1,n} + \frac{1}{n} f\left(t_i, Y_{t_i}^{\varepsilon,p,n}, Z_{t_i}^{\varepsilon,p,n}\right) + \frac{1}{\varepsilon}\left(S - Y_{t_i}^{\varepsilon,p,n}\right)^+ \\ &- \frac{1}{\sqrt{n}} Z_{t_i}^{\varepsilon,p+1,n} \zeta_{i+1} \qquad (8) \end{aligned}$$

Lemma

There exists $\alpha_\varepsilon > 1$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, for all $p \in \mathbb{N}^*$,

$$\left\|\left(Y^{\varepsilon,p+1,n} - Y^{\varepsilon,p,n}, Z^{\varepsilon,p+1,n} - Z^{\varepsilon,p,n}\right)\right\|_{\alpha_\varepsilon}^2 \leq \frac{1}{4}\left\|\left(Y^{\varepsilon,p,n} - Y^{\varepsilon,p-1,n}, Z^{\varepsilon,p,n} - Z^{\varepsilon,p-1,n}\right)\right\|_{\alpha_\varepsilon}^2$$ for $p \in \mathbb{N}$,

$$\left\|\left(Y^{\varepsilon,p+1,n} - Y^{\varepsilon,p,n}, Z^{\varepsilon,p+1,n} - Z^{\varepsilon,p,n}\right)\right\|_{\alpha_\varepsilon}^2 :=$$

$$\mathbb{E}\left[\sup_{0\leq k\leq n} {\alpha_\varepsilon}^{k/n}|Y^{\varepsilon,p+1,n} - Y^{\varepsilon,p,n}|^2\right] + \frac{1}{n}\mathbb{E}\left[\sum_{k=0}^{n-1} {\alpha_\varepsilon}^{k/n}|Z^{\varepsilon,p+1,n} - Z^{\varepsilon,p,n}|^2\right]$$

Main Result

Under the assumptions **(A1), (A2), (A3)** and **(H)**, the triplet $(\xi^n, Y^{\varepsilon,\infty,n}, Z^{\varepsilon,\infty,n}, K^{\varepsilon,\infty,n})$ converges in the Skorohod topology towards the solution (ξ, Y, Z, K) of the RBSDE (1).

Idea of the Proof The main idea of the proof is the following decomposition of the error:

$$Y_t - Y_t^{\varepsilon,\infty,n} = (Y_t - Y_t^{\varepsilon}) + (Y_t^{\varepsilon} - Y_t^{\varepsilon,p}) + (Y_t^{\varepsilon,p} - Y_t^{\varepsilon,p,n}) + (Y_t^{\varepsilon,p,n} - Y_t^{\varepsilon,\infty,n}),$$

the first term corresponds to penalization term in the continuous setting, the second one is the Picard's iteration procedure for the continuous BSDE, the third term is the discretization of a BSDE by using a random walk instead of the Brownian motion, and the last term is related to a Picard's iteration procedure in the discrete case.

Main Idea of the Proof

The main idea of the proof is the intermediate result:

Proposición 1 *Let the assumptions* **(A1), (A2), (A3)**, *and hypothesis* **(H)**. *Let us consider the scaled random walks* W^n. *We have that for each fixed* $\varepsilon \in]0,1]$,

$$\sup_{0\le t\le 1}\left|Y_{t-}^{\varepsilon,p,n}-Y_t^{\varepsilon,p}\right|+\int_0^1\left|Z_{s-}^{\varepsilon,p,n}-Z_s^{\varepsilon,p}\right|^2 ds\to 0. \qquad (9)$$

as $n\to+\infty$; *in probability.*

A new procedure: Ma and Zhang's method

The 2-step scheme in the discrete case:

- $Y_1^n := \xi^n$.
- for $i = n, n-1, \ldots 1$, and $t \in [t_{i-1}, t_i[$, let $\left(\tilde{Y}^n, Z^n\right)$ be the solution of the BSDE:

$$\tilde{Y}_t^n = Y_{t_i}^n + \int_t^{t_i} f\left(s, \tilde{Y}_s^n, Z_s^n\right) ds - \int_t^{t_i} Z_s^n dW_s^n. \quad (10)$$

  where, for each $i = n-1, \ldots 0$, we defined

$$Y_{t_{i+1}}^n := \tilde{Y}_{t_{i+1}}^n \vee S = \tilde{Y}_{t_{i+1}}^n + \left(S - \tilde{Y}_{t_{i+1}}^n\right)^+$$

- Comparison with penalization procedure?

Ma and Zhang's method

We define a **Modified Picard's iteration procedure** for a penalization discrete BSDE $(Y^{\varepsilon,\infty,n}, Z^{\varepsilon,\infty,n})$.
We define $\ddot{Y}^{\varepsilon,p+1,n}$ for $i = 0, \dots, n-1$, by

$$\ddot{Y}^{\varepsilon,p+1,n}_{t_i} = \ddot{Y}^{\varepsilon,p+1,n}_{t_{i+1}} + \frac{1}{n} f\left(t_i, \ddot{Y}^{\varepsilon,p,n}_{t_i}, \ddot{Z}^{\varepsilon,p,n}_{t_i}\right)$$

$$-\frac{1}{\sqrt{n}} \ddot{Z}^{\varepsilon,p+1,n}_{t_i} \zeta^n_{i+1} + \left(\ddot{K}^{\varepsilon,p+1,n}_{t_{i+1}} - \ddot{K}^{\varepsilon,p+1,n}_{t_i}\right)$$

$$\ddot{Y}^{\varepsilon,p,n}_1 := \xi^n$$

$$\ddot{K}^{\varepsilon,p+1,n}_{t_{i+1}} - \ddot{K}^{\varepsilon,p+1,n}_{t_i} := \frac{1}{n\varepsilon}\left(S - \ddot{Y}^{\varepsilon,p+1,n}_{t_i}\right)^+ .$$

The main difference between this approximation and the Picard's iteration procedure is that instead of p we use $p+1$ in the last two terms.

Main Result

$(\tilde{Y}_{t_i}^{p,n}, Z_{t_i}^{p,n})$ denotes the the Picard iteration procedure for the couple of processes $(\tilde{Y}_{t_i}^{n}, Z_{t_i}^{n})$ defined as the solution of the BSDE equation (10)

Theorem Assume $(\mathbf{A1})$-$(\mathbf{A3})$ and $\mathbf{H}$. Then, for all $p \in \mathbb{N}$,

$$\lim_{\varepsilon \to 0} \lim_{n \to +\infty} \mathbb{E}\left[\sup_{0 \le i \le n} \left\{ \left| \tilde{Y}_{t_i}^{p,n} - \ddot{Y}_{t_i}^{\varepsilon,p,n} \right|^2 + \frac{1}{n} \sum_{i=0}^{n} \left| Z_{t_i}^{p,n} - \ddot{Z}_{t_i}^{\varepsilon,p,n} \right|^2 \right\}\right] = 0.$$

Picard modified method

Lemma There exists λ fixed in $]0,1[$ such that the map $\Phi : \left(\tilde{Y}^{p,n}, Z^{p,n}\right) \mapsto \left(\tilde{Y}^{p+1,n}, Z^{p+1,n}\right)$ is contractive on $[\lambda, 1]$ for the norm $|||\,.\,|||$ defined by

$$|||\left(\tilde{Y}, Z\right)||| := \left\{ \frac{1}{n} \sum_{t_i \geq \lambda}^{1} \frac{I\!\!E\left[|\tilde{Y}_{t_i}|^2 + |Z_{t_i}|^2\right]}{(1-4/n)^i} \right\}^{1/2}, \quad (11)$$

uniformly on n.

Idea of the proof : a fundamental Lemma

For all $n \in \mathbb{N}, \;\; p \in \mathbb{N}, \;\; 0 \leq i \leq n$,

$$|S - \tilde{Y}_{t_i}^{p+1,n}| \mathbf{1}_{\tilde{Y}_{t_i}^{p+1,n} \leq S} \leq \frac{1}{n} ||f||_\infty$$

American Options

An American option is a one that can be exercised at any time between the purchase date and the expiration date T. We consider the price of the risk asset $S = (S_t)_{0 \leq t \leq T}$ and the wealth process $Y = (Y_t)_{0 \leq t \leq T}$. We assume that the rate interest r is constant. The aim is to obtain Y_0, the value of the American Option.

The equation that describes the evolution of Y is given by a linear reflected BSDE coupled with the forward equation for S.

$$
\begin{aligned}
Y_t &= (K - S_1)^+ - \int_t^1 (rY_s + (\mu - r)Z_s)\, ds + K_1 - K_t - \int_t^1 Z_s dB_s, \\
S_t &= S_0 + \int_0^t \mu S_s ds + \int_0^t \sigma S_s dB_s.
\end{aligned}
$$

American Options

The increasing process K keeps the process Y above the barrier $L_t = (S_t - K)^+$ (for a call option) in a minimal way, that is $Y_t \geq L_t,\ dK_t \geq 0$ and

$$\int_0^1 (Y_t - L_t)dK_t = 0.$$

Simulation

n	$S_0 = 80$	$S_0 = 100$	$S_0 = 120$
1	20	11.2773	4.1187
2	22.1952	10.0171	3.8841
3	21.8707	10.7979	3.1489
4	22.8245	10.1496	3.9042
5	22.4036	10.9673	3.4262
14	22.6062	10.5968	3.5636
15	22.6775	10.8116	3.7119
16	22.6068	10.6171	3.6070
17	22.7144	10.7798	3.6811
18	22.6271	10.6125	3.6364
Real Values	**21.6059**	**9.9458**	**4.0611**

Simulation

						Node 7.1 260,88728
					Node 6.1 222,35356	
				Node 5.1 189,51137		Node 7.2 188,266912
			Node 4.1 161,520055		Node 6.2 160,459406	
		Node 3.1 137,663129		Node 5.2 136,759141		Node 7.3 135,861089
	Node 2.1 117,3299316		Node 4.2 116,559465		Node 6.3 115,794058	
Node 1.1 100		Node 3.2 99,3433333		Node 5.3 98,6909788		Node 7.4 98,042908
	Node 2.2 84,67006838		Node 4.3 84,1140683		Node 6.4 83,5617192	
		Node 3.3 71,6902048		Node 5.4 71,2194391		Node 7.5 70,7517648
			Node 4.4 60,7001454		Node 6.5 60,3015478	
				Node 5.5 51,3948546		Node 7.6 51,0573618
					Node 6.6 43,5160586	
						Node 7.7 35,8450765

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